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Research Article

DEGREE-BASED INVARIANTS OF MYCIELSKI CONSTRUCTION: IRREGULARITY, TOTAL IRREGULARITY, VARIANCE

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ABSTRACT

The degree-based graph invariants are parameters defined by degrees of vertices. A graph is *regular* if all of its vertices have the same degree. Otherwise a graph is *irregular*. To measure how irregular a graph is, graph topological indices were proposed including the *irregularity of a graph*, *total irregularity of a graph*, and the *variance of the vertex degrees*. In this paper, the above mentioned irregularity measures for Mycielski constructions of any underlying graph are considered and exact formulae are derived.
Keywords: Irregularity of a graph, total irregularity of a graph, variance of the vertex degrees, Mycielski construction.
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1. INTRODUCTION

Let G be a simple undirected graph with vertex set V and edge set E. The order of G is the number of vertices in G. If $|V| = n$ and $|E| = m,$ we say that G is an (n,m)-graph. The *open neighborhood* of a vertex v in a graph $G,$ denoted by $N_G(v),$ is the set of all vertices of G which are adjacent to v. The *degree* of a vertex v in $G,$ denoted by $d_G(v)$ is the cardinality of $N_G(v)$. A graph G is *regular* if all of its vertices have the same degree, otherwise it is *irregular*.

Graph theoretical methods are all used in the characterization of molecular structure and prediction of properties, especially in chemical graph theory. Many of the problems, especially in computer network design can be easily handled if the related graphs are regular or close to regular. It is of interest to measure the irregularity of chemical graphs both for descriptive purposes and for QSAR/QSPR studies. Therefore, it is of great importance to know how irregular a given graph is in many applications and problems such as analyzing the structure of deterministic and random networks and systems occurring in chemistry, biology and social networks [6,7]. For that purpose, several graph topological indices have been proposed.

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Irregularity me asures are expected to be of practical value in QSAR/QSPR studies. Among the most investigated ones are the irregularity of a graph introduced by Albertson [1], the total irregularity of a graph [2], and the variance of vertex degrees [3].

The irregularity of a graph can be defined by different graph topological indices. In this paper, three graph topological indices that quantify the irregularity of a graph are considered. The irregularities of graphs are investigated with respect to the irregularity of a graph, the total irregularity of a graph, and the variance of the vertex degrees. It is known that these irregularity measures are not always compatible.

The three irregularity measures of interest in this paper are presented next.

Albertson [1] defines the *imbalance* of an edge $e = uv \in E$ as $\left|d_G(u) - d_G(v)\right|$ and the *irregularity* of G as

$$irr(G) = \sum_{uv \in E(G)} \left|d_G(u) - d_G(v)\right|.$$

Upper bounds on irregularity for bipartite graphs, triangle-free graphs, and a sharp upper bound for trees were presented in [1]. For general graphs with n vertices, Albertson [1] has obtained an asymptotically tight upper bound on the irregularity, and in [18], for general graphs with n vertices a sharp upper bound was presented. The graphs with maximal irregularity were characterized in [8].

In [2], a new measure of irregularity, so-called the *total irregularity*, was recently defined as

$$irr_t(G) = \frac{1}{2} \sum_{u,v \in V(G)} \left|d_G(u) - d_G(v)\right|.$$

The upper bound of the total irregularity among all graphs was obtained in [2], and it was shown that the star graph is the tree with maximal total irregularity among all trees. In [9], the unicyclic graphs with maximal total irregularity among all unicyclic graphs were determined. The bicyclic graphs with maximal total irregularity among all bicyclic graphs were characterized in [10]. In [11], the graph with the minimal, the second minimal, and the third minimal total irregularity among trees, unicyclic or bicyclic graphs was characterized.

The relation between $irr(G)$ and $irr_t(G)$ for a connected graph G was derived in [12]. $irr_t(G)$ can be computed directly from the sequence of the vertex degrees of G. $irr_t(G)$ has a property that the graphs with the same degree sequences have the same total irregularity, while $irr(G)$ does not have. The most irregular graphs with respect to $irr(G)$ are graphs that have only two degrees. The most irregular graphs with respect to $irr_t(G)$ are graphs with maximal number of different vertex degrees [2].

A sequence of non-negative integers $d_1, \ldots, d_n$ is a degree sequence, if there exists a graph G with $V(G) = \{v_1, \ldots, v_n\}$ such that $d(v_i) = d_i$. Let n_i denote the number of vertices of degree i for $1 \le i \le n-1$ and let $d_1, \ldots, d_n$ denote the degree sequence of the graph G, where n_i is the number of vertices of degree i for $1 \le i \le n-1$. The variance $Var(G)$ of the vertex degrees [3] of the graph G is

$$Var(G)=\frac{1}{n}\sum_{i=1}^{n}d_i^{\,2}-\frac{1}{n^2}\left(\sum_{i=1}^{n}d_i\right)^2=\frac{1}{n}\sum_{i=1}^{n-1}n_i\left(i-\frac{2m}{n}\right)^2.$$

Bell [3] characterized the most irregular graphs in some classes and obtained upper and lower bounds of $Var(G)$ as functions of n and m.

These irregularity measures as well as other attempts to measure the irregularity of a graph were studied in several works [4,5,8,9,10,12,13,14,15,16,17,18,19,20,21,22,23].

In [24], Mycielski developed a graph transformation that transforms a graph G into a new graph $\mu(G)$ called *Mycielskian of* G. For a given graph G with $V=V(G)=\{v_1,v_2,\ldots,v_n\}$, denote $V'=\{v_1{}',v_2{}',\ldots,v_n{}'\}$ to be the corresponding set of V, the *Mycielski graph* $\mu(G)$ of G is defined with vertex set consisting of the disjoint union $V(\mu(G))=V\cup V'\cup\{u\}$, and edge set $E(\mu(G))=E(G)\cup\{v_iv_j{}':v_iv_j\in E(G)\}\cup\{uv_i{}':1\le i\le n\}$. We call v' the twin of v in $\mu(G)$, and vice versa, and call u the root of $\mu(G)$.

In Section 2, the irregularity, total irregularity, and the variance of the vertex degrees of Mycielski graphs are determined in terms of the same parameters over G.

Remark 1.1. In [17], exact formulae are given for the irregularity, total irregularity, and the variance of the vertex degrees of Mycielskian of paths and cycles. Motivated from the results in [17], in this paper, exact general formulae are derived for those three irregularity measures of Mycielskian of any underlying graph G.

2. MAIN RESULTS

An obvious inference from the definition of $\mu(G)$, the order of $\mu(G)$ is $2n+1$, and

$$\mathrm{d}_{\mu(G)}(v_i)=2d_G(v_i),\ \mathrm{d}_{\mu(G)}(v_i')=d_G(v_i)+1,\ \text{and}\ \mathrm{d}_{\mu(G)}(u)=n.$$

The vertex set of $\mu(G)$ can be partitioned into three subsets as

$$V_1=\{v_i\in V(\mu(G)):v_i\in V(G),1\le i\le n\},$$

$$V_2=\{v_i'\in V(\mu(G)):v_i'\in V',1\le i\le n\},\ \text{and}$$

$$V_3=\{u\in V(\mu(G)):u\ \text{is the root of}\ \mu(G)\}.$$

The edge set of $\mu(G)$ can be partitioned into three subsets as

$$E_1 = \{xy \in E(\mu(G)) : x, y \in V(G)\},$$

$$E_2 = \{xy \in E(\mu(G)) : x \in V(G), y = y' \in V'\}, \text{ and}$$

$$E_3 = \{xy \in E(\mu(G)) : x = x' \in V', y \text{ is the root of } \mu(G)\}.$$

Theorem 2.1. *Let* G *be* (n,m)*-graph. Then*

$$irr(\mu(G)) = 2irr(G) + n(n-1) - 2m + \sum_{i=1}^{n} \sum_{\substack{u_j \in N_G(v_i) \\ 1 \le j \le d_G(v_i)}} \left|2d_G(v_i) - (d_G(u_j)+1)\right| \cdot$$

Proof. By the definition of graph irregularity, it follows that

$$irr(\mu(G)) = \sum_{uv \in E(\mu(G))} \left|d_{\mu(G)}(u) - d_{\mu(G)}(v)\right| = \sum_{xy \in E_i} \left|d_{\mu(G)}(x) - d_{\mu(G)}(y)\right|$$

for $1 \le i \le 3$.

The contribution of the edges in E_1 to $irr(\mu(G))$ is given by

$$irr_1(\mu(G)) = \sum_{xy \in E_1} \left|d_{\mu(G)}(x) - d_{\mu(G)}(y)\right| = \sum_{xy \in E_1} \left|2d_G(x) - 2d_G(y)\right|$$
$$= 2\sum_{xy \in E_1} \left|d_G(x) - d_G(y)\right| = 2\sum_{xy \in E(G)} \left|d_G(x) - d_G(y)\right| = 2irr(G).$$

The contribution of the edges in E_2 to $irr(\mu(G))$ is given by

$$irr_2(\mu(G)) = \sum_{xy \in E_2} \left|d_{\mu(G)}(x) - d_{\mu(G)}(y)\right| = \sum_{xy \in E_2} \left|2d_G(x) - d_{\mu(G)}(y')\right|$$
$$= \sum_{xy \in E_2} \left|2d_G(x) - (d_G(y)+1)\right| = \sum_{i=1}^{n} \sum_{\substack{u_j \in N_G(v_i) \\ 1 \le j \le d_G(v_i)}} \left|2d_G(v_i) - (d_G(u_j)+1)\right|.$$

The contribution of the edges in E_3 to $irr(\mu(G))$ is given by

$$irr_3(\mu(G)) = \sum_{xy \in E_3} \left|d_{\mu(G)}(x) - d_{\mu(G)}(y)\right| = \sum_{xy \in E_3} \left|d_{\mu(G)}(x') - n\right|$$
$$= \sum_{xy \in E_3} \left|d_G(x) + 1 - n\right|.$$

Under the constraints $0 \le d_G(x) \le n-1 \ \ \forall x \in V(G)$ and $n \ge 1$, it holds that $d_G(x) + 1 - n \le 0$. Thus,

$$irr_3(\mu(G)) = \sum_{xy \in E_3} n - d_G(x) - 1 = \sum_{i=1}^{n} n - d_G(x_i) - 1 = n(n-1) - \sum_{i=1}^{n} d_G(x_i).$$

By using the equality $\sum_{v\in V(G)} d_G(v) = 2|E(G)|$, we compute the summation as follows:

$$irr_3(\mu(G)) = n(n-1) - 2m.$$

The desired formula for $irr(\mu(G))$ is obtained by summing the above three expressions. ■

Theorem 2.2. *Let* G *be* (n,m)*-graph. Then*

$$irr_t(\mu(G)) = 3irr_t(G) + n(n-2) + \sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\left|2d_G(v_i) - \left(d_G(v_j)+1\right)\right| + \sum_{i=1}^{n}\left|2d_G(u_i) - n\right| \cdot$$

Proof. By the definition of graph total irregularity, it follows that

$$irr_t(\mu(G)) = \frac{1}{2}\sum_{\substack{u\in V_i, v\in V_j\\ 1\le i,j\le 3}}\left|d_{\mu(G)}(u) - d_{\mu(G)}(v)\right|.$$

The contribution of the vertices in V_1 to $irr_t(\mu(G))$ is given by

$$irr_{t_1}(\mu(G)) = \frac{1}{2}\sum_{\substack{u\in V_1, v\in V_i\\ (1\le i\le 3)}}\left|d_{\mu(G)}(u) - d_{\mu(G)}(v)\right|.$$

$$\frac{1}{2}\sum_{u,v\in V_1}\left|d_{\mu(G)}(u) - d_{\mu(G)}(v)\right| = \frac{1}{2}\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\left|d_{\mu(G)}(v_i) - d_{\mu(G)}(v_j)\right|$$

$$= \frac{1}{2}\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\left|2d_G(v_i) - 2d_G(v_j)\right|$$

$$= \sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\left|d_G(v_i) - d_G(v_j)\right| = \sum_{u,v\in V(G)}\left|d_G(u) - d_G(v)\right| = 2irr_t(G). \quad (1)$$

$$\frac{1}{2}\sum_{u\in V_1, v\in V_2}\left|d_{\mu(G)}(u) - d_{\mu(G)}(v)\right| = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left|d_{\mu(G)}(v_i) - d_{\mu(G)}(v_j')\right|$$

$$= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left|2d_G(v_i) - \left(d_G(v_j)+1\right)\right|$$

$$= \frac{1}{2}\sum_{i=1}^{n}\left(d_G(v_i) - 1\right) + \frac{1}{2}\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\left|2d_G(v_i) - \left(d_G(v_j)+1\right)\right|$$

$$= m - \frac{n}{2} + \frac{1}{2}\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\left|2d_G(v_i) - \left(d_G(v_j)+1\right)\right|. \quad (2)$$

$$\frac{1}{2}\sum\nolimits_{u\in V_1,v\in V_3}\left|d_{\mu(G)}(u)-d_{\mu(G)}(v)\right|=\frac{1}{2}\sum_{i=1}^{n}\left|2d_G(u_i)-n\right|. \tag{3}$$

From (1), (2), and (3), we get

$$irr_{t_1}(\mu(G))=2irr_t(G)+m-\frac{n}{2}+\frac{1}{2}\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\left|2d_G(v_i)-\left(d_G(v_j)+1\right)\right|+\frac{1}{2}\sum_{i=1}^{n}\left|2d_G(u_i)-n\right|\cdot$$

The contribution of the vertices in V_2 to $irr_t(\mu(G))$ is given by

$$irr_{t_2}(\mu(G))=\frac{1}{2}\sum\nolimits_{\substack{u\in V_2,v\in V_i\\(1\leq i\leq 3)}}\left|d_{\mu(G)}(u)-d_{\mu(G)}(v)\right|.$$

For the case $u\in V_2$, $v\in V_1$, we receive the same equality as in (2).

$$\begin{aligned}\frac{1}{2}\sum\nolimits_{u,v\in V_2}\left|d_{\mu(G)}(u)-d_{\mu(G)}(v)\right|&=\frac{1}{2}\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\left|d_{\mu(G)}(u_i')-d_{\mu(G)}(v_j')\right|\\
&=\frac{1}{2}\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\left|\left(d_G(u_i)+1\right)-\left(d_G(v_j)+1\right)\right|=\frac{1}{2}\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\left|d_G(u_i)-d_G(v_j)\right|\\
&=\frac{1}{2}\sum\nolimits_{u,v\in V(G)}\left|d_G(u)-d_G(v)\right|\\
&=irr_t(G).\end{aligned} \tag{4}$$

$$\begin{aligned}\frac{1}{2}\sum\nolimits_{u\in V_2,v\in V_3}\left|d_{\mu(G)}(u)-d_{\mu(G)}(v)\right|&=\frac{1}{2}\sum_{i=1}^{n}\left|d_{\mu(G)}(u_i')-n\right|\\
&=\frac{1}{2}\sum_{i=1}^{n}\left|\left(d_G(u_i)+1\right)-n\right|.\end{aligned}$$

Under the constraint $0\leq d_G(v)\leq n-1 \quad \forall v\in V(G)$, we have the equality $1-n\leq\left(d_G(v)+1\right)-n\leq 0$ yielding

$$\frac{1}{2}\sum_{i=1}^{n}\left|\left(d_G(u_i)+1\right)-n\right|=\frac{1}{2}\sum_{i=1}^{n}\left|n-d_G(u_i)-1\right|=\frac{n}{2}(n-1)-m. \tag{5}$$

Summing (2), (4), and (5), we receive

$$irr_{t_2}(\mu(G))=irr_t(G)+n\left(\frac{n}{2}-1\right)+\frac{1}{2}\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\left|2d_G(v_i)-\left(d_G(v_j)+1\right)\right|.$$

Similarly, the contribution of the root vertex to $irr_t(\mu(G))$ is given by

$$irr_{t_3}(\mu(G)) = \frac{1}{2}\sum_{\substack{u\in V_3, v\in V_i \\ (1\le i\le 2)}} \left| d_{\mu(G)}(u) - d_{\mu(G)}(v) \right|.$$

For the two necessary cases $u \in V_3$, $v \in V_1$ and $u \in V_3$, $v \in V_2$, the same equalities as in (3) and (5) are obtained yielding

$$irr_{t_3}(\mu(G)) = \frac{n}{2}(n-1) - m + \frac{1}{2}\sum_{i=1}^{n}\left|2d_G(u_i) - n\right|.$$

The desired equality for $irr_t(\mu(G))$ is obtained by summing the above three contributions. ■

Theorem 2.3. *Let* G *be* (n,m)*-graph. Then*

$$Var(\mu(G)) = \left((2n+1)\left(5M_1(G) + 4m + n(n+1)\right) - 4(3m+n)^2\right) \Big/ (2n+1)^2 .$$

Proof. By the definition of the variance of vertex degrees of a graph, we receive

$$Var(\mu(G)) = \frac{1}{(2n+1)}\left(\sum_{\substack{v\in V_i \\ (1\le i\le 3)}} d_{\mu(G)}(v)^2\right) - \frac{1}{(2n+1)^2}\left(\sum_{\substack{v\in V_i \\ (1\le i\le 3)}} d_{\mu(G)}(v)\right)^2$$

$$= \frac{1}{(2n+1)}\left(\sum_{i=1}^{n} d_{\mu(G)}(v_i)^2 + \sum_{i=1}^{n} d_{\mu(G)}(v_i')^2 + n^2\right) - \frac{1}{(2n+1)^2}\left(\sum_{i=1}^{n} d_{\mu(G)}(v_i) + \sum_{i=1}^{n} d_{\mu(G)}(v_i') + n\right)^2$$

$$= \frac{1}{(2n+1)}\left(\sum_{i=1}^{n}(2d_G(v_i))^2 + \sum_{i=1}^{n}(d_G(v_i)+1)^2 + n^2\right) - \frac{1}{(2n+1)^2}\left(\sum_{i=1}^{n}2d_G(v_i) + \sum_{i=1}^{n}(d_G(v_i)+1) + n\right)^2.$$

By the definition of the first Zagreb index [25], that is $M_1(G) = \sum_{v\in V(G)} d_G(v)^2$, and by the equality $\sum_{v\in V(G)} d_G(v) = 2|E(G)|$, we proceed as

$$Var(\mu(G)) = \frac{1}{(2n+1)}\left(4M_1(G) + \left(M_1(G) + 2(2m) + n\right) + n^2\right) - \frac{1}{(2n+1)^2}\left(2(2m) + (2m+n) + n\right)^2.$$

Thus, the proof of the theorem holds. ■

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