Sigma J Eng & Nat Sci 37 (2), 2019, 529-540

Sigma Journal of Engineering and Natural Sciences
Sigma Mühendislik ve Fen Bilimleri Dergisi

Research Article

n-TIMES DIFFERENTIABLE PREINVEX AND PREQUASIINVEX FUNCTIONS

Huriye KADAKAL*[1]

[1]*Ministry of Education, Bulancak Bahçelievler Anatolian High School, GIRESUN;*
ORCID: 0000-0002-0304-7192

Received: 09.04.2019 Revised: 03.05.2019 Accepted: 14.05.2019

ABSTRACT

In this manuscript, a new identity for functions defined on an open invex subset of set of real numbers is established. We present new type integral inequalities for functions whose powers of nth derivatives in absolute value are preinvex and prequasiinvex functions. This paper is a generalization of studies being done for functions whose first, second, third and fourth derivatives are preinvex and prequasiinvex. Moreover, the results we obtained in this article coincide with the previous ones in special cases.
Keywords: Invex set, preinvex function, prequasiinvex function, Hölder integral inequality, power-mean integral inequality.
AMS classification: 26A51, 26D10, 26D15.

1. PRELIMINARIES AND FUNDAMENTALS

Definition 1. *A function $f: I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if the inequality*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

is valid for all $x, y \in I$ and $t \in [0,1]$. If this inequality reverses, then f is said to be concave on interval $I \neq \emptyset$.

Convexity theory has appeared as a powerful technique to study a wide class of unrelated problems in pure and applied sciences.

Definition 2. $f: I \subseteq \mathbb{R} \to \mathbb{R}$ *be a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$. The following celebrated double inequality*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}$$

is well known in the literature as Hermite-Hadamard's inequality for convex functions.

Both inequalities hold in the reserved direction if f is concave. Some of the classical inequalities for means can be derived from Hermite-Hadamard inequality for particular choices of the function f.

Definition 3. *A function $f: I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be quasi-convex if the inequality*

* Corresponding Author: e-mail: huriyekadakal@hotmail.com, tel: (454) 318 20 77

$f(tx+(1-t)y) \leq max\{f(x), f(y)\}$

holds for all $x, y \in I$ *and* $t \in [0,1]$.

Clearly, any convex function is a quasi-convex function. Furthermore, there exist quasi-convex functions which are not convex [5].

Let us recall the notions of preinvexity and prequasiinvexity which are signicant generalizations of the notions of convexity and qusi-convexity respectively, and some related results.

Definition 4. *Let* K *be a non-empty subset in* $\mathbb{R}^n$ *and* $\eta: K \times K \to \mathbb{R}^n$. *Let* $x \in K$, *then the set* K *is said to be invex at* x *with respect to* $\eta(\cdot,\cdot)$, *if* $x + t\eta(y,x) \in K, \forall x, y \in Kt \in [0,1]$. K *is said to be an invex set with respect to if* K *is invex at each* $x \in K$. *The invex set* K *is also called* η-*connected set.*

Note that, if we demand that y should be an end point of the path for every pair of points $x, y \in K$, then $\eta(y,x) = y - x$, and consequently invexity reduces to convexity. Thus, it is true that every convex set is also an invex set with respect to $\eta(y,x) = y - x$, but the converse is not necessarily true, see [14, 18] and the references therein. For the sake of simplicity, we always assume that $K = [x, x + t\eta(y,x)]$, unless otherwise specified [1].

Definition 5. [17] *A function* $f: K \to \mathbb{R}$ *on an invex set* $K \subseteq \mathbb{R}$ *is said to be preinvex with respect to* η, *if*

$$f(u + t\eta(v,u)) \leq (1-t)f(u) + tf(v), \forall u, v \in K, t \in [0,1].$$

The function f *is said to be preconcave if and only if* $-f$ *is preinvex.*

It is to be noted that every convex function is preinvex with respect to the map $\eta(y,x) = x - y$ but the converse is not true see for instance.

Definition 6. [2] *A function* $f: K \to \mathbb{R}$ *on an invex set* $K \subseteq \mathbb{R}$ *is said to be prequasiinvex with respect to* η, *if*

$$f(u + t\eta(v,u)) \leq max\{f(u), f(v)\}, \forall u, v \in K, t \in [0,1].$$

Also every quasi-convex function is a prequasiinvex with respect to the map $\eta(v,u) = v - u$ but the converse does not hold, see for example [2].

Mohan and Neogy [14] introduced Condition C defined as follows:

Definition 7. [14] *Let* $S \subseteq \mathbb{R}$ *be an open invex subset with respect to the map* $\eta: S \times S \to \mathbb{R}$. *We say that the function satisfies the Condition* C *if, for any* $x, y \in S$ *and any* $t \in [0,1]$,

$$\eta(y, y + t\eta(x,y)) = -t\eta(x,y) \tag{1.1}$$

$$\eta(x, y + t\eta(x,y)) = (1-t)\eta(x,y). \tag{1.2}$$

Note that, from the Condition C, we have $\eta(y + t_2\eta(x,y)y, y + t_1\eta(x,y)) = (t_2 - t_1)\eta(x,y)$ for any $x, y \in S$ and any $t_1, t_2 \in [0,1]$.

Many mathematicians have been studying about preinvexity and types of preinvexity. A lot of researchs have been made by many mathematicians to generalize the classical convexity. These studies include, among others, the work of [4, 16, 17, 19]. In this papers have been studied the basic properties of the preinvex functions and their role in optimization, variational inequalities and equilibrium problems.

Noor has obtained the following Hermite-Hadamard type inequalities for the preinvex functions [15].

Theorem 1. [15] *Let* $f: [a, a + \eta(b,a)] \to (0, \infty)$ *be a preinvex function on the interval of the real numbers* K° *(the interior of* K*) and* $a, b \in K^\circ$ *with* $\eta(b,a) > 0$. *Then the following inequalities holds*

$$f\left(\frac{2a+\eta(b,a)}{2}\right) \leq \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.3}$$

For several recent results on inequalities for preinvex and prequasiinvex functions which are connected to (1.3), we refer the interested reader to [3, 6, 12, 13] and the references therein. Let $0 < a < b$, throughout this paper we will use

$$A = A(a,b) = \frac{a+b}{2}$$

$$L_p(a,b) = \left(\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}}, a \neq b, p \in \mathbb{R}, \qquad p \neq -1,0$$

for the arithmetic and generalized logarithmic mean, respectively. Moreover, for shortness, the following notations will be used:

$$\delta(n) = \delta(a,b,\eta) = a + \frac{\eta(b,a)}{n}, \quad \delta_t(n) = \delta_t(a,b,\eta) = a + t\frac{\eta(b,a)}{n},$$

$$I_f(a,b,\eta) := \sum_{k=1}^{n} \frac{(-1)^{k-1}\eta^{k-1}(b,a)}{(k-1)!}\left(a + \frac{\eta(b,a)}{k}\right) f^{(k-1)}(a + t\eta(b,a)) - af(a) - \int_a^{a+\eta(b,a)} f(x)dx$$

In this paper, using a general integral identity for a n-times differentiable functions, we establish some new type integral inequalities for mappings whose nth derivative in absolute value at certain powers are preinvex and prequasiinvex.

2. MAIN RESULTS FOR OUR LEMMA

We will use the following Lemma for obtain our main results about the generalisation of the integral inequalities related to preinvexity and prequasiinvexity.

Lemma 1. *Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to mapping $\eta(\cdot,\cdot): K \times K \to \mathbb{R}, n \in \mathbb{N}$ and $a, b \in K$ with $\eta(b,a) > 0$. Suppose that the function $f: K \to \mathbb{R}$ is a n-times differentiable function on K such that $f^{(n)} \in L[a, a + \eta(b,a)]$. Then the following identity hold:*

$$\sum_{k=1}^{n} \frac{(-1)^{k-1}\eta^{k-1}(b,a)}{(k-1)!}\left(a + \frac{\eta(b,a)}{k}\right) f^{(k-1)}(a + t\eta(b,a)) - af(a) - \int_a^{a+\eta(b,a)} f(x)dx \tag{2.1}$$
$$= (-1)^{n-1}\eta^n(b,a) \int_0^1 \frac{t^{n-1}}{(n-1)!}\left(a + t\frac{\eta(b,a)}{n}\right) f^{(n)}(a + t\eta(b,a))dt.$$

Proof. To prove, we shall use the induction method. For $n = 1$, by integration by parts and changing the variable, we have

$$(a + \eta(b,a))f(a + \eta(b,a)) - af(a) - \int_a^{a+\eta(b,a)} f(x)dx$$
$$= \eta(b,a) \int_0^1 (a + t\eta(b,a))f'(a + t\eta(b,a))dt.$$

This coincides with the equality (2.1) for $n = 1$. Suppose (2.1) holds for $n = m$. That is,

$$\sum_{k=1}^{m} \frac{(-1)^{k-1}\eta^{k-1}(b,a)}{(k-1)!}\left(a + \frac{\eta(b,a)}{k}\right) f^{(k-1)}(a + t\eta(b,a)) - af(a) - \int_a^{a+\eta(b,a)} f(x)dx$$
$$= (-1)^{m-1}\eta^m(b,a) \int_0^1 \frac{t^{m-1}}{(m-1)!}\left(a + t\frac{\eta(b,a)}{m}\right) f^{(m)}\big(a + t\eta(b,a)\big)dt. \tag{2.2}$$

Using the integration by parts in the right hand side of (2.2) we get

$$(-1)^{m-1}\eta^m(b,a)\int_0^1 \frac{t^{m-1}}{(m-1)!}\left(a+t\frac{\eta(b,a)}{m}\right)f^{(m)}\big(a+t\eta(b,a)\big)dt \tag{2.3}$$
$$=\frac{(-1)^{m-1}\eta^m(b,a)}{m!}\left(a+t\frac{\eta(b,a)}{m+1}\right)f^{(m)}(a+t\eta(b,a))$$
$$+(-1)^m\eta^{m+1}(b,a)\int_0^1 \frac{t^m}{m!}\left(a+t\frac{\eta(b,a)}{m+1}\right)f^{(m+1)}(a+t\eta(b,a))dt.$$

Substituting (2.3) in (2.2) we obtain

$$\sum_{k=1}^{m}\frac{(-1)^{k-1}\eta^{k-1}(b,a)}{(k-1)!}\left(a+\frac{\eta(b,a)}{k}\right)f^{(k-1)}(a+t\eta(b,a))-af(a)-\int_a^{a+\eta(b,a)}f(x)dx$$
$$=\frac{(-1)^{m-1}\eta^m(b,a)}{m!}\left(a+t\frac{\eta(b,a)}{m+1}\right)f^{(m)}\big(a+t\eta(b,a)\big)$$
$$+(-1)^m\eta^{m+1}(b,a)\int_0^1 \frac{t^m}{m!}\left(a+t\frac{\eta(b,a)}{m+1}\right)f^{(m+1)}(a+t\eta(b,a))dt,$$

that is,

$$\sum_{k=1}^{m+1}\frac{(-1)^{k}\eta^{k}(b,a)}{k!}\left(a+\frac{\eta(b,a)}{k}\right)f^{(k)}(a+t\eta(b,a))-af(a)-\int_a^{a+\eta(b,a)}f(x)dx$$
$$=(-1)^m\eta^{m+1}(b,a)\int_0^1 \frac{t^m}{m!}\left(a+t\frac{\eta(b,a)}{m+1}\right)f^{(m+1)}(a+t\eta(b,a))dt.$$

Theorem 2. *Let $K\subseteq\mathbb{R}$ be an open invex subset with respect to mapping $\eta(\cdot,\cdot):K\times K\to\mathbb{R}$ and $a,b\in K$ with $\eta(b,a)>0$. Suppose that $f:K\to\mathbb{R}$ is a n-times differentiable function on K such that $f^{(n)}\in L[a,a+\eta(b,a)]$. If $\left|f^{(n)}\right|^q$ is preinvex on K for $q>1$, then the following holds:*

$$\left|I_f(a,b,\eta)\right|\le\frac{n^{\frac{2}{q}}\eta^{n-\frac{2}{q}}(b,a)}{(n-1)![(n-1)p+1]^{\frac{1}{p}}}\left[\left|f^{(n)}(b)\right|^q C_{1,\eta}(a,b)+\left|f^{(n)}(a)\right|^q C_{2,\eta}(a,b)\right]^{\frac{1}{q}}. \tag{2.4}$$

where

$$C_{1,\eta}(a,b,n):=\begin{cases}\frac{\eta(b,a)}{n}\left[L_{q+1}^{q+1}(\delta(n),a)-aL_q^q(\delta(n),a)\right], & a>0,\delta(n)>0,\\ [\delta(n)+a]L_{q+1}^{q+1}(\delta(n),-a)-\frac{2a}{q+1}A([\delta(n)]^{q+1},(-a)^{q+1}), & a<0,\delta(n)>0,\\ -\frac{\eta(b,a)}{n}\left[L_{q+1}^{q+1}(-a,-\delta(n))+aL_q^q(-a,-\delta(n))\right], & a<0,\delta(n)<0.\end{cases}$$

$$C_{2,\eta}(a,b,n):=\begin{cases}-\frac{\eta(b,a)}{n}\left[L_{q+1}^{q+1}(\delta(n),a)-\delta(n)L_q^q(\delta(n),a)\right], & a>0,\delta(n)>0,\\ -[\delta(n)+a]L_{q+1}^{q+1}(\delta(n),-a)+\frac{2\delta(n)}{q+1}A([\delta(n)]^{q+1},(-a)^{q+1}), & a<0,\delta(n)>0,\\ \frac{\eta(b,a)}{n}\left[L_{q+1}^{q+1}(-a,-\delta(n))+\delta(n)L_q^q(-a,-\delta(n))\right], & a<0,\delta(n)<0.\end{cases}$$

Proof. If $\left|f^{(n)}\right|^q$ for $q>1$ is preinvex on $[a,a+\eta(b,a)]$, using Lemma 1, the Hölder integral inequality and

$$\left|f^{(n)}(a+t\eta(b,a))\right|^q\le t\left|f^{(n)}(b)\right|^q+(1-t)\left|f^{(n)}(a)\right|^q$$

we get

$$\left|I_f(a,b,\eta)\right| \le \frac{\eta^n(b,a)}{(n-1)!}\int_0^1 t^{n-1}\ |\delta_t(n)|\left|f^{(n)}(a+t\eta(b,a))\right|dt$$

$$\le \frac{\eta^n(b,a)}{(n-1)!}\left(\int_0^1 t^{(n-1)p}dt\right)^{\frac{1}{p}}\left(\int_0^1 |\delta_t(n)|^q\left|f^{(n)}(a+t\eta(b,a))\right|^q dt\right)^{\frac{1}{q}}$$

$$\le \frac{\eta^n(b,a)}{(n-1)!}\left(\int_0^1 t^{(n-1)p}dt\right)^{\frac{1}{p}}\left(\int_0^1 |\delta_t(n)|^q\left[t\left|f^{(n)}(b)\right|^q+(1-t)\left|f^{(n)}(a)\right|^q\right]dt\right)^{\frac{1}{q}}$$

$$=\frac{\eta^n(b,a)}{(n-1)!\,[(n-1)p+1]^{\frac{1}{p}}}\left(\left|f^{(n)}(b)\right|^q\int_0^1 t|\delta_t(n)|^q dt+\left|f^{(n)}(a)\right|^q\int_0^1(1-t)|\delta_t(n)|^q dt\right)^{\frac{1}{q}}$$

$$=\frac{\eta^n(b,a)}{(n-1)![(n-1)p+1]^{\frac{1}{p}}}$$

$$\times\left(\frac{n^2\left|f^{(n)}(b)\right|^q}{\eta^2(b,a)}\int_a^{\delta(n)}(x-a)|x|^q dx+\frac{n^2\left|f^{(n)}(a)\right|^q}{\eta^2(b,a)}\int_a^{\delta(n)}(\delta(n)-x)|x|^q dx\right)^{\frac{1}{q}}$$

$$=\frac{\eta^{n-\frac{2}{q}}(b,a)n^{\frac{2}{q}}}{(n-1)![(n-1)p+1]^{\frac{1}{p}}}\left(\left|f^{(n)}(b)\right|^q\int_a^{\delta(n)}(x-a)|x|^q dx+\left|f^{(n)}(a)\right|^q\int_a^{\delta(n)}(\delta(n)-x)|x|^q dx\right)^{\frac{1}{q}}$$

$$=\frac{n^{\frac{2}{q}}\eta^{n-\frac{2}{q}}(b,a)}{(n-1)![(n-1)p+1]^{\frac{1}{p}}}\left[\left|f^{(n)}(b)\right|^q C_{1,\eta}(a,b)+\left|f^{(n)}(a)\right|^q C_{2,\eta}(a,b)\right]^{\frac{1}{q}}.$$

Corollary 1. *Suppose that all the assumptions of Theorem 2 are satisfied. If we choose $\eta(b,a)=b-a$ then when $\left|f^{(n)}\right|^q$ is convex on K for $q>1$ we obtain following inequality:*

$$\left|\frac{I_f(a,b,\eta)}{b-a}\right|\le\frac{n^{\frac{2}{q}}(b-a)^{n-1-\frac{2}{q}}}{(n-1)!\,[(n-1)p+1]^{\frac{1}{p}}}\left[\left|f^{(n)}(b)\right|^q C_1(a,b,n)+\left|f^{(n)}(a)\right|^q C_2(a,b,n)\right]^{\frac{1}{q}},$$

where

$$C_1(a,b,n)$$

$$=\begin{cases}\frac{b-a}{n}\left[L_{q+1}^{q+1}\left(\frac{(n-1)a+b}{n},a\right)-aL_q^q\left(\frac{(n-1)a+b}{n},a\right)\right], & a>0,\frac{(n-1)a+b}{n}>0,\\ \left[\frac{(2n-1)a+b}{n}\right]L_{q+1}^{q+1}\left(\frac{(n-1)a+b}{n},-a\right)-\frac{2a}{q+1}A\left(\left[\frac{(n-1)a+b}{n}\right]^{q+1},(-a)^{q+1}\right), & a<0,\frac{(n-1)a+b}{n}>0,\\ -\frac{b-a}{n}\left[L_{q+1}^{q+1}\left(-a,-\frac{(n-1)a+b}{n}\right)+aL_q^q\left(-a,-\frac{(n-1)a+b}{n}\right)\right], & a<0,\frac{(n-1)a+b}{n}<0.\end{cases}$$

$$C_2(a,b,n)=\begin{cases}-\frac{b-a}{n}\left[L_{q+1}^{q+1}\left(\frac{(n-1)a+b}{n},a\right)-\delta(n)L_q^q\left(\frac{(n-1)a+b}{n},a\right)\right], \\ \qquad\qquad a>0,\frac{(n-1)a+b}{n}>0, \\ -\left[\frac{(2n-1)a+b}{n}\right]L_{q+1}^{q+1}\left(\frac{(n-1)a+b}{n},-a\right)+\frac{2}{q+1}\left[\frac{(n-1)a+b}{n}\right] \\ \times A\left(\left[\frac{(n-1)a+b}{n}\right]^{q+1},(-a)^{q+1}\right), a<0,\frac{(n-1)a+b}{n}>0, \\ \frac{b-a}{n}\left[L_{q+1}^{q+1}\left(-a,-\frac{(n-1)a+b}{n}\right)+\frac{(n-1)a+b}{n}L_q^q\left(-a,-\frac{(n-1)a+b}{n}\right)\right], \\ \qquad\qquad a<0,\frac{(n-1)a+b}{n}<0.\end{cases}$$

Remark 1. *If the mapping η satisfies condition C then by use of the preinvexity of $\left|f^{(n)}\right|^q$ we obtain following inequality for every $t\in[0,1]$:*

$$\left|f^{(n)}(a+t\eta(b,a))\right|^q=\left|f^{(n)}(a+\eta(b,a)+(1-t)\eta(a,a+\eta(b,a)))\right|^q$$
$$\le t\left|f^{(n)}\big(a+\eta(b,a)\big)\right|^q+(1-t)\left|f^{(n)}(a)\right|^q. \tag{2.5}$$

If we use (2.5) in the proof of Theorem 2, then (2.4) becomes the following inequality:

$$I_f(a,b,\eta)\le\frac{n^{\frac{2}{q}}\eta^{n-\frac{2}{q}}(b,a)}{(n-1)![(n-1)p+1]^{\frac{1}{p}}}\left[\left|f^{(n)}\big(a+\eta(b,a)\big)\right|^q C_{1,\eta}(a,b)+\left|f^{(n)}(a)\right|^q C_{2,\eta}(a,b)\right]^{\frac{1}{q}}. \tag{2.6}$$

We note that by use of the preinvexity of $\left|f^{(n)}\right|^q$ we get $\left|f^{(n)}(a+\eta(b,a))\right|^q\le\left|f^{(n)}(b)\right|^q$. Therefore, the inequality (2.6) is better than the inequality (2.4).

Remark 2. *The following results are remarkable for the Theorem 2.*

i. The results obtained in this paper reduces to the results of [9] for $n=1$.
ii. The results obtained in this paper reduces to the results of [8] for $n=2$.
iii. The results obtained in this paper reduces to the results of [10] for $n=3$.
iv. The results obtained in this paper reduces to the results of [7] for $n=4$.

Theorem 3. *Let $K\subseteq\mathbb{R}$ be an open invex subset with respect to mapping $\eta(\cdot,\cdot):K\times K\to\mathbb{R}$ and $a,b\in K$ with $\eta(b,a)>0$. Suppose that $f:K\to\mathbb{R}$ is a n-times differentiable function on K such that $f^{(n)}\in L[a,a+\eta(b,a)]$. If $\left|f^{(n)}\right|^q$ is preinvex on K for $q>1$, then the following holds:*

$$\left|I_f(a,b,\eta)\right|\le\frac{n^{\frac{1}{p}}\eta^{n-\frac{1}{p}}(b,a)}{(n-1)!}\left(\int_a^{\delta(n)}|x|^p dx\right)^{\frac{1}{p}}\left[\frac{[(n-1)q+1]\left|f^{(n)}(b)\right|^q+\left|f^{(n)}(a)\right|^q}{[(n-1)q+1][(n-1)q+2]}\right]^{\frac{1}{q}} \tag{2.7}$$

where

$$C_{3,\eta}(a,b):=\begin{cases}\frac{\eta(b,a)}{n}L_p^p(\delta(n),a), & a>0,\delta(n)>0,\\ \frac{2}{p+1}A\left(\big(\delta(n)\big)^{p+1},(-a)^{p+1}\right), & a<0,\delta(n)>0,\\ \frac{\eta(b,a)}{n}L_p^p\big(-a,-\delta(n)\big), & a<0,\delta(n)<0.\end{cases}$$

Proof. If $\left|f^{(n)}\right|^q$ for $q>1$ is preinvex on $[a,a+\eta(b,a)]$, using Lemma 1, the Hölder integral inequality and $\left|f^{(n)}(a+t\eta(b,a))\right|^q\le t\left|f^{(n)}(b)\right|^q+(1-t)\left|f^{(n)}(a)\right|^q$, we obtain the following inequality:

$$\left|I_f(a,b,\eta)\right|\le\frac{\eta^n(b,a)}{(n-1)!}\int_0^1 t^{n-1}\ |\delta_t(n)|\left|f^{(n)}(a+t\eta(b,a))\right|dt$$
$$\le\frac{\eta^n(b,a)}{(n-1)!}\left(\int_0^1|\delta_t(n)|^p\right)^{\frac{1}{p}}\left(\int_0^1 t^{(n-1)q}\left|f^{(n)}(a+t\eta(b,a))\right|^q dt\right)^{\frac{1}{q}}$$

$$\leq \frac{\eta^n(b,a)}{(n-1)!}\left(\int_0^1 |\delta_t(n)|^p\right)^{\frac{1}{p}}\left(\int_0^1 t^{(n-1)q}\left[t\left|f^{(n)}(b)\right|^q+(1-t)\left|f^{(n)}(a)\right|^q\right]dt\right)^{\frac{1}{q}}$$

$$=\frac{\eta^n(b,a)}{(n-1)!}\left(\int_0^1 |\delta_t(n)|^p\right)^{\frac{1}{p}}\left(\left|f^{(n)}(b)\right|^q\int_0^1 t^{(n-1)q+1}dt+\left|f^{(n)}(a)\right|^q\int_0^1 \left(t^{(n-1)q}-t^{(n-1)q+1}\right)dt\right)^{\frac{1}{q}}$$

$$=\frac{n^{\frac{1}{p}}\eta^{n-\frac{1}{p}}(b,a)}{(n-1)!}\left(\int_a^{\delta(n)}|x|^p dx\right)^{\frac{1}{p}}\left[\frac{\left|f^{(n)}(b)\right|^q}{(n-1)q+2}+\left|f^{(n)}(a)\right|^q\left(\frac{1}{(n-1)q+1}-\frac{1}{(n-1)q+2}\right)\right]^{\frac{1}{q}}$$

$$=\frac{n^{\frac{1}{p}}\eta^{n-\frac{1}{p}}(b,a)}{(n-1)!}\left(\int_a^{\delta(n)}|x|^p dx\right)^{\frac{1}{p}}\left[\frac{[(n-1)q+1]\left|f^{(n)}(b)\right|^q+\left|f^{(n)}(a)\right|^q}{[(n-1)q+1][(n-1)q+2]}\right]^{\frac{1}{q}}.$$

Corollary 2. *Suppose that all the assumptions of Theorem 3 are satisfied. If we choose $\eta(b,a)=b-a$ then when $\left|f^{(n)}\right|^q$ is convex on K for $q>1$ we have the following inequality:*

$$\left|\frac{I_f(a,b,\eta)}{b-a}\right|\leq\frac{n^{\frac{1}{p}}(b-a)^{n-1-\frac{1}{p}}}{(n-1)!}\left(\int_a^{\delta(n)}|x|^p dx\right)^{\frac{1}{p}}\left[\frac{[(n-1)q+1]\left|f^{(n)}(b)\right|^q+\left|f^{(n)}(a)\right|^q}{[(n-1)q+1][(n-1)q+2]}\right]^{\frac{1}{q}},$$

where

$$C_3(a,b)=\begin{cases}\frac{b-a}{n}L_p^p\left(\frac{(n-1)a+b}{n},a\right), & a>0,\frac{(n-1)a+b}{n}>0,\\ \frac{2}{p+1}A\left(\left(\frac{(n-1)a+b}{n}\right)^{p+1},(-a)^{p+1}\right), & a<0,\frac{(n-1)a+b}{n}>0,\\ \frac{b-a}{n}L_p^p\left(-a,-\left(\frac{(n-1)a+b}{n}\right)\right), & a<0,\frac{(n-1)a+b}{n}<0.\end{cases}$$

Remark 3. *If the mapping η satisfies condition C then using the inequality (2.5) in the proof of Theorem 3, then the inequality (2.7) becomes the following inequality:*

$$\left|I_f(a,b,\eta)\right|\leq\frac{n^{\frac{1}{p}}\eta^{n-\frac{1}{p}}(b,a)}{(n-1)!}\left(\int_a^{\delta(n)}|x|^p dx\right)^{\frac{1}{p}}\left[\frac{[(n-1)q+1]\left|f^{(n)}(a+\eta(b,a))\right|^q+\left|f^{(n)}(a)\right|^q}{[(n-1)q+1][(n-1)q+2]}\right]^{\frac{1}{q}}. \tag{2.8}$$

We note that by use of the preinvexity of $\left|f^{(n)}\right|^q$ we get $\left|f^{(n)}(a+\eta(b,a))\right|^q\leq\left|f^{(n)}(b)\right|^q$. Therefore, the inequality (2.8) is better than the inequality (2.7).

Remark 4. *The following results are remarkable for the Theorem 3.*

i. The results obtained in this paper reduces to the results of [9] for $n=1$.
ii. The results obtained in this paper reduces to the results of [8] for $n=2$.
iii. The results obtained in this paper reduces to the results of [10] for $n=3$.
iv. The results obtained in this paper reduces to the results of [7] for $n=4$.

Theorem 4. *Let $K\subseteq\mathbb{R}$ be an open invex subset with respect to mapping $\eta(\cdot,\cdot):K\times K\to\mathbb{R}$ and $a,b\in K$ with $\eta(b,a)>0$. Suppose that $f:K\to\mathbb{R}$ is a n-times differentiable function on K such that $f^{(n)}\in L[a,a+\eta(b,a)]$. If $\left|f^{(n)}\right|^q$ is preinvex on K for $q\geq 1$, then the following holds:*

$$\left|I_f(a,b,\eta)\right|\leq\frac{n^{n+\frac{1}{q}}}{(n-1)!}\eta^{-\frac{1}{q}}(b,a)D_{1,\eta}^{1-\frac{1}{q}}(a,b)\left[\left|f^{(n)}(b)\right|^q D_{2,\eta}(a,b)+\left|f^{(n)}(a)\right|^q D_{3,\eta}(a,b)\right]^{\frac{1}{q}} \tag{2.9}$$

where

$$D_{1,\eta}(a,b):=\begin{cases}\left(\frac{\eta(b,a)}{n}\right)^n\frac{\eta(b,a)+(n+1)a}{n(n+1)}, & a>0,\delta(n)>0\\ \left(\frac{\eta(b,a)}{n}\right)^n\frac{\eta(b,a)+(n+1)a}{n(n+1)}+2\frac{(-a)^{n+1}}{n(n+1)}, & a<0,\delta(n)>0\\ -\left(\frac{\eta(b,a)}{n}\right)^n\frac{\eta(b,a)+(n+1)a}{n(n+1)}, & a<0,\delta(n)<0\end{cases},$$

$$D_{2,\eta}(a,b):=\begin{cases}\left(\frac{\eta(b,a)}{n}\right)^{n+1}\frac{(n+1)\eta(b,a)+n(n+2)a}{n(n+1)(n+2)}, & a>0,\delta(n)>0\\ \left(\frac{\eta(b,a)}{n}\right)^{n+1}\frac{(n+1)\eta(b,a)+n(n+2)a}{n(n+1)(n+2)}+2\frac{(-a)^{n+2}}{(n+1)(n+2)}, & a<0,\delta(n)>0\\ -\left(\frac{\eta(b,a)}{n}\right)^{n+1}\frac{(n+1)\eta(b,a)+n(n+2)a}{n(n+1)(n+2)}, & a<0,\delta(n)<0\end{cases},$$

$$D_{3,\eta}(a,b):=\begin{cases}\left(\frac{\eta(b,a)}{n}\right)^{n+1}\frac{\eta(b,a)+(n+2)a}{n(n+1)(n+2)}, & a>0,\delta(n)>0\\ \left(\frac{\eta(b,a)}{n}\right)^{n+1}\frac{\eta(b,a)+(n+2)a}{n(n+1)(n+2)}+2\frac{\eta(b,a)}{n}\frac{(-a)^{n+1}}{n(n+1)}-2\frac{(-a)^{n+2}}{(n+1)(n+2)}, & a<0,\delta(n)>0\\ -\left(\frac{\eta(b,a)}{n}\right)^{n+1}\frac{\eta(b,a)+(n+2)a}{n(n+1)(n+2)}, & a<0,\delta(n)<0.\end{cases}$$

Proof. Using Lemma 1 and Power-mean integral inequality, we obtain

$$\left|I_f(a,b,\eta)\right|\leq\frac{\eta^n(b,a)}{(n-1)!}\int_0^1 t^{n-1}\,|\delta_t(n)|\left|f^{(n)}(a+t\eta(b,a))\right|dt$$

$$\leq\frac{\eta^n(b,a)}{(n-1)!}\left(\int_0^1 t^{n-1}|\delta_t(n)|dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{n-1}|\delta_t(n)|\left|f^{(n)}(a+t\eta(b,a))\right|^q dt\right)^{\frac{1}{q}}$$

$$\leq\frac{\eta^n(b,a)}{(n-1)!}\left(\int_0^1 t^{n-1}|\delta_t(n)|dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{n-1}|\delta_t(n)|\left[t\left|f^{(n)}(b)\right|^q+(1-t)\left|f^{(n)}(a)\right|^q\right]dt\right)^{\frac{1}{q}}$$

$$=\frac{\eta^n(b,a)}{(n-1)!}\left(\int_0^1 t^{n-1}|\delta_t(n)|dt\right)^{1-\frac{1}{q}}$$

$$\times\left(\left|f^{(n)}(b)\right|^q\int_0^1 t^n|\delta_t(n)|dt+\left|f^{(n)}(a)\right|^q\int_0^1 t^{n-1}(1-t)|\delta_t(n)|dt\right)^{\frac{1}{q}}$$

$$=\frac{\eta^n(b,a)}{(n-1)!}\left(\frac{n^n}{\eta^n(b,a)}\right)^{1-\frac{1}{q}}\left(\frac{n^{n+1}}{\eta^{n+1}(b,a)}\right)^{\frac{1}{q}}\left(\int_a^{\delta(n)}(x-a)^{n-1}|x|dx\right)^{1-\frac{1}{q}}$$

$$\times\left(\left|f^{(n)}(b)\right|^q\int_a^{\delta(n)}(x-a)^n|x|dx+\left|f^{(n)}(a)\right|^q\int_a^{\delta(n)}(x-a)^{n-1}[\delta(n)-x]|x|dx\right)^{\frac{1}{q}}$$

$$=\frac{n^{n+\frac{1}{q}}}{(n-1)!}\eta^{-\frac{1}{q}}(b,a)\left(\int_a^{\delta(n)}(x-a)^{n-1}|x|dx\right)^{1-\frac{1}{q}}$$

$$\times\left(\left|f^{(n)}(b)\right|^q\int_a^{\delta(n)}(x-a)^n|x|dx+\left|f^{(n)}(a)\right|^q\int_a^{\delta(n)}(x-a)^{n-1}[\delta(n)-x]|x|dx\right)^{\frac{1}{q}}$$

$$=\frac{n^{n+\frac{1}{q}}}{(n-1)!}\eta^{-\frac{1}{q}}(b,a)D_{1,\eta}^{1-\frac{1}{q}}(a,b)\left[\left|f^{(n)}(b)\right|^q D_{2,\eta}(a,b)+\left|f^{(n)}(a)\right|^q D_{3,\eta}(a,b)\right]^{\frac{1}{q}}.$$

Corollary 3. *Suppose that all the assumptions of Theorem 3 are satisfied. If we choose* $\eta(b,a)=b-a$ *then when* $\left|f^{(n)}\right|^q$ *is convex on* K *for* $q\geq 1$ *we get*

$$\left|\frac{I_f(a,b,\eta)}{b-a}\right|\leq\frac{n^{n+\frac{1}{q}}}{(n-1)!}(b-a)^{-1-\frac{1}{q}}D_{1,\eta}^{1-\frac{1}{q}}(a,b)\left[\left|f^{(n)}(b)\right|^q D_{2,\eta}(a,b)+\left|f^{(n)}(a)\right|^q D_{3,\eta}(a,b)\right]^{\frac{1}{q}},$$

where

$$D_1(a,b)=\begin{cases}\left(\frac{b-a}{n}\right)^n\frac{b+na}{n(n+1)}, & a>0,\frac{(n-1)a+b}{n}>0\\ \left(\frac{b-a}{n}\right)^n\frac{b+na}{n(n+1)}+2\frac{(-a)^{n+1}}{n(n+1)}, & a<0,\frac{(n-1)a+b}{n}>0\\ -\left(\frac{b-a}{n}\right)^n\frac{b+na}{n(n+1)}, & a<0,\frac{(n-1)a+b}{n}<0\end{cases}$$

$$D_2(a,b)=\begin{cases}\left(\frac{b-a}{n}\right)^{n+1}\frac{(n+1)b+(n^2+n-1)a}{n(n+1)(n+2)}, & a>0,\frac{(n-1)a+b}{n}>0\\ \left(\frac{b-a}{n}\right)^{n+1}\frac{(n+1)b+(n^2+n-1)a}{n(n+1)(n+2)}+2\frac{(-a)^{n+2}}{(n+1)(n+2)}, & a<0,\frac{(n-1)a+b}{n}>0\\ -\left(\frac{b-a}{n}\right)^{n+1}\frac{(n+1)b+(n^2+n-1)a}{n(n+1)(n+2)}, & a<0,\frac{(n-1)a+b}{n}<0\end{cases}$$

$$D_3(a,b)=\begin{cases}\left(\frac{b-a}{n}\right)^{n+1}\frac{b+(n+1)a}{n(n+1)(n+2)}, & a>0,\frac{(n-1)a+b}{n}>0\\ \left(\frac{b-a}{n}\right)^{n+1}\frac{b+(n+1)a}{n(n+1)(n+2)}+2\frac{b-a}{n}\frac{(-a)^{n+1}}{n(n+1)}-2\frac{(-a)^{n+2}}{(n+1)(n+2)}, & a<0,\frac{(n-1)a+b}{n}>0\\ -\left(\frac{b-a}{n}\right)^{n+1}\frac{b+(n+1)a}{n(n+1)(n+2)}, & a<0,\frac{(n-1)a+b}{n}<0.\end{cases}$$

Remark 5. *If the mapping η satisfies condition C then using the inequality (2.5) in the proof of Theorem 4, then the inequality (2.9) becomes the following inequality:*

$$\left|I_f(a,b,\eta)\right|\le\frac{n^{n+\frac{1}{q}}}{(n-1)!}\eta^{-\frac{1}{q}}(b,a)D_{1,\eta}^{1-\frac{1}{q}}(a,b)\left[\left|f^{(n)}(b)\right|^qD_{2,\eta}(a,b)+\left|f^{(n)}(a)\right|^qD_{3,\eta}(a,b)\right]^{\frac{1}{q}}. \quad (2.10)$$

We note that by use of the preinvexity of $\left|f^{(n)}\right|^q$ we get $\left|f^{(n)}(a+\eta(b,a))\right|^q\le\left|f^{(n)}(b)\right|^q$. Therefore, the inequality (2.10) is better than the inequality (2.9).

Corollary 4. *If we take $q=1$ in Theorem 4, then we have the following inequality:*

$$\left|I_f(a,b,\eta)\right|\le\frac{n^{n+1}}{(n-1)!}\eta^{-1}(b,a)\left[\left|f^{(n)}(b)\right|D_{2,\eta}(a,b)+\left|f^{(n)}(a)\right|D_{3,\eta}(a,b)\right].$$

Remark 6. *The following results are remarkable for the Theorem 4.*

i. The results obtained in this paper reduces to the results of [9] for $n=1$.
ii. The results obtained in this paper reduces to the results of [8] for $n=2$.
iii. The results obtained in this paper reduces to the results of [10] for $n=3$.
iv. The results obtained in this paper reduces to the results of [7] for $n=4$.

Theorem 5. *Let $K\subseteq\mathbb{R}$ be an open invex subset with respect to mapping $\eta(\cdot,\cdot):K\times K\to\mathbb{R}$ and $a,b\in K$ with $\eta(b,a)>0$. Suppose that the function $f:K\to\mathbb{R}$ is a n-times differentiable function on K such that $f^{(n)}\in L[a,a+\eta(b,a)]$. If $\left|f^{(n)}\right|^q$ is prequasiinvex on K for $q>1$, then the following inequality holds:*

$$\left|I_f(a,b,\eta)\right|\le\frac{n^{\frac{1}{q}}\eta^{n-\frac{1}{q}}(b,a)}{(n-1)!}\left[\frac{1}{(n-1)p+1}\right]^{\frac{1}{p}}max\left\{\left|f^{(n)}(a)\right|^q,\left|f^{(n)}(b)\right|^q\right\}^{\frac{1}{q}}C_\eta^{\frac{1}{q}}(q,a,b) \quad (2.11)$$

where

$$C_\eta(q,a,b):=\begin{cases}\frac{\eta(b,a)}{n}L_q^q(\delta(n),a), & a>0,\delta(n)>0,\\ \frac{2}{q+1}A[(\delta(n))^{q+1},(-a)^{q+1}], & a<0,\delta(n)>0,\\ \frac{\eta(b,a)}{n}L_q^q(-a,-\delta(n)), & a<0,\delta(n)<0.\end{cases}$$

Proof. If $\left|f^{(n)}\right|^q$ for $q>1$ is prequasiinvex on $[a,a+\eta(b,a)]$, using Lemma 1, the Hölder integral inequality and

$$\left|f^{(n)}(a+t\eta(b,a))\right|^q\le max\left\{\left|f^{(n)}(a)\right|^q,\left|f^{(n)}(b)\right|^q\right\}$$

we obtain

$$\begin{aligned}
\left|I_f(a,b,\eta)\right| &\leq \frac{\eta^n(b,a)}{(n-1)!}\int_0^1 t^{n-1}\ |\delta_t(n)|\left|f^{(n)}(a+t\eta(b,a))\right|dt \\
&\leq \frac{\eta^n(b,a)}{(n-1)!}\left(\int_0^1 t^{(n-1)p}dt\right)^{\frac{1}{p}}\left(\int_0^1 |\delta_t(n)|^q\left|f^{(n)}(a+t\eta(b,a))\right|^q dt\right)^{\frac{1}{q}} \\
&\leq \frac{\eta^n(b,a)}{(n-1)!}\left(\int_0^1 t^{(n-1)p}dt\right)^{\frac{1}{p}}\left(\int_0^1 |\delta_t(n)|^q max\left\{\left|f^{(n)}(a)\right|^q,\left|f^{(n)}(b)\right|^q\right\}dt\right)^{\frac{1}{q}} \\
&= \frac{\eta^n(b,a)}{(n-1)!}\left[\frac{1}{(n-1)p+1}\right]^{\frac{1}{p}} max\left\{\left|f^{(n)}(a)\right|^q,\left|f^{(n)}(b)\right|^q\right\}^{\frac{1}{q}}\left(\int_0^1 |\delta_t(n)|^q dt\right)^{\frac{1}{q}} \\
&= \frac{n^{\frac{1}{q}}\eta^{n-\frac{1}{q}}(b,a)}{(n-1)!}\left[\frac{1}{(n-1)p+1}\right]^{\frac{1}{p}} max\left\{\left|f^{(n)}(a)\right|^q,\left|f^{(n)}(b)\right|^q\right\}^{\frac{1}{q}}\left(\int_a^{\delta(n)} |x|^q dx\right)^{\frac{1}{q}} \\
&= \frac{n^{\frac{1}{q}}\eta^{n-\frac{1}{q}}(b,a)}{(n-1)!}\left[\frac{1}{(n-1)p+1}\right]^{\frac{1}{p}} max\left\{\left|f^{(n)}(a)\right|^q,\left|f^{(n)}(b)\right|^q\right\}^{\frac{1}{q}} C_\eta^{\frac{1}{q}}(q,a,b)
\end{aligned}$$

Corollary 5. *Suppose that all the assumptions of Theorem 5 are satisfied. If we choose $\eta(b,a)=b-a$ then when $\left|f^{(n)}\right|^q$ is prequasiinvex on K for $q>1$ we have*

$$\left|\frac{I_f(a,b,\eta)}{b-a}\right| \leq \frac{n^{\frac{1}{q}}(b-a)^{n-1-\frac{1}{q}}}{(n-1)!}\left[\frac{1}{(n-1)p+1}\right]^{\frac{1}{p}} \max\left\{\left|f^{(n)}(a)\right|^q,\left|f^{(n)}(b)\right|^q\right\}^{\frac{1}{q}} C_\eta^{\frac{1}{q}}(q,a,b)$$

where

$$C(q,a,b)=\begin{cases}\frac{\eta(b,a)}{n}L_q^q\left(\frac{(n-1)a+b}{n},a\right), & a>0, \frac{(n-1)a+b}{n}>0, \\ \frac{2}{q+1}A\left[\left(\frac{(n-1)a+b}{n}\right)^{q+1},(-a)^{q+1}\right], & a<0, \frac{(n-1)a+b}{n}>0, \\ \frac{b-a}{n}L_q^q\left(-a,-\frac{(n-1)a+b}{n}\right), & a<0, \frac{(n-1)a+b}{n}<0.\end{cases}$$

Remark 7. *If the mapping η satisfies condition C then by use of the prequasiinvexity of $\left|f^{(n)}\right|^q$ we get*

$$\begin{aligned}
&\left|f^{(n)}(a+t\eta(b,a))\right|^q = \left|f^{(n)}(a+\eta(b,a)+(1-t)\eta(a,a+\eta(b,a)))\right|^q \\
&\leq max\left\{\left|f^{(n)}(a)\right|^q,\left|f^{(n)}\big(a+\eta(b,a)\big)\right|^q\right\} \qquad (2.12)
\end{aligned}$$

for every $t\in[0,1]$. If we use (2.12) in the proof of Theorem 5, then the inequality (2.11) becomes the following inequality:

$$\begin{aligned}
&\left|I_f(a,b,\eta)\right| \\
&\leq \frac{n^{\frac{1}{q}}\eta^{n-\frac{1}{q}}(b,a)}{(n-1)!}\left[\frac{1}{(n-1)p+1}\right]^{\frac{1}{p}} max\left\{\left|f^{(n)}(a)\right|^q,\left|f^{(n)}\big(a+\eta(b,a)\big)\right|^q\right\}^{\frac{1}{q}} C_\eta^{\frac{1}{q}}(q,a,b) \qquad (2.13)
\end{aligned}$$

We note that by use of the prequasiinvexity of $\left|f^{(n)}\right|^q$ we have

$$\left|f^{(n)}(a+\eta(b,a))\right|^q \leq max\left\{\left|f^{(n)}(a)\right|^q,\left|f^{(n)}(a+\eta(b,a))\right|^q\right\}.$$

Therefore, the inequality (2.13) is better than the inequality (2.11).

Remark 8. *The following results are remarkable for the Theorem 5.*

i. The results obtained in this paper reduces to the results of [9] for $n=1$.
ii. The results obtained in this paper reduces to the results of [8] for $n=2$.
iii. The results obtained in this paper reduces to the results of [10] for $n=3$.
iv. The results obtained in this paper reduces to the results of [11] for $n=4$.

Theorem 6. *Let $K\subseteq\mathbb{R}$ be an open invex subset with respect to mapping $\eta(\cdot,\cdot): K\times K\to\mathbb{R}$ and $a,b\in K$ with $\eta(b,a)>0$. Suppose that the function $f: K\to\mathbb{R}$ is a n-times differentiable function* *on K such that* $f^{(n)} \in L[a, a+\eta(b,a)]$. *If* $\left|f^{(n)}\right|^q$ *is prequasiinvex on K for* $q \geq 1$, *then the following inequality holds:*

$$\left|I_f(a,b,\eta)\right| \leq \frac{n^n}{(n-1)!}\left(\max\left\{\left|f^{(n)}(a)\right|^q, \left|f^{(n)}(b)\right|^q\right\}\right)^{\frac{1}{q}} D_{1,\eta}(a,b) \tag{2.14}$$

where

$$D_{1,\eta}(a,b) := \begin{cases} \left(\frac{\eta(b,a)}{n}\right)^n \frac{\eta(b,a)+(n+1)a}{n(n+1)}, & a>0, \delta(n)>0 \\ \left(\frac{\eta(b,a)}{n}\right)^n \frac{\eta(b,a)+(n+1)a}{n(n+1)} + 2\frac{(-a)^{n+1}}{n(n+1)}, & a<0, \delta(n)>0, \\ -\left(\frac{\eta(b,a)}{n}\right)^n \frac{\eta(b,a)+(n+1)a}{n(n+1)}, & a<0, \delta(n)<0 \end{cases}$$

Proof. From Lemma 1 and Power-mean integral inequality, we obtain

$$\left|I_f(a,b,\eta)\right| \leq \frac{\eta^n(b,a)}{(n-1)!}\int_0^1 t^{n-1}\ |\delta_t(n)|\left|f^{(n)}(a+t\eta(b,a))\right|dt$$

$$\leq \frac{\eta^n(b,a)}{(n-1)!}\left(\int_0^1 t^{n-1}\ |\delta_t(n)|dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{n-1}|\delta_t(n)|\left|f^{(n)}(a+t\eta(b,a))\right|^q dt\right)^{\frac{1}{q}}$$

$$\leq \frac{\eta^n(b,a)}{(n-1)!}\left(\int_0^1 t^{n-1}\ |\delta_t(n)|dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{n-1}|\delta_t(n)| max\left\{\left|f^{(n)}(a)\right|^q, \left|f^{(n)}(b)\right|^q\right\}dt\right)^{\frac{1}{q}}$$

$$= \frac{\eta^n(b,a)}{(n-1)!}\left(max\left\{\left|f^{(n)}(a)\right|^q, \left|f^{(n)}(b)\right|^q\right\}\right)^{\frac{1}{q}}\int_0^1 t^{n-1}\ |\delta_t(n)|dt$$

$$= \frac{n^n}{(n-1)!}\left(max\left\{\left|f^{(n)}(a)\right|^q, \left|f^{(n)}(b)\right|^q\right\}\right)^{\frac{1}{q}}\int_a^{\delta(n)}(x-a)^{n-1}|x|dx$$

$$= \frac{n^n}{(n-1)!}\left(max\left\{\left|f^{(n)}(a)\right|^q, \left|f^{(n)}(b)\right|^q\right\}\right)^{\frac{1}{q}} D_{1,\eta}(a,b).$$

Corollary 6. *Suppose that all the assumptions of Theorem 6 are satisfied. If we choose* $\eta(b,a) = b-a$ *then when* $\left|f^{(n)}\right|^q$ *is prequasiinvex on K for* $q \geq 1$ *we have*

$$\left|\frac{I_f(a,b,\eta)}{b-a}\right| \leq \frac{n^n(b-a)^{-1}}{(n-1)!}\left(max\left\{\left|f^{(n)}(a)\right|^q, \left|f^{(n)}(b)\right|^q\right\}\right)^{\frac{1}{q}} D_{1,\eta}(a,b)$$

where

$$D_1(a,b) = \begin{cases} \left(\frac{b-a}{n}\right)^n \frac{b+na}{n(n+1)}, & a>0, \frac{(n-1)a+b}{n}>0 \\ \left(\frac{b-a}{n}\right)^n \frac{b+na}{n(n+1)} + 2\frac{(-a)^{n+1}}{n(n+1)}, & a<0, \frac{(n-1)a+b}{n}>0 \\ -\left(\frac{b-a}{n}\right)^n \frac{b+na}{n(n+1)}, & a<0, \frac{(n-1)a+b}{n}<0 \end{cases}$$

Remark 9. *If we use the inequality (2.12) in the proof of Theorem 6, then (2.14) becomes the following inequality:*

$$\left|I_f(a,b,\eta)\right| \leq \frac{n^n}{(n-1)!}\left(max\left\{\left|f^{(n)}(a)\right|^q, \left|f^{(n)}(a+\eta(b,a))\right|^q\right\}\right)^{\frac{1}{q}} D_{1,\eta}(a,b).$$

This inequality is better than the inequality (2.14).

Corollary 7. *If we take* $q = 1$ *in Theorem 7, then we have the following inequality:*

$$\left|I_f(a,b,\eta)\right| \leq \frac{n^n}{(n-1)!} max\left\{\left|f^{(n)}(a)\right|, \left|f^{(n)}(b)\right|\right\} D_{1,\eta}(a,b).$$

Remark 10. *The following results are remarkable for the Theorem 6.*

i. The results obtained in this paper reduces to the results of [9] for $n = 1$.
ii. The results obtained in this paper reduces to the results of [8] for $n = 2$.
iii. The results obtained in this paper reduces to the results of [10] for $n = 3$.
iv. The results obtained in this paper reduces to the results of [11] for $n = 4$.

REFERENCES

[1] Antczak T., (2005) Mean value in invexity analysis, *Nonl. Anal.*, 60, 1473-1484.
[2] Barani A., (2011) Ghazanfari AG, Dragomir SS. Hermite-Hadamard inequality through prequasiinvex functions, *RGMIA Research Report Collection*; 14, Article 48, 7 pp.
[3] Barani A., (2012) Ghazanfari AG, Dragomir SS. Hermite-Hadamard inequality for functions whose derivatives absolute values are preinvex, *J. Inequal*., 2012:247.
[4] Israel AB. and Mond B. What is invexity? *J. Aust. Math. Soc. Ser*. B 1986; 28(1), 1-9.
[5] Ion D.A., (2007) Some estimates on the Hermite-Hadamard inequality through quasi-convex functions, *Annals of University of Craiova, Math. Comp. Sci. Ser*., Volume 34, Pages 82–87.
[6] İşcan İ., (2014) Ostrowski type inequalities for functions whose derivatives are preinvex, *Bulletin of the Iranian Mathematical Society*, 40 (2), 373-386.
[7] Kadakal H., and İşcan İ., (2019) New Integral Inequalities for Preinvex Functions, *Mathematical Sciences and Applications, E-Notes*, 7 (1) 54-61 (2019).
[8] İşcan İ., Kadakal M. and Kadakal H., (2019) On Two Times Differentiable Preinvex and Prequasiinvex Functions, *Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat*., Volume 68, Number 1, Pages 950–963.
[9] Kadakal H., (2019) Differentiable Preinvex and Prequasiinvex Functions, *Miskolc Mathematical Notes*, (Submitted to journal).
[10] Kadakal H., Kadakal M. and İşcan İ., (2018) New Type Integral Inequalities for Three Times Differentiable Preinvex and Prequasiinvex Functions, *Open Journal of Mathematical Analysis*, Vol. 2, Issue 1, pp. 33–46.
[11] Kadakal H. and Maden S., (2019). Some Inequalities for Prequasiinvex Functions, *Communication in Mathematical Modelling and Applications*, CMMA 4, No. 1, 18-24.
[12] Latif M.A. and Dragomir S.S., (2013) Some Hermite-Hadamard type inequalities for functions whose partial derivatives in absloute value are preinvex on the co-oordinates, *Facta Universitatis (NIŠ) Ser. Math. Inform*., Vol. 28, No 3, 257-270.
[13] Matloka M., (2014) On some new inequalities for differentiable $(h_1;h_2)$-preinvex functions on the co-ordinates, *Mathematics and Statistics*, 2(1), 6-14.
[14] Mohan S.R. and Neogy S.K., (1995) On invex sets and preinvex functions, *J. Math. Anal. Appl*., 189, 901-908.
[15] Noor M.A., (2007) Hermite-Hadamard integral inequalities for log-preinvex functions, *J. Math. Anal. Approx. Theory*, 2, 126-131.
[16] Noor M.A., (2005) Invex equilibrium problems. *J. Math. Anal. Appl*, 302, 463-475.
[17] Weir T. and Mond B., (1998) Preinvex functions in multiple objective optimization, *J. Math Anal Appl*., 136, 29-38.
[18] Yang X.M., Yang X.Q. and Teo K.L., (2003) Generalized invexity and generalized invariant monotonicity, *J. Optim. Theory. Appl*., 117, 607-625.
[19] Yang X.M. and Li D., (2001) On properties of preinvex functions. *J. Math. Anal. Appl*., 256, 229-241.