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Research Article

QUANTUM HERMITE-HADAMARD TYPE INEQUALITY AND SOME ESTIMATES OF QUANTUM MIDPOINT TYPE INEQUALITIES FOR DOUBLE INTEGRALS

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ABSTRACT

In this paper, we give the correct quantum Hermite-Hadamard type inequality for the functions of two variables over finite rectangles. We provide some quantum estimates between the middle and the leftmost terms in correct quantum Hermite-Hadamard inequalities of functions of two variables using convexity and quasi-convexity on the co-ordinates.
Keywords: Hermite-Hadamard inequality, Hermite-Hadamard type inequality on co-ordinates, Quantum Hermite-Hadamard type inequality, Quantum Hermite-Hadamard type inequality on co-ordinates, Convexity on co-ordinates, Quasi-convexity on co-ordinates.

1. INTRODUCTION

Quantum calculus (named q-calculus) is the study of calculus without limits. In [5], Jackson started study of q-calculus and presented q-definite integrals. The topic of q-calculus has wide applications in different areas of mathematics and physics. Some recent developments in the theory of q-calculus and theory of inequalities in q-calculus see [3, 4, 6] .The most recently, many authors started developing quantum integral inequalities using classical convexity and quasi-convexity (see [1, 10, 11, 12, 13, 15, 16, 17, 18]).

In, [10], Latif et al. develop quantum integral inequalities theory for functions of two variables and provide some q-Hermite-Hadamard type inequality of functions of two variables over .nite rectangles. Also, Latif et al. provide some quantum estimates for the rightmost terms of

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the q-Hermite-Hadamard type inequalities of functions of two variables using convexity and quasi-convexity on the co-ordinates.

Throughout this paper, for the conciseness we will suppose that $a,b,c,d \in \mathbb{R}$ with $a<b$ and $c<d$, $0<q<1$, $0<q_1<1$ and $0<q_2<1$ are constants, $\Delta := [a,b]\times[c,d] \subset \mathbb{R}^2$ is a rectangle and $\Delta^\circ := (a,b)\times(c,d)$ is the interior of Δ.

Let real function f be defined on some non-empty interval I of real line $\mathbb{R}$. The function f said to be convex on I, if the inequality

$$f(ta+(1-t)b) \leq tf(a)+(1-t)f(b)$$

holds for all $a,b \in I$ and $t \in [0,1]$.

The function f said to be quasi-convex on I, if the inequality

$$f(ta+(1-t)b) \leq \sup\{f(a),f(b)\}$$

holds for all $a,b \in I$ and $t \in [0,1]$.

The most important integral inequality for convex functions is the Hermite-Hadamard inequality, which is stated as follows:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)\,dt \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

A function $f\colon \Delta \to \mathbb{R}$ is called convex (quasi-convex) on the co-ordinates if the partial mappings $f_y\colon[a,b]\to\mathbb{R}$, $f_y(u) := f(u,y)$ and $f_x\colon[c,d]\to\mathbb{R}$, $f_x(v) := f(x,v)$ convex (quasi-convex) where defined for all $y \in [c,d]$ and $x \in [a,b]$ (see [2, 14]).

A formal definition for the co-ordinated convex functions is given in [7, Definition 1] as follows:

Definition 1. [7] *A function $f\colon \Delta \to \mathbb{R}$ is said to be co-ordinated convex on Δ, for all $t,s \in [0,1]$ and $(x,y),(u,v) \in \Delta$, if the following inequality holds:*

$$f(tx+(1-t)u, sy+(1-s)v)$$
$$\leq tsf(x,y)+s(1-t)f(x,v)+t(1-s)f(u,y)+(1-t)(1-s)f(u,v).$$

Similarly, a formal definition for the co-ordinated quasi-convex functions is given in [9, Definition 2] as follows:

Definition 2. [9] *A function $f\colon \Delta \to \mathbb{R}$ is said to be quasi-convex on the co-ordinates on Δ, for all $t,s \in [0,1]$ and $(x,y),(u,v) \in \Delta$, if the following inequality holds:*

$$f(ty+(1-t)x, sv+(1-s)u) \leq \sup\{f(x,y),f(x,v),f(u,y),f(u,v)\}.$$

In [2], Dragomir proves the Hermite-Hadamard type inequality for co-ordinated convex functions as follows:

Theorem 1. *Suppose that $f\colon \Delta \to \mathbb{R}$ is the co-ordinated convex on Δ. Then one has the following inequalities*

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \leq \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)\,dx + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)\,dy\right] \leq$$
$$\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx \leq \frac{1}{4}\left[\begin{array}{c}\frac{1}{b-a}\int_a^b f(x,c)\,dx + \frac{1}{b-a}\int_a^b f(x,d)\,dx \\ +\frac{1}{d-c}\int_c^d f(a,y)\,dy + \frac{1}{d-c}\int_c^d f(b,y)\,dy\end{array}\right] \leq$$
$$\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}. \tag{1.2}$$

The above inequalities are sharp.

In [8], Latif and Dragomir prove some new inequalities which give estimate between the middle and the leftmost terms in (1:2) for differentiable the co-ordinated convex functions, by using the following identity.

Lemma 1. *Let $f\colon \Delta \to \mathbb{R}$ be a partial diff erentiable mapping. If $\frac{\partial^2 f}{\partial s \partial t} \in L(\Delta)$, then the following identity holds:*

$$\mu(a,b,c,d) = (b-a)(d-c)\int_0^1\int_0^1 K(t,s)\frac{\partial^2 f(ta+(1-t)b, sc+(1-s)d)}{\partial s \partial t}dsdt \qquad (1.3)$$

where

$$\mu(a,b,c,d) = \left[\begin{array}{l} \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx + f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \\ -\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx - \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy \end{array}\right],$$

and

$$K(t,s) = \begin{cases} ts &, \quad (t,s) \in \left[0,\frac{1}{2}\right]\times\left[0,\frac{1}{2}\right] \\ t(s-1) &, \quad (t,s) \in \left[0,\frac{1}{2}\right]\times\left(\frac{1}{2},1\right] \\ s(t-1) &, \quad (t,s) \in \left(\frac{1}{2},1\right]\times\left[0,\frac{1}{2}\right] \\ (t-1)(s-1) &, \quad (t,s) \in \left(\frac{1}{2},1\right]\times\left(\frac{1}{2},1\right] \end{cases}.$$

In [1], Alp et al. prove the correct q-Hermite-Hadamard inequality as follows:

Theorem 2. *Let $f\colon [a,b] \to \mathbb{R}$ be a convex differentiable function on (a,b) and $0 < q < 1$. Then we have*

$$f\left(\frac{qa+b}{1+q}\right) \le \frac{1}{b-a}\int_a^b f(x)\ {}_a d_q x \le \frac{qf(a)+f(b)}{1+q}. \qquad (1.4)$$

In the same paper, Alp et al. obtain inequalities for q-differentiable convex and quasi-convex mappings which are connected with the left hand part of the inequality (1.4).

2. PRELIMINARIES AND DEFINITIONS OF q-CALCULUS IN THE PLANE

The following definitions and properties for partial q_1, q_2, q_1q_2-derivatives and q_1q_2-integral of a function f on Δ are given in [10].

Definition 3. [10] *Let $f\colon \Delta \to \mathbb{R}$ be a continuous function of two variables, the partial q_1-derivatives, q_2-derivatives and q_1q_2-derivatives of f at $(x,y) \in \Delta$ can be defined as follows:*

$$\frac{{}_a\partial_{q_1} f(x,y)}{{}_a\partial_{q_1} x} = \frac{f(q_1x+(1-q_1)a, y) - f(x,y)}{(1-q_1)(x-a)},\ x \neq a, \qquad (2.1)$$

$$\frac{{}_c\partial_{q_2} f(x,y)}{{}_c\partial_{q_2} x} = \frac{f(x, q_2y+(1-q_2)c) - f(x,y)}{(1-q_2)(y-c)},\ y \neq c, \qquad (2.2)$$

$$\frac{{}_{a,c}\partial_{q_1,q_2} f(x,y)}{{}_a\partial_{q_1} x\ {}_c\partial_{q_2} x} = \frac{\left[\begin{array}{c} f(q_1x+(1-q_1)a, q_2y+(1-q_2)c) \\ -f(q_1x+(1-q_1)a, y) \\ -f(x, q_2y+(1-q_2)c) + f(x,y) \end{array}\right]}{(1-q_1)(1-q_2)(x-a)(y-c)},\ x \neq a, y \neq c. \qquad (2.3)$$

The function $f\colon \Delta \to \mathbb{R}$ is said to be partially q_1, q_2, q_1q_2-differentiable on Δ° if $\frac{{}_a\partial_{q_1} f(x,y)}{{}_a\partial_{q_1} x}$, $\frac{{}_c\partial_{q_2} f(x,y)}{{}_c\partial_{q_2} y}$ and $\frac{{}_{a,c}\partial^2_{q_1,q_2} f(x,y)}{{}_a\partial_{q_1} x\ {}_c\partial_{q_2} y}$ exist for all $(x,y) \in \Delta^\circ$ respectively.

Definition 4. [10] *Let $f\colon \Delta \to \mathbb{R}$ be a continuous function of two variables. Then the definite q_1q_2-integral on Δ is defined by*

$$\int_c^y \int_a^x f(t,s)\ {}_a d_{q_1} t\ {}_c d_{q_2} s = (1-q_1)(1-q_2)(x-a)(y-c) \times \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} q_1^n q_2^m f(q_1^n x + (1-q_1^n)a, q_2^m y + (1-q_2^m)c) \quad (2.4)$$

for $(x,y) \in \Delta$. *If* $(x_1,y_1) \in \Delta^\circ$, *then*

$$\int_{y_1}^y \int_{x_1}^x f(t,s)\ {}_a d_{q_1} t\ {}_c d_{q_2} s = \int_{y_1}^y \int_a^x f(t,s)\ {}_a d_{q_1} t\ {}_c d_{q_2} s - \int_{y_1}^y \int_a^{x_1} f(t,s)\ {}_a d_{q_1} t\ {}_c d_{q_2} s = \int_c^y \int_a^x f(t,s)\ {}_a d_{q_1} t\ {}_c d_{q_2} s - \int_c^{y_1} \int_a^x f(t,s)\ {}_a d_{q_1} t\ {}_c d_{q_2} s - \int_c^y \int_a^{x_1} f(t,s)\ {}_a d_{q_1} t\ {}_c d_{q_2} s + \int_c^{y_1} \int_a^{x_1} f(t,s)\ {}_a d_{q_1} t\ {}_c d_{q_2} s. \quad (2.5)$$

In [10], Latif et al. prove the quantum Hermite-Hadamard inequality of functions of two variables over finite rectangles as follows:

Theorem 3. *Suppose that* $f: \Delta \to \mathbb{R}$ *is the co-ordinated convex on* Δ. *Then one has the following inequalities*

$$f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \le \frac{1}{2(b-a)} \int_a^b f\left(x, \frac{c+d}{2}\right)\ {}_a d_{q_1} x + \frac{1}{2(d-c)} \int_c^d f\left(\frac{a+b}{2}, y\right)\ {}_c d_{q_2} y \quad (2.6)$$

$$\le \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)\ {}_c d_{q_2} y\ {}_a d_{q_1} x$$

$$\le \left[\begin{array}{l} \frac{q_1}{2(1+q_1)(d-c)} \int_c^d f(a,y)\ {}_c d_{q_2} y + \frac{q_1}{2(1+q_1)(d-c)} \int_c^d f(b,y)\ {}_c d_{q_2} y \\ + \frac{q_2}{2(1+q_2)(b-a)} \int_a^b f(x,c)\ {}_a d_{q_1} x + \frac{q_2}{2(1+q_2)(b-a)} \int_a^b f(x,d)\ {}_a d_{q_1} x \end{array}\right]$$

$$\le \frac{q_1 q_2 f(a,c) + q_1 f(a,d) + q_2 f(b,c) + f(b,d)}{(1+q_1)(1+q_2)}.$$

The first and second inequalities in (2.6) are not correct. In this paper, our aim is to give correct quantum Hermite-Hadamard type inequality of functions of two variables over finite rectangles and provide some quantum estimates between the middle and the leftmost terms in correct quantum Hermite-Hadamard inequalities of functions of two variables using convexity and quasi-convexity on the co-ordinates.

3. NEW QUANTUM HERMITE-HADAMARD TYPE INEQUALITIES ON THE CO-ORDINATES

Throughout this section, we will take

$$\mu_{q_1,q_2}(a,b,c,d)(f) := f\left(\frac{q_1 a + b}{1+q_1}, \frac{q_2 c + d}{1+q_2}\right) - \frac{1}{d-c} \int_c^d f\left(\frac{q_1 a + b}{1+q_1}, y\right)\ {}_c d_{q_2} y - \frac{1}{b-a} \int_a^b f\left(x, \frac{q_2 c + d}{1+q_2}\right)\ {}_a d_{q_1} x + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)\ {}_c d_{q_2} y\ {}_a d_{q_1} x.$$

Theorem 4. (Quantum Hermite-Hadamard inequalities on the co-ordinates) *Let* $f: \Delta \to \mathbb{R}$ *is the co-ordinated convex and partially differentiable function on* Δ. *Then we have the following inequalities*

$$f\left(\frac{q_1 a + b}{1+q_1}, \frac{q_2 c + d}{1+q_2}\right) \quad (3.1)$$

$$\le \frac{1}{2(b-a)} \int_a^b f\left(x, \frac{q_2 c + d}{1+q_2}\right)\ {}_a d_{q_1} x + \frac{1}{2(d-c)} \int_c^d f\left(\frac{q_1 a + b}{1+q_1}, y\right)\ {}_c d_{q_2} y$$

$$\leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\ {}_cd_{q_2}y\ {}_ad_{q_1}x \leq$$
$$\left[\begin{array}{l}\frac{q_1}{2(1+q_1)(d-c)}\int_c^d f(a,y)\ {}_cd_{q_2}y+\frac{q_1}{2(1+q_1)(d-c)}\int_c^d f(b,y)\ {}_cd_{q_2}y\\ +\frac{q_2}{2(1+q_2)(b-a)}\int_a^b f(x,c)\ {}_ad_{q_1}x+\frac{q_2}{2(1+q_2)(b-a)}\int_a^b f(x,d)\ {}_ad_{q_1}x\end{array}\right]\leq$$
$$\frac{q_1q_2f(a,c)+q_1f(a,d)+q_2f(b,c)+f(b,d)}{(1+q_1)(1+q_2)}.$$

The above inequalities are sharp.

Proof. Since f is the co-ordinated convex function on Δ and partially differentiable function on Δ°, it follows that the function $f_x:[c,d]\to\mathbb{R}$, $f_x(y):=f(x,y)$ is convex on $[c,d]$ and differentiable function on (c,d) for all $x\in[a,b]$. Then by using the q-Hermite-Hadamard inequality (1.4), one has

$$f_x\left(\frac{q_2c+d}{1+q_2}\right)\leq\frac{1}{d-c}\int_c^d f_x(y)\ {}_cd_{q_2}y\leq\frac{q_2f_x(c)+f_c(d)}{1+q_2},\quad x\in[a,b]. \tag{3.2}$$

That is

$$f\left(x,\frac{q_2c+d}{1+q_2}\right)\leq\frac{1}{d-c}\int_c^d f(x,y)\ {}_cd_{q_2}y\leq\frac{q_2f(x,c)+f(x,d)}{1+q_2},\quad x\in[a,b]. \tag{3.3}$$

By q_1-integrating of (3.3) on $[a,b]$, we have

$$\frac{1}{b-a}\int_a^b f\left(x,\frac{q_2c+d}{1+q_2}\right)\ {}_ad_{q_1}x\leq\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\ {}_cd_{q_2}y\ {}_ad_{q_1}x\leq$$
$$\frac{1}{1+q_2}\left[\frac{q_2}{b-a}\int_a^b f(x,c)\ {}_ad_{q_1}x+\frac{1}{b-a}\int_a^b f(x,d)\ {}_ad_{q_1}x\right]. \tag{3.4}$$

By a similar argument applied for the function $f_y:[a,b]\to\mathbb{R}$, $f_y(x):=f(x,y)$, we get

$$\frac{1}{d-c}\int_c^d f\left(\frac{q_1a+b}{1+q_1},y\right)\ {}_cd_{q_2}y\leq\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\ {}_cd_{q_2}y\ {}_ad_{q_1}x\leq$$
$$\frac{1}{1+q_1}\left[\frac{q_1}{d-c}\int_c^d f(a,y)\ {}_cd_{q_2}y+\int_c^d f(b,y)\ {}_cd_{q_2}y\right]. \tag{3.5}$$

Summing the inequalities (3.4) and (3.5), we have the following inequalities

$$\frac{1}{2(b-a)}\int_a^b f\left(x,\frac{q_2c+d}{1+q_2}\right)\ {}_ad_{q_1}x+\frac{1}{2(d-c)}\int_c^d f\left(\frac{q_1a+b}{1+q_1},y\right)\ {}_cd_{q_2}y\leq$$
$$\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\ {}_cd_{q_2}y\ {}_ad_{q_1}x\leq$$
$$\frac{q_2}{2(1+q_2)(b-a)}\int_a^b f(x,c)\ {}_ad_{q_1}x+\frac{1}{2(1+q_2)(b-a)}\int_a^b f(x,d)\ {}_ad_{q_1}x+\frac{q_1}{2(1+q_1)(d-c)}\int_c^d f(a,y)\ {}_cd_{q_2}y+$$
$$\frac{1}{2(1+q_1)(d-c)}\int_c^d f(b,y)\ {}_cd_{q_2}y. \tag{3.6}$$

Also by using the q-Hermite-Hadamard inequality (1.4), we have

$$f\left(\frac{q_1a+b}{1+q_1},\frac{q_2c+d}{1+q_2}\right)\leq\frac{1}{b-a}\int_a^b f\left(x,\frac{q_2c+d}{1+q_2}\right)\ {}_ad_{q_1}x \tag{3.7}$$

and

$$f\left(\frac{q_1a+b}{1+q_1},\frac{q_2c+d}{1+q_2}\right)\leq\frac{1}{d-c}\int_c^d f\left(\frac{q_1a+b}{1+q_1},y\right)\ {}_cd_{q_2}y. \tag{3.8}$$

Summing the inequalities (3.7) and (3.8), we have the following inequality

$$f\left(\frac{q_1a+b}{1+q_1},\frac{q_2c+d}{1+q_2}\right)\leq\frac{1}{2(b-a)}\int_a^b f\left(x,\frac{q_2c+d}{1+q_2}\right)\ {}_ad_{q_1}x \tag{3.9}$$
$$+\frac{1}{2(d-c)}\int_c^d f\left(\frac{q_1a+b}{1+q_1},y\right)\ {}_cd_{q_2}y.$$

Finally by using the q-Hermite-Hadamard inequality (1.4), we have

$$\frac{q_2}{2(1+q_2)}\left(\frac{1}{b-a}\int_a^b f(x,c)\ {}_ad_{q_1}x\right)\leq\frac{q_2}{2(1+q_2)}\left(\frac{q_1f(a,c)+f(b,c)}{1+q_1}\right), \tag{3.10}$$

$$\frac{q_2}{2(1+q_2)}\left(\frac{1}{b-a}\int_a^b f(x,d)\ {}_ad_{q_1}x\right) \leq \frac{q_2}{2(1+q_2)}\left(\frac{q_1 f(a,d)+f(b,d)}{1+q_1}\right), \tag{3.11}$$

$$\frac{q_1}{2(1+q_1)}\left(\frac{1}{d-c}\int_c^d f(a,y)\ {}_cd_{q_2}y\right) \leq \frac{q_1}{2(1+q_1)}\left(\frac{q_2 f(a,c)+f(a,d)}{1+q_2}\right), \tag{3.12}$$

$$\frac{q_1}{2(1+q_1)}\left(\frac{1}{d-c}\int_c^d f(b,y)\ {}_cd_{q_2}y\right) \leq \frac{q_1}{2(1+q_1)}\left(\frac{q_2 f(b,c)+f(b,d)}{1+q_2}\right). \tag{3.13}$$

Summing the inequalities (3.10)-(3.13), we have the following inequality

$$\begin{aligned}&\frac{q_2}{2(1+q_2)}\left(\frac{1}{b-a}\int_a^b f(x,c)\ {}_ad_{q_1}x\right)+\frac{q_2}{2(1+q_2)}\left(\frac{1}{b-a}\int_a^b f(x,d)\ {}_ad_{q_1}x\right)\\&+\frac{q_1}{2(1+q_1)}\left(\frac{1}{d-c}\int_c^d f(a,y)\ {}_cd_{q_2}y\right)+\frac{q_1}{2(1+q_1)}\left(\frac{1}{d-c}\int_c^d f(b,y)\ {}_cd_{q_2}y\right)\\&\leq \frac{q_1q_2 f(a,c)+q_1 f(a,d)+q_2 f(b,c)+f(b,d)}{(1+q_1)(1+q_2)}.\end{aligned} \tag{3.14}$$

By combining (3.6), (3.9) and (3.14), we have (3.1). Thus the proof is accomplished. ■

Remark 1. *In the Theorem 4, if one takes limit $q_1^- \to 1$ and $q_2^- \to 1$, then one has the Theorem 1.*

Lemma 2. *Let $f\colon \Delta \to \mathbb{R}$ be a twice partially q_1q_2-differentiable function on Δ°. If partial q_1q_2-derivative $\frac{{}_{a,c}\partial^2_{q_1,q_2} f}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}$ is continuous and integrable on Δ, then the following equality holds:*

$$\begin{aligned}&\mu_{q_1,q_2}(a,b,c,d)(f) = q_1q_2(b-a)(d-c)\\&\times\left[\int_0^1\int_0^1 \kappa(t,s)\frac{{}_{a,c}\partial^2_{q_1,q_2} f(tb+(1-t)a,sd+(1-s)c)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\ {}_0d_{q_2}s\ {}_0d_{q_1}t\right]\end{aligned} \tag{3.15}$$

where

$$\kappa(t,s)=\begin{cases} ts &, \quad (t,s)\in\left[0,\frac{1}{1+q_1}\right]\times\left[0,\frac{1}{1+q_2}\right]\\ t\left(s-\frac{1}{q_2}\right) &, \quad (t,s)\in\left[0,\frac{1}{1+q_1}\right]\times\left(\frac{1}{1+q_2},1\right]\\ s\left(t-\frac{1}{q_1}\right) &, \quad (t,s)\in\left(\frac{1}{1+q_1},1\right]\times\left[0,\frac{1}{1+q_2}\right]\\ \left(t-\frac{1}{q_1}\right)\left(s-\frac{1}{q_2}\right) &, \quad (t,s)\in\left(\frac{1}{1+q_1},1\right]\times\left(\frac{1}{1+q_2},1\right]\end{cases}.$$

Proof. It is clear that

$$\begin{aligned}&q_1q_2(b-a)(d-c)\left[\int_0^1\int_0^1 \kappa(t,s)\frac{{}_{a,c}\partial^2_{q_1,q_2} f(tb+(1-t)a,sd+(1-s)c)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\ {}_0d_{q_2}s\ {}_0d_{q_1}t\right]\\
&= q_1q_2(b-a)(d-c)\left[\int_0^{\frac{1}{1+q_1}}\int_0^{\frac{1}{1+q_2}} ts\frac{{}_{a,c}\partial^2_{q_1,q_2} f(tb+(1-t)a,sd+(1-s)c)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\ {}_0d_{q_2}s\ {}_0d_{q_1}t\right.\\
&+\int_{\frac{1}{1+q_1}}^1\int_0^{\frac{1}{1+q_2}} s\left(t-\frac{1}{q_1}\right)\frac{{}_{a,c}\partial^2_{q_1,q_2} f(tb+(1-t)a,sd+(1-s)c)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\ {}_0d_{q_2}s\ {}_0d_{q_1}t\\
&+\int_0^{\frac{1}{1+q_1}}\int_{\frac{1}{1+q_2}}^1 t\left(s-\frac{1}{q_2}\right)\frac{{}_{a,c}\partial^2_{q_1,q_2} f(tb+(1-t)a,sd+(1-s)c)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\ {}_0d_{q_2}s\ {}_0d_{q_1}t\\
&\left.+\int_{\frac{1}{1+q_1}}^1\int_{\frac{1}{1+q_2}}^1 \left(t-\frac{1}{q_1}\right)\left(s-\frac{1}{q_2}\right)\frac{{}_{a,c}\partial^2_{q_1,q_2} f(tb+(1-t)a,sd+(1-s)c)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\ {}_0d_{q_2}s\ {}_0d_{q_1}t\right]\\
&= q_1q_2(b-a)(d-c)\left[\int_0^{\frac{1}{1+q_1}}\int_0^{\frac{1}{1+q_2}} ts\frac{{}_{a,c}\partial^2_{q_1,q_2} f(tb+(1-t)a,sd+(1-s)c)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\ {}_0d_{q_2}s\ {}_0d_{q_1}t\right.\\
&+\int_{\frac{1}{1+q_1}}^1\int_0^{\frac{1}{1+q_2}} ts\frac{{}_{a,c}\partial^2_{q_1,q_2} f(tb+(1-t)a,sd+(1-s)c)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\ {}_0d_{q_2}s\ {}_0d_{q_1}t\\
&+\int_0^{\frac{1}{1+q_1}}\int_{\frac{1}{1+q_2}}^1 ts\frac{{}_{a,c}\partial^2_{q_1,q_2} f(tb+(1-t)a,sd+(1-s)c)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\ {}_0d_{q_2}s\ {}_0d_{q_1}t\end{aligned} \tag{3.16}$$

$$+\int_{\frac{1}{1+q_1}}^{1}\int_{\frac{1}{1+q_2}}^{1} ts\frac{{}_{a,c}\partial^2_{q_1,q_2}f(tb+(1-t)a,sd+(1-s)c)}{{}_a\partial_{q_1}t\,{}_c\partial_{q_2}s}\,{}_0d_{q_2}s\,{}_0d_{q_1}t$$
$$-\frac{1}{q_1}\int_{\frac{1}{1+q_1}}^{1}\int_0^{\frac{1}{1+q_2}} s\frac{{}_{a,c}\partial^2_{q_1,q_2}f(tb+(1-t)a,sd+(1-s)c)}{{}_a\partial_{q_1}t\,{}_c\partial_{q_2}s}\,{}_0d_{q_2}s\,{}_0d_{q_1}t$$
$$-\frac{1}{q_2}\int_0^{\frac{1}{1+q_1}}\int_{\frac{1}{1+q_2}}^{1} t\frac{{}_{a,c}\partial^2_{q_1,q_2}f(tb+(1-t)a,sd+(1-s)c)}{{}_a\partial_{q_1}t\,{}_c\partial_{q_2}s}\,{}_0d_{q_2}s\,{}_0d_{q_1}t$$
$$-\frac{1}{q_2}\int_{\frac{1}{1+q_1}}^{1}\int_{\frac{1}{1+q_2}}^{1} t\frac{{}_{a,c}\partial^2_{q_1,q_2}f(tb+(1-t)a,sd+(1-s)c)}{{}_a\partial_{q_1}t\,{}_c\partial_{q_2}s}\,{}_0d_{q_2}s\,{}_0d_{q_1}t$$
$$-\frac{1}{q_1}\int_{\frac{1}{1+q_1}}^{1}\int_{\frac{1}{1+q_2}}^{1} s\frac{{}_{a,c}\partial^2_{q_1,q_2}f(tb+(1-t)a,sd+(1-s)c)}{{}_a\partial_{q_1}t\,{}_c\partial_{q_2}s}\,{}_0d_{q_2}s\,{}_0d_{q_1}t$$
$$+\frac{1}{q_1q_2}\int_{\frac{1}{1+q_1}}^{1}\int_{\frac{1}{1+q_2}}^{1} \frac{{}_{a,c}\partial^2_{q_1,q_2}f(tb+(1-t)a,sd+(1-s)c)}{{}_a\partial_{q_1}t\,{}_c\partial_{q_2}s}\,{}_0d_{q_2}s\,{}_0d_{q_1}t\Bigg]$$
$$=\Bigg[q_1q_2(b-a)(d-c)\int_0^1\int_0^1 ts\frac{{}_{a,c}\partial^2_{q_1,q_2}f(tb+(1-t)a,sd+(1-s)c)}{{}_a\partial_{q_1}t\,{}_c\partial_{q_2}s}\,{}_0d_{q_2}s\,{}_0d_{q_1}t$$
$$-q_2(b-a)(d-c)\int_0^1\int_0^1 s\frac{{}_{a,c}\partial^2_{q_1,q_2}f(tb+(1-t)a,sd+(1-s)c)}{{}_a\partial_{q_1}t\,{}_c\partial_{q_2}s}\,{}_0d_{q_2}s\,{}_0d_{q_1}t$$
$$-q_1(b-a)(d-c)\int_0^1\int_0^1 t\frac{{}_{a,c}\partial^2_{q_1,q_2}f(tb+(1-t)a,sd+(1-s)c)}{{}_a\partial_{q_1}t\,{}_c\partial_{q_2}s}\,{}_0d_{q_2}s\,{}_0d_{q_1}t$$
$$+q_2(b-a)(d-c)\int_0^{\frac{1}{1+q_1}}\int_0^1 s\frac{{}_{a,c}\partial^2_{q_1,q_2}f(tb+(1-t)a,sd+(1-s)c)}{{}_a\partial_{q_1}t\,{}_c\partial_{q_2}s}\,{}_0d_{q_2}s\,{}_0d_{q_1}t$$
$$+q_1(b-a)(d-c)\int_0^1\int_0^{\frac{1}{1+q_1}} t\frac{{}_{a,c}\partial^2_{q_1,q_2}f(tb+(1-t)a,sd+(1-s)c)}{{}_a\partial_{q_1}t\,{}_c\partial_{q_2}s}\,{}_0d_{q_2}s\,{}_0d_{q_1}t$$

$$+(b-a)(d-c)\int_0^1\int_0^1 \frac{{}_{a,c}\partial^2_{q_1,q_2}f(tb+(1-t)a,sd+(1-s)c)}{{}_a\partial_{q_1}t\,{}_c\partial_{q_2}s}\,{}_0d_{q_2}s\,{}_0d_{q_1}t$$
$$-(b-a)(d-c)\int_0^{\frac{1}{1+q_1}}\int_0^1 \frac{{}_{a,c}\partial^2_{q_1,q_2}f(tb+(1-t)a,sd+(1-s)c)}{{}_a\partial_{q_1}t\,{}_c\partial_{q_2}s}\,{}_0d_{q_2}s\,{}_0d_{q_1}t$$
$$-(b-a)(d-c)\int_0^1\int_0^{\frac{1}{1+q_1}} \frac{{}_{a,c}\partial^2_{q_1,q_2}f(tb+(1-t)a,sd+(1-s)c)}{{}_a\partial_{q_1}t\,{}_c\partial_{q_2}s}\,{}_0d_{q_2}s\,{}_0d_{q_1}t$$
$$+(b-a)(d-c)\int_0^{\frac{1}{1+q_1}}\int_0^{\frac{1}{1+q_2}} \frac{{}_{a,c}\partial^2_{q_1,q_2}f(tb+(1-t)a,sd+(1-s)c)}{{}_a\partial_{q_1}t\,{}_c\partial_{q_2}s}\,{}_0d_{q_2}s\,{}_0d_{q_1}t\,.$$

We use Definition 3 and Definition 4 to calculate the appearing last nine integrals in (3.16). In terms of brevity, we will omit the details.

$$q_1q_2(b-a)(d-c)\int_0^1\int_0^1 ts\frac{{}_{a,c}\partial^2_{q_1,q_2}f(tb+(1-t)a,sd+(1-s)c)}{{}_a\partial_{q_1}t\,{}_c\partial_{q_2}s}\,{}_0d_{q_2}s\,{}_0d_{q_1}t \tag{3.17}$$
$$=$$
$$-f(b,d)-\frac{1}{b-a}\int_a^b f(x,d)\,{}_0d_{q_1}x-\frac{1}{d-c}\int_c^d f(b,y)\,{}_0d_{q_2}y+$$
$$\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,{}_cd_{q_2}y\,{}_ad_{q_1}x\,,$$

$$q_2(b-a)(d-c)\int_0^1\int_0^1 s\frac{{}_{a,c}\partial^2_{q_1,q_2}f(tb+(1-t)a,sd+(1-s)c)}{{}_a\partial_{q_1}t\,{}_c\partial_{q_2}s}\,{}_0d_{q_2}s\,{}_0d_{q_1}t \tag{3.18}$$
$$=-f(b,d)-\frac{1}{d-c}\int_c^d f(b,y)\,{}_0d_{q_2}y\,,$$

$$q_1(b-a)(d-c)\int_0^1\int_0^1 t\frac{{}_{a,c}\partial^2_{q_1,q_2}f(tb+(1-t)a,sd+(1-s)c)}{{}_a\partial_{q_1}t\,{}_c\partial_{q_2}s}\,{}_0d_{q_2}s\,{}_0d_{q_1}t \tag{3.19}$$
$$=-f(b,d)-\frac{1}{b-a}\int_a^b f(x,d)\,{}_0d_{q_1}x\,,$$

$$q_2(b-a)(d-c)\int_0^{\frac{1}{1+q_1}}\int_0^1 s\frac{{}_{a,c}\partial^2_{q_1,q_2}f(tb+(1-t)a,sd+(1-s)c)}{{}_a\partial_{q_1}t\,{}_c\partial_{q_2}s}\,{}_0d_{q_2}s\,{}_0d_{q_1}t \tag{3.20}$$

$= -f\left(\frac{q_1a+b}{1+q_1}, d\right) - \frac{1}{d-c}\int_c^d f\left(\frac{q_1a+b}{1+q_1}, y\right) \,_0d_{q_2}y\,,$

$q_1(b-a)(d-c)\int_0^1\int_0^{\frac{1}{1+q_1}} t\,\frac{{}_{a,c}\partial^2_{q_1,q_2}f(tb+(1-t)a, sd+(1-s)c)}{{}_a\partial_{q_1}t\,{}_c\partial_{q_2}s}\,_0d_{q_2}s\,_0d_{q_1}t$
(3.21)
$= -f\left(b, \frac{q_2c+d}{1+q_2}\right) - \frac{1}{b-a}\int_a^b f\left(x, \frac{q_2c+d}{1+q_2}\right) \,_0d_{q_1}x\,,$

$$(b-a)(d-c)\int_0^1\int_0^1 \frac{{}_{a,c}\partial^2_{q_1,q_2}f(tb+(1-t)a, sd+(1-s)c)}{{}_a\partial_{q_1}t\,{}_c\partial_{q_2}s}\,_0d_{q_2}s\,_0d_{q_1}t = -f(b,d), \tag{3.22}$$

$$(b-a)(d-c)\int_0^{\frac{1}{1+q_1}}\int_0^1 \frac{{}_{a,c}\partial^2_{q_1,q_2}f(tb+(1-t)a, sd+(1-s)c)}{{}_a\partial_{q_1}t\,{}_c\partial_{q_2}s}\,_0d_{q_2}s\,_0d_{q_1}t = -f\left(\frac{q_1a+b}{1+q_1}, d\right), \tag{3.23}$$

$$(b-a)(d-c)\int_0^1\int_0^{\frac{1}{1+q_1}} \frac{{}_{a,c}\partial^2_{q_1,q_2}f(tb+(1-t)a, sd+(1-s)c)}{{}_a\partial_{q_1}t\,{}_c\partial_{q_2}s}\,_0d_{q_2}s\,_0d_{q_1}t = -f\left(b, \frac{q_2c+d}{1+q_2}\right), \tag{3.24}$$

$$(b-a)(d-c)\int_0^{\frac{1}{1+q_1}}\int_0^{\frac{1}{1+q_2}} \frac{{}_{a,c}\partial^2_{q_1,q_2}f(tb+(1-t)a, sd+(1-s)c)}{{}_a\partial_{q_1}t\,{}_c\partial_{q_2}s}\,_0d_{q_2}s\,_0d_{q_1}t = f\left(\frac{q_1a+b}{1+q_1}, \frac{q_2c+d}{1+q_2}\right). \tag{3.25}$$

A combination of (3.16)-(3.25), we have (3.15). Thus the proof is accomplished. ■

Remark 2. *In Lemma 2, if one takes limit $q_1^- \to 1$ and $q_2^- \to 1$, then one has the Lemma 1.*

Theorem 5. *Let $f\colon \Delta \to \mathbb{R}$ be a twice partially q_1q_2-differentiable function on Δ°. If partial q_1q_2-derivative $\frac{{}_{a,c}\partial^2_{q_1,q_2}f}{{}_a\partial_{q_1}t\,{}_c\partial_{q_2}s}$ is continuous and integrable on Δ and $\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f}{{}_a\partial_{q_1}t\,{}_c\partial_{q_2}s}\right|^r$ is convex on the co-ordinates on Δ for $r \geq 0$, then the following inequality holds:*

$$\begin{aligned}\left|\mu_{q_1,q_2}(a,b,c,d)(f)\right| &\leq q_1q_2(b-a)(d-c)\,C_1^{1-\frac{1}{r}}(q_1,q_2)\\ &\times\Bigg[\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(b,d)}{{}_a\partial_{q_1}t\,{}_c\partial_{q_2}s}\right|^r C_2(q_1,q_2) + \left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(a,d)}{{}_a\partial_{q_1}t\,{}_c\partial_{q_2}s}\right|^r C_3(q_1,q_2)\\ &+\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(b,c)}{{}_a\partial_{q_1}t\,{}_c\partial_{q_2}s}\right|^r C_4(q_1,q_2) + \left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(a,c)}{{}_a\partial_{q_1}t\,{}_c\partial_{q_2}s}\right|^r C_5(q_1,q_2)\Bigg]^{\frac{1}{r}}\end{aligned} \tag{3.26}$$

where

$$C_1(q_1,q_2) = \frac{4}{(1+q_1)^3(1+q_2)^3},$$

$$C_2(q_1,q_2) = \frac{9}{(1+q_1)^3(1+q_2)^3(1+q_1+q_1^2)(1+q_2+q_2^2)},$$

$$C_3(q_1,q_2) = \frac{\begin{pmatrix}-q_1^3q_2^3 - q_1^3q_2^2 + 5q_1^3q_2 - q_1^2q_2^3 + 6q_1^2q_2 + q_1^2\\ -q_1q_2^3 - 3q_1q_2 + q_1 + q_2^2 + q_2 + 1\end{pmatrix}}{q_1q_2(1+q_1)^3(1+q_2)^3(1+q_1+q_1^2)(1+q_2+q_2^2)},$$

$$C_4(q_1,q_2) = \frac{\begin{pmatrix}-q_2^3q_1^3 - q_2^3q_1^2 + 5q_2^3q_1 - q_2^2q_1^3 + 6q_2^2q_1 + q_2^2\\ -q_2q_1^3 - 3q_2q_1 + q_2 + q_1^2 + q_1 + 1\end{pmatrix}}{q_1q_2(1+q_1)^3(1+q_2)^3(1+q_1+q_1^2)(1+q_2+q_2^2)},$$

$$C_4(q_1,q_2)=\frac{\begin{pmatrix}-2q_1^5q_2^3-2q_1^5q_2-2q_1^5+2q_1^4q_2^3-4q_1^4q_2^2-4q_1^4q_2-6q_1^4+2q_1^3q_2^4+16q_1^3q_2^3\\+10q_1^3q_2^2+2q_1^3q_2+6q_1^3-2q_1^2q_2^5-4q_1^2q_2^4+10q_1^2q_2^3+8q_1^2q_2^2+4q_1^2q_2-4q_1^2\\-2q_1q_2^5-4q_1q_2^4+2q_1q_2^3+4q_1q_2^2+9q_1q_2-2q_2^5-6q_2^4-6q_2^3-4q_2^2\end{pmatrix}}{q_1q_2(1+q_1)^3(1+q_2)^3(1+q_1+q_1^2)(1+q_2+q_2^2)}.$$

Proof. Taking the absolute value on both sides of the equality (3.15), using q_1q_2-power mean inequality for functions of two variables (see [10, Theorem 5]) and the convexity of $\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|^r$ (see Definition 1) on the co-ordinates on Δ, we have

$$\left|\mu_{q_1,q_2}(a,b,c,d)(f)\right|\le q_1q_2(b-a)(d-c)$$

$$\times\left[\int_0^1\int_0^1|\kappa(t,s)|\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(tb+(1-t)a,sd+(1-s)c)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|{}_0d_{q_2}s\ {}_0d_{q_1}t\right]$$

$$\le q_1q_2(b-a)(d-c)\left(\int_0^1\int_0^1|\kappa(t,s)|\,{}_0d_{q_2}s\ {}_0d_{q_1}t\right)^{1-\frac{1}{r}}$$

$$\times\left(\int_0^1\int_0^1|\kappa(t,s)|\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(tb+(1-t)a,sd+(1-s)c)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|^r{}_0d_{q_2}s\ {}_0d_{q_1}t\right)^{\frac{1}{r}}$$

$$=q_1q_2(b-a)(d-c)\left(\begin{array}{l}\displaystyle\int_0^{\frac{1}{1+q_1}}\int_0^{\frac{1}{1+q_2}}ts\,{}_0d_{q_2}s\ {}_0d_{q_1}t\\ \displaystyle+\int_0^{\frac{1}{1+q_1}}\int_{\frac{1}{1+q_2}}^1t\left(\frac{1}{q_2}-s\right){}_0d_{q_2}s\ {}_0d_{q_1}t\\ \displaystyle+\int_{\frac{1}{1+q_1}}^1\int_0^{\frac{1}{1+q_2}}s\left(\frac{1}{q_1}-t\right){}_0d_{q_2}s\ {}_0d_{q_1}t\\ \displaystyle+\int_{\frac{1}{1+q_1}}^1\int_{\frac{1}{1+q_2}}^1\left(\frac{1}{q_1}-t\right)\left(\frac{1}{q_2}-s\right){}_0d_{q_2}s\ {}_0d_{q_1}t\end{array}\right)$$

$$\times\left[\int_0^{\frac{1}{1+q_1}}\int_0^{\frac{1}{1+q_2}}ts\begin{bmatrix}ts\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(b,d)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|^r\\ s(1-t)\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(a,d)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|^r\\ t(1-s)\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(b,c)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|^r\\ (1-t)(1-s)\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(a,c)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|^r\end{bmatrix}{}_0d_{q_2}s\ {}_0d_{q_1}t\right.$$

$$+\int_0^{\frac{1}{1+q_1}}\int_{\frac{1}{1+q_2}}^{1} t\left(\frac{1}{q_2}-s\right)\begin{bmatrix} ts\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(b,d)}{{}_a\partial_{q_1}t\;{}_c\partial_{q_2}s}\right|^r \\ s(1-t)\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(a,d)}{{}_a\partial_{q_1}t\;{}_c\partial_{q_2}s}\right|^r \\ t(1-s)\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(b,c)}{{}_a\partial_{q_1}t\;{}_c\partial_{q_2}s}\right|^r \\ (1-t)(1-s)\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(a,c)}{{}_a\partial_{q_1}t\;{}_c\partial_{q_2}s}\right|^r \end{bmatrix} {}_0d_{q_2}s\;{}_0d_{q_1}t$$

$$+\int_{\frac{1}{1+q_1}}^{1}\int_0^{\frac{1}{1+q_2}} s\left(\frac{1}{q_1}-t\right)\begin{bmatrix} ts\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(b,d)}{{}_a\partial_{q_1}t\;{}_c\partial_{q_2}s}\right|^r \\ s(1-t)\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(a,d)}{{}_a\partial_{q_1}t\;{}_c\partial_{q_2}s}\right|^r \\ t(1-s)\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(b,c)}{{}_a\partial_{q_1}t\;{}_c\partial_{q_2}s}\right|^r \\ (1-t)(1-s)\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(a,c)}{{}_a\partial_{q_1}t\;{}_c\partial_{q_2}s}\right|^r \end{bmatrix} {}_0d_{q_2}s\;{}_0d_{q_1}t$$

$$\left.+\int_{\frac{1}{1+q_1}}^{1}\int_{\frac{1}{1+q_2}}^{1} \left(\frac{1}{q_1}-t\right)\left(\frac{1}{q_2}-s\right)\begin{bmatrix} ts\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(b,d)}{{}_a\partial_{q_1}t\;{}_c\partial_{q_2}s}\right|^r \\ s(1-t)\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(a,d)}{{}_a\partial_{q_1}t\;{}_c\partial_{q_2}s}\right|^r \\ t(1-s)\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(b,c)}{{}_a\partial_{q_1}t\;{}_c\partial_{q_2}s}\right|^r \\ (1-t)(1-s)\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(a,c)}{{}_a\partial_{q_1}t\;{}_c\partial_{q_2}s}\right|^r \end{bmatrix} {}_0d_{q_2}s\;{}_0d_{q_1}t\right]^{\frac{1}{r}}$$

$$= q_1q_2(b-a)(d-c)\,C_1^{1-\frac{1}{r}}(q_1,q_2)$$
$$\times\left[\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(b,d)}{{}_a\partial_{q_1}t\;{}_c\partial_{q_2}s}\right|^r C_2(q_1,q_2)+\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(a,d)}{{}_a\partial_{q_1}t\;{}_c\partial_{q_2}s}\right|^r C_3(q_1,q_2)\right.$$
$$\left.+\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(b,c)}{{}_a\partial_{q_1}t\;{}_c\partial_{q_2}s}\right|^r C_4(q_1,q_2)+\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(a,c)}{{}_a\partial_{q_1}t\;{}_c\partial_{q_2}s}\right|^r C_5(q_1,q_2)\right]^{\frac{1}{r}}.$$

Note that, in order to compute the coefficients $C_1(q_1,q_2)$, $C_2(q_1,q_2)$, $C_3(q_1,q_2)$, $C_4(q_1,q_2)$ and $C_5(q_1,q_2)$, we calculate 16, 16, 25, 25 and 36 double quantum integrals respectively. In terms of brevity, we omit the details. Thus the proof is accomplished. ■

Corollary 1. *In Theorem 5,*

(1) If one takes limit $q_1^- \to 1$ and $q_2^- \to 1$, then one has

$$\left|\begin{matrix}\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dy\,dx+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\ -\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)\,dy-\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)\,dx\end{matrix}\right|\leq\frac{(b-a)(d-c)}{16} \tag{3.27}$$

$$\times\left[\frac{\left|\frac{\partial^2 f(b,d)}{\partial t\partial s}\right|^r+\left|\frac{\partial^2 f(a,d)}{\partial t\partial s}\right|^r+\left|\frac{\partial^2 f(b,c)}{\partial t\partial s}\right|^r+\left|\frac{\partial^2 f(a,c)}{\partial t\partial s}\right|^r}{4}\right]^{\frac{1}{r}},$$

(2) If one takes $r=1$, limit $q_1^- \to 1$ and $q_2^- \to 1$, then one has

$$\left| \begin{array}{l} \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)\, dy\, dx + f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\ -\frac{1}{d-c} \int_c^d f\left(\frac{a+b}{2}, y\right) dy - \frac{1}{b-a} \int_a^b f\left(x, \frac{c+d}{2}\right) dx \end{array} \right| \le \frac{(b-a)(d-c)}{16} \tag{3.28}$$

$$\times \left[\frac{\left|\frac{\partial^2 f(b,d)}{\partial t \partial s}\right| + \left|\frac{\partial^2 f(a,d)}{\partial t \partial s}\right| + \left|\frac{\partial^2 f(b,c)}{\partial t \partial s}\right| + \left|\frac{\partial^2 f(a,c)}{\partial t \partial s}\right|}{4} \right]^{\frac{1}{r}}$$

Remark 3. *In (3.27) we recapture [8, Theorem 4, inequality (2.13)], in (3.28) we recapture [8, Theorem 4, inequality (2.4)].*

Theorem 6. *Let $f\colon \Delta \to \mathbb{R}$ be a twice partially $q_1 q_2$-differentiable function on Δ°. If partial $q_1 q_2$-derivative $\frac{{}_{a,c}\partial^2_{q_1,q_2} f}{{}_a\partial_{q_1} t\, {}_c\partial_{q_2} s}$ is continuous and integrable on Δ and $\left|\frac{{}_{a,c}\partial^2_{q_1,q_2} f}{{}_a\partial_{q_1} t\, {}_c\partial_{q_2} s}\right|^r$ is convex on the co-ordinates on Δ for $r>0$, then the following inequality holds:*

$$\left|\mu_{q_1,q_2}(a,b,c,d)(f)\right| \le q_1 q_2 (b-a)(d-c) \left(\int_0^1 \int_0^1 |\kappa(t,s)|^p\, {}_0d_{q_2}s\, {}_0d_{q_1}t\right)^{\frac{1}{p}} \tag{3.29}$$

$$\times \left[\frac{\left|\frac{{}_{a,c}\partial^2_{q_1,q_2} f(b,d)}{{}_a\partial_{q_1} t\, {}_c\partial_{q_2} s}\right|^r + q_1 \left|\frac{{}_{a,c}\partial^2_{q_1,q_2} f(a,d)}{{}_a\partial_{q_1} t\, {}_c\partial_{q_2} s}\right|^r + q_2 \left|\frac{{}_{a,c}\partial^2_{q_1,q_2} f(b,c)}{{}_a\partial_{q_1} t\, {}_c\partial_{q_2} s}\right|^r + q_1 q_2 \left|\frac{{}_{a,c}\partial^2_{q_1,q_2} f(a,c)}{{}_a\partial_{q_1} t\, {}_c\partial_{q_2} s}\right|^r}{(1+q_1)(1+q_2)} \right]^{\frac{1}{r}}.$$

where $\kappa(t,s)$ is the same in Lemma 2 and $\frac{1}{p}+\frac{1}{r}=1$.

Proof. Taking the absolute value on both sides of the equality (3.15), using $q_1 q_2$-Hölder inequality for functions of two variables (see [10, Theorem 5]) and the convexity of $\left|\frac{{}_{a,c}\partial^2_{q_1,q_2} f}{{}_a\partial_{q_1} t\, {}_c\partial_{q_2} s}\right|^r$ (see Definition 1) on the co-ordinates on Δ, we have

$$\left|\mu_{q_1,q_2}(a,b,c,d)(f)\right| \le q_1 q_2 (b-a)(d-c)$$

$$\times \left[\int_0^1 \int_0^1 |\kappa(t,s)| \left| \frac{{}_{a,c}\partial^2_{q_1,q_2} f(tb+(1-t)a, sd+(1-s)c)}{{}_a\partial_{q_1} t\, {}_c\partial_{q_2} s} \right| {}_0d_{q_2}s\, {}_0d_{q_1}t \right]$$

$$\le q_1 q_2 (b-a)(d-c) \left[\left(\int_0^1 \int_0^1 |\kappa(t,s)|^p\, {}_0d_{q_2}s\, {}_0d_{q_1}t \right)^{\frac{1}{p}} \right.$$

$$\left. \times \left(\int_0^1 \int_0^1 |\kappa(t,s)| \left| \frac{{}_{a,c}\partial^2_{q_1,q_2} f(tb+(1-t)a, sd+(1-s)c)}{{}_a\partial_{q_1} t\, {}_c\partial_{q_2} s} \right|^r {}_0d_{q_2}s\, {}_0d_{q_1}t \right)^{\frac{1}{r}} \right]$$

$$\le q_1 q_2 (b-a)(d-c) \left[\left(\int_0^1 \int_0^1 |\kappa(t,s)|^p\, {}_0d_{q_2}s\, {}_0d_{q_1}t \right)^{\frac{1}{p}} \right.$$

$$\times\left[\begin{array}{c}\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(b,d)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|^r\int_0^1\int_0^1 ts\ {}_0d_{q_2}s\ {}_0d_{q_1}t\\+\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(a,d)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|^r\int_0^1\int_0^1 s(1-t)\ {}_0d_{q_2}s\ {}_0d_{q_1}t\\+\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(b,c)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|^r\int_0^1\int_0^1 t(1-s)\ {}_0d_{q_2}s\ {}_0d_{q_1}t\\+\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(a,c)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|^r\int_0^1\int_0^1(1-t)(1-s)\ {}_0d_{q_2}s\ {}_0d_{q_1}t\end{array}\right]^{\frac{1}{r}}$$

$$=q_1q_2(b-a)(d-c)\left(\int_0^1\int_0^1|\kappa(t,s)|^p\ {}_0d_{q_2}s\ {}_0d_{q_1}t\right)^{\frac{1}{p}}$$

$$\times\left[\frac{\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(b,d)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|^r+q_1\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(a,d)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|^r+q_2\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(b,c)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|^r+q_1q_2\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(a,c)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|^r}{(1+q_1)(1+q_2)}\right]^{\frac{1}{r}}.$$

Thus the proof is accomplished. ■

Remark 4. *In theorem 6, if one takes limit $q_1^-\to 1$ and $q_2^-\to 1$, then one has [8, Theorem 3].*
In terms of brevity, we will use the following notations

$$L=\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(a,c)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|,\ M=\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(a,d)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|,\ N=\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(b,c)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|,\ O=\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(b,d)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|,$$

$$P=\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f\left(a,\frac{q_2c+d}{1+q_2}\right)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|,\ Q=\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f\left(b,\frac{q_2c+d}{1+q_2}\right)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|,\ R=\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f\left(\frac{q_1a+b}{1+q_1},c\right)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|,$$

$$S=\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f\left(\frac{q_1a+b}{1+q_1},d\right)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|\ \text{and}\ T=\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f\left(\frac{q_1a+b}{1+q_1},\frac{q_2c+d}{1+q_2}\right)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|.$$

Theorem 7. *Let $f\colon\ \Delta\to\mathbb{R}$ be a twice partially q_1q_2-differentiable function on Δ°. If partial q_1q_2-derivative $\frac{{}_{a,c}\partial^2_{q_1,q_2}f}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}$ is continuous and integrable on Δ and $\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|^r$ is quasi-convex on the co-ordinates on Δ for $r\geq 0$, then the following inequality holds:*

$$\left|\mu_{q_1,q_2}(a,b,c,d)(f)\right|\leq(b-a)(d-c)\frac{q_1q_2}{(1+q_1)^3(1+q_2)^3}(A+B+C+D)\tag{3.30}$$

where $A=\sup\{L,P,R,T\}$, $B=\sup\{N,R,Q,T\}$, $C=\sup\{M,P,S,T\}$ and $D=\sup\{O,Q,S,T\}$.
Proof. Taking the absolute value on both sides of the equality (3.15), using q_1q_2-power mean inequality for functions of two variables (see [10, Theorem 5]) and the quasi-convexity of $\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|^r$ (see Definition 1) on the co-ordinates on Δ, we have

$$\left|\mu_{q_1,q_2}(a,b,c,d)(f)\right|\leq q_1q_2(b-a)(d-c)$$

$$\times\left[\int_0^1\int_0^1|\kappa(t,s)|\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(tb+(1-t)a,sd+(1-s)c)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|{}_0d_{q_2}s\ {}_0d_{q_1}t\right]$$

$$=q_1q_2(b-a)(d-c)$$

$$\times\left[\begin{array}{c}\int_0^{\frac{1}{1+q_1}}\int_0^{\frac{1}{1+q_2}} ts\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(tb+(1-t)a,sd+(1-s)c)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right| {}_0d_{q_2}s\ {}_0d_{q_1}t\\ +\int_0^{\frac{1}{1+q_1}}\int_{\frac{1}{1+q_2}}^1 t\left(\frac{1}{q_2}-s\right)\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(tb+(1-t)a,sd+(1-s)c)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right| {}_0d_{q_2}s\ {}_0d_{q_1}t\\ +\int_{\frac{1}{1+q_1}}^1\int_0^{\frac{1}{1+q_2}} s\left(\frac{1}{q_1}-t\right)\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(tb+(1-t)a,sd+(1-s)c)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right| {}_0d_{q_2}s\ {}_0d_{q_1}t\\ +\int_{\frac{1}{1+q_1}}^1\int_{\frac{1}{1+q_2}}^1 \left(\frac{1}{q_1}-t\right)\left(\frac{1}{q_2}-s\right)\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(tb+(1-t)a,sd+(1-s)c)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right| {}_0d_{q_2}s\ {}_0d_{q_1}t\end{array}\right]$$

$$\leq q_1q_2(b-a)(d-c)$$

$$\times\Bigg[\left(\begin{array}{c}\left[\int_0^{\frac{1}{1+q_1}}\int_0^{\frac{1}{1+q_2}} ts\,{}_0d_{q_2}s\ {}_0d_{q_1}t\right]^{1-\frac{1}{r}}\\ \left[\int_0^{\frac{1}{1+q_1}}\int_0^{\frac{1}{1+q_2}} ts\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(tb+(1-t)a,sd+(1-s)c)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|^r {}_0d_{q_2}s\ {}_0d_{q_1}t\right]^{\frac{1}{r}}\end{array}\right)$$

$$+\left(\begin{array}{c}\left[\int_0^{\frac{1}{1+q_1}}\int_{\frac{1}{1+q_2}}^1 t\left(\frac{1}{q_2}-s\right) {}_0d_{q_2}s\ {}_0d_{q_1}t\right]^{1-\frac{1}{r}}\\ \left[\int_0^{\frac{1}{1+q_1}}\int_{\frac{1}{1+q_2}}^1 t\left(\frac{1}{q_2}-s\right)\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(tb+(1-t)a,sd+(1-s)c)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|^r {}_0d_{q_2}s\ {}_0d_{q_1}t\right]^{\frac{1}{r}}\end{array}\right)$$

$$+\left(\begin{array}{c}\left[\int_{\frac{1}{1+q_1}}^1\int_0^{\frac{1}{1+q_2}} s\left(\frac{1}{q_1}-t\right) {}_0d_{q_2}s\ {}_0d_{q_1}t\right]^{1-\frac{1}{r}}\\ \left[\int_{\frac{1}{1+q_1}}^1\int_0^{\frac{1}{1+q_2}} s\left(\frac{1}{q_1}-t\right)\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(tb+(1-t)a,sd+(1-s)c)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|^r {}_0d_{q_2}s\ {}_0d_{q_1}t\right]^{\frac{1}{r}}\end{array}\right)$$

$$+\left(\begin{array}{c}\left[\int_{\frac{1}{1+q_1}}^1\int_{\frac{1}{1+q_2}}^1 \left(\frac{1}{q_1}-t\right)\left(\frac{1}{q_2}-s\right) {}_0d_{q_2}s\ {}_0d_{q_1}t\right]^{1-\frac{1}{r}}\\ \left[\int_{\frac{1}{1+q_1}}^1\int_{\frac{1}{1+q_2}}^1 \left(\frac{1}{q_1}-t\right)\left(\frac{1}{q_2}-s\right)\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(tb+(1-t)a,sd+(1-s)c)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|^r {}_0d_{q_2}s\ {}_0d_{q_1}t\right]^{\frac{1}{r}}\end{array}\right)\Bigg]$$

$$\leq q_1q_2(b-a)(d-c)\Bigg[\sup\{L,P,R,T\}\left[\int_0^{\frac{1}{1+q_1}}\int_0^{\frac{1}{1+q_2}} ts\,{}_0d_{q_2}s\ {}_0d_{q_1}t\right]$$

$$+\sup\{N,R,Q,T\}\left[\int_{\frac{1}{1+q_1}}^1\int_0^{\frac{1}{1+q_2}} s\left(\frac{1}{q_1}-t\right) {}_0d_{q_2}s\ {}_0d_{q_1}t\right]$$

$$+\sup\{M,P,S,T\}\left[\int_0^{\frac{1}{1+q_1}}\int_{\frac{1}{1+q_2}}^1 t\left(\frac{1}{q_2}-s\right) {}_0d_{q_2}s\ {}_0d_{q_1}t\right]$$

$$+\sup\{O,Q,S,T\}\left[\int_{\frac{1}{1+q_1}}^1\int_{\frac{1}{1+q_2}}^1 \left(\frac{1}{q_1}-t\right)\left(\frac{1}{q_2}-s\right) {}_0d_{q_2}s\ {}_0d_{q_1}t\right]\Bigg]$$

$$=(b-a)(d-c)\frac{q_1q_2}{(1+q_1)^3(1+q_2)^3}(A+B+C+D).$$

Thus the proof is accomplished. ■

Corollary 2. *Suppose the conditions of the Theorem 7 are satisfied. Additionally if*

(1) $\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|$ *is increasing on the co-ordinates on* $\triangle$*, then*

$$\left|\mu_{q_1,q_2}(a,b,c,d)(f)\right| \leq (b-a)(d-c)\frac{q_1q_2}{(1+q_1)^3(1+q_2)^3}(O+Q+S+T), \tag{3.31}$$

(2) $\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|$ *is decreasing on the co-ordinates on* Δ, *then*

$$\left|\mu_{q_1,q_2}(a,b,c,d)(f)\right| \le (b-a)(d-c)\frac{q_1q_2}{(1+q_1)^3(1+q_2)^3}(L+R+P+T), \tag{3.32}$$

(3) $T = R = S = P = Q = 0$, *then*

$$\left|\mu_{q_1,q_2}(a,b,c,d)(f)\right| \le (b-a)(d-c)\frac{q_1q_2}{(1+q_1)^3(1+q_2)^3}(L+M+N+O), \tag{3.33}$$

(4) $L = M = N = O = P = Q = R = S = 0$, *then*

$$\left|\mu_{q_1,q_2}(a,b,c,d)(f)\right| \le (b-a)(d-c)\frac{q_1q_2}{(1+q_1)^3(1+q_2)^3}T. \tag{3.34}$$

Remark 5. *In Theorem 7, if we take limit* $q_1^- \to 1$ *and* $q_2^- \to 1$, *then we recapture [9, Theorem 2 and Theorem 4], in Corollary 2, if we take limit* $q_1^- \to 1$ *and* $q_2^- \to 1$, *then we recapture [9, Corollary 1 and Corollary 3].*

Theorem 8. *Let* $f\colon \Delta \to \mathbb{R}$ *be a twice partially* q_1q_2*-differentiable function on* Δ°. *If partial* q_1q_2*-derivative* $\frac{{}_{a,c}\partial^2_{q_1,q_2}f}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}$ *is continuous and integrable on* Δ *and* $\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|^r$ *is quasi-convex on the co-ordinates on* Δ *for* $r > 0$, *then the following inequality holds:*

$$\left|\mu_{q_1,q_2}(a,b,c,d)(f)\right| \le q_1q_2(b-a)(d-c) \tag{3.35}$$
$$\times\begin{bmatrix} C_6(q_1,q_2,p)\left(\frac{1}{(1+q_1)(1+q_2)}\right)^{\frac{1}{r}}A + C_7(q_1,q_2,p)\left(\frac{q_1}{(1+q_1)(1+q_2)}\right)^{\frac{1}{r}}B \\ +C_8(q_1,q_2,p)\left(\frac{q_2}{(1+q_1)(1+q_2)}\right)^{\frac{1}{r}}C + C_9(q_1,q_2,p)\left(\frac{q_1q_2}{(1+q_1)(1+q_2)}\right)^{\frac{1}{r}}D\end{bmatrix}.$$

where A, B, C, D *are the same in Theorem 7*

$$C_6(q_1,q_2,p) = \left[\int_0^{\frac{1}{1+q_1}}\int_0^{\frac{1}{1+q_2}} t^p s^p\ {}_0d_{q_2}s\ {}_0d_{q_1}t\right]^{\frac{1}{p}},$$

$$C_7(q_1,q_2,p) = \left[\int_{\frac{1}{1+q_1}}^1\int_0^{\frac{1}{1+q_2}} s^p\left(\frac{1}{q_1}-t\right)^p\ {}_0d_{q_2}s\ {}_0d_{q_1}t\right]^{\frac{1}{p}},$$

$$C_8(q_1,q_2,p) = \left[\int_0^{\frac{1}{1+q_1}}\int_{\frac{1}{1+q_2}}^1 t^p\left(\frac{1}{q_2}-s\right)^p\ {}_0d_{q_2}s\ {}_0d_{q_1}t\right]^{\frac{1}{p}},$$

$$C_9(q_1,q_2,p) = \left[\int_{\frac{1}{1+q_1}}^1\int_{\frac{1}{1+q_2}}^1 \left(\frac{1}{q_1}-t\right)^p\left(\frac{1}{q_2}-s\right)^p\ {}_0d_{q_2}s\ {}_0d_{q_1}t\right]^{\frac{1}{p}}.$$

and $\frac{1}{p}+\frac{1}{r} = 1.$

Proof. Taking the absolute value on both sides of the equality (3.15), using q_1q_2-Hölder inequality for functions of two variables (see [10, Theorem 5]) and the quasi-convexity of $\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|^r$ (see Definition 1) on the co-ordinates on Δ, we have

$$\left|\mu_{q_1,q_2}(a,b,c,d)(f)\right| \le q_1q_2(b-a)(d-c)$$
$$\times\left[\int_0^1\int_0^1|\kappa(t,s)|\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(tb+(1-t)a,sd+(1-s)c)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|\ {}_0d_{q_2}s\ {}_0d_{q_1}t\right]$$
$$= q_1q_2(b-a)(d-c)$$
$$\times\left[\int_0^{\frac{1}{1+q_1}}\int_0^{\frac{1}{1+q_2}} ts\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(tb+(1-t)a,sd+(1-s)c)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|\ {}_0d_{q_2}s\ {}_0d_{q_1}t\right.$$
$$+\int_0^{\frac{1}{1+q_1}}\int_{\frac{1}{1+q_2}}^1 t\left(\frac{1}{q_2}-s\right)\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(tb+(1-t)a,sd+(1-s)c)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|\ {}_0d_{q_2}s\ {}_0d_{q_1}t$$

$$+\int_{\frac{1}{1+q_1}}^{1}\int_{0}^{\frac{1}{1+q_2}} s\left(\frac{1}{q_1}-t\right)\left|\frac{{}_{a,c}\partial^2_{q_1,q_2} f(tb+(1-t)a, sd+(1-s)c)}{{}_{a}\partial_{q_1}t\ {}_{c}\partial_{q_2}s}\right|\ {}_0d_{q_2}s\ {}_0d_{q_1}t$$

$$+\int_{\frac{1}{1+q_1}}^{1}\int_{\frac{1}{1+q_2}}^{1}\left(\frac{1}{q_1}-t\right)\left(\frac{1}{q_2}-s\right)\left|\frac{{}_{a,c}\partial^2_{q_1,q_2} f(tb+(1-t)a, sd+(1-s)c)}{{}_{a}\partial_{q_1}t\ {}_{c}\partial_{q_2}s}\right|\ {}_0d_{q_2}s\ {}_0d_{q_1}t\ \Bigg]$$

$$\le q_1q_2(b-a)(d-c)$$

$$\times\Bigg[\begin{pmatrix}\left[\int_0^{\frac{1}{1+q_1}}\int_0^{\frac{1}{1+q_2}} t^p s^p\ {}_0d_{q_2}s\ {}_0d_{q_1}t\right]^{\frac{1}{p}} \\ \left[\int_0^{\frac{1}{1+q_1}}\int_0^{\frac{1}{1+q_2}}\left|\frac{{}_{a,c}\partial^2_{q_1,q_2} f(tb+(1-t)a, sd+(1-s)c)}{{}_{a}\partial_{q_1}t\ {}_{c}\partial_{q_2}s}\right|^r\ {}_0d_{q_2}s\ {}_0d_{q_1}t\right]^{\frac{1}{r}}\end{pmatrix}$$

$$+\begin{pmatrix}\left[\int_0^{\frac{1}{1+q_1}}\int_{\frac{1}{1+q_2}}^{1} t^p \left(\frac{1}{q_2}-s\right)^p\ {}_0d_{q_2}s\ {}_0d_{q_1}t\right]^{\frac{1}{p}} \\ \left[\int_0^{\frac{1}{1+q_1}}\int_{\frac{1}{1+q_2}}^{1}\left|\frac{{}_{a,c}\partial^2_{q_1,q_2} f(tb+(1-t)a, sd+(1-s)c)}{{}_{a}\partial_{q_1}t\ {}_{c}\partial_{q_2}s}\right|^r\ {}_0d_{q_2}s\ {}_0d_{q_1}t\right]^{\frac{1}{r}}\end{pmatrix}$$

$$+\begin{pmatrix}\left[\int_{\frac{1}{1+q_1}}^{1}\int_0^{\frac{1}{1+q_2}} s^p \left(\frac{1}{q_1}-t\right)^p\ {}_0d_{q_2}s\ {}_0d_{q_1}t\right]^{\frac{1}{p}} \\ \left[\int_{\frac{1}{1+q_1}}^{1}\int_0^{\frac{1}{1+q_2}}\left|\frac{{}_{a,c}\partial^2_{q_1,q_2} f(tb+(1-t)a, sd+(1-s)c)}{{}_{a}\partial_{q_1}t\ {}_{c}\partial_{q_2}s}\right|^r\ {}_0d_{q_2}s\ {}_0d_{q_1}t\right]^{\frac{1}{r}}\end{pmatrix}$$

$$+\begin{pmatrix}\left[\int_{\frac{1}{1+q_1}}^{1}\int_{\frac{1}{1+q_2}}^{1} \left(\frac{1}{q_1}-t\right)^p\left(\frac{1}{q_2}-s\right)^p\ {}_0d_{q_2}s\ {}_0d_{q_1}t\right]^{\frac{1}{p}} \\ \left[\int_{\frac{1}{1+q_1}}^{1}\int_{\frac{1}{1+q_2}}^{1}\left|\frac{{}_{a,c}\partial^2_{q_1,q_2} f(tb+(1-t)a, sd+(1-s)c)}{{}_{a}\partial_{q_1}t\ {}_{c}\partial_{q_2}s}\right|^r\ {}_0d_{q_2}s\ {}_0d_{q_1}t\right]^{\frac{1}{r}}\end{pmatrix}\Bigg]$$

$$\le q_1q_2(b-a)(d-c)$$

$$\times\begin{bmatrix} C_6(q_1,q_2,p)\left(\frac{1}{(1+q_1)(1+q_2)}\right)^{\frac{1}{r}}A + C_7(q_1,q_2,p)\left(\frac{q_1}{(1+q_1)(1+q_2)}\right)^{\frac{1}{r}}B \\ +C_8(q_1,q_2,p)\left(\frac{q_2}{(1+q_1)(1+q_2)}\right)^{\frac{1}{r}}C + C_9(q_1,q_2,p)\left(\frac{q_1q_2}{(1+q_1)(1+q_2)}\right)^{\frac{1}{r}}D\end{bmatrix}.$$

Thus the proof is accomplished. ■

Corollary 3. *Suppose the conditions of the Theorem 8 are satisfied. Additionally if*

(1) $\left|\frac{{}_{a,c}\partial^2_{q_1,q_2} f}{{}_{a}\partial_{q_1}t\ {}_{c}\partial_{q_2}s}\right|$ *is increasing on the co-ordinates on* Δ*, then*

$$\left|\mu_{q_1,q_2}(a,b,c,d)(f)\right| \le q_1q_2(b-a)(d-c) \tag{3.36}$$

$$\times\begin{bmatrix} C_6(q_1,q_2,p)\left(\frac{1}{(1+q_1)(1+q_2)}\right)^{\frac{1}{r}}O + C_7(q_1,q_2,p)\left(\frac{q_1}{(1+q_1)(1+q_2)}\right)^{\frac{1}{r}}Q \\ +C_8(q_1,q_2,p)\left(\frac{q_2}{(1+q_1)(1+q_2)}\right)^{\frac{1}{r}}S + C_9(q_1,q_2,p)\left(\frac{q_1q_2}{(1+q_1)(1+q_2)}\right)^{\frac{1}{r}}T\end{bmatrix},$$

(2) $\left|\frac{{}_{a,c}\partial^2_{q_1,q_2} f}{{}_{a}\partial_{q_1}t\ {}_{c}\partial_{q_2}s}\right|$ *is decreasing on the co-ordinates on* Δ*, then*

$$\left|\mu_{q_1,q_2}(a,b,c,d)(f)\right| \le q_1q_2(b-a)(d-c) \tag{3.37}$$

$$\times\left[\begin{array}{l} C_6(q_1,q_2,p)\left(\frac{1}{(1+q_1)(1+q_2)}\right)^{\frac{1}{r}}L + C_7(q_1,q_2,p)\left(\frac{q_1}{(1+q_1)(1+q_2)}\right)^{\frac{1}{r}}R \\ +C_8(q_1,q_2,p)\left(\frac{q_2}{(1+q_1)(1+q_2)}\right)^{\frac{1}{r}}P + C_9(q_1,q_2,p)\left(\frac{q_1 q_2}{(1+q_1)(1+q_2)}\right)^{\frac{1}{r}}T \end{array}\right],$$

(3) $T = R = S = P = Q = 0$, *then*

$$\left|\mu_{q_1,q_2}(a,b,c,d)(f)\right| \leq q_1 q_2 (b-a)(d-c) \tag{3.38}$$

$$\times\left[\begin{array}{l} C_6(q_1,q_2,p)\left(\frac{1}{(1+q_1)(1+q_2)}\right)^{\frac{1}{r}}L + C_7(q_1,q_2,p)\left(\frac{q_1}{(1+q_1)(1+q_2)}\right)^{\frac{1}{r}}M \\ +C_8(q_1,q_2,p)\left(\frac{q_2}{(1+q_1)(1+q_2)}\right)^{\frac{1}{r}}N + C_9(q_1,q_2,p)\left(\frac{q_1 q_2}{(1+q_1)(1+q_2)}\right)^{\frac{1}{r}}O \end{array}\right],$$

(4) $L = M = N = O = P = Q = R = S = 0$, *then*

$$\left|\mu_{q_1,q_2}(a,b,c,d)(f)\right| \leq q_1 q_2 (b-a)(d-c)T \tag{3.39}$$

$$\times\left[\begin{array}{l} C_6(q_1,q_2,p)\left(\frac{1}{(1+q_1)(1+q_2)}\right)^{\frac{1}{r}} + C_7(q_1,q_2,p)\left(\frac{q_1}{(1+q_1)(1+q_2)}\right)^{\frac{1}{r}} \\ +C_8(q_1,q_2,p)\left(\frac{q_2}{(1+q_1)(1+q_2)}\right)^{\frac{1}{r}} + C_9(q_1,q_2,p)\left(\frac{q_1 q_2}{(1+q_1)(1+q_2)}\right)^{\frac{1}{r}} \end{array}\right].$$

Remark 6. *In Theorem 8, if we take limit* $q_1^- \to 1$ *and* $q_2^- \to 1$, *then we recapture [9, Theorem 3], in Corollary 3, if we take limit* $q_1^- \to 1$ *and* $q_2^- \to 1$, *then we recapture [9, Corollary 2].*

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