Sigma J Eng & Nat Sci 37 (1), 2019, 187-205

Sigma Journal of Engineering and Natural Sciences
Sigma Mühendislik ve Fen Bilimleri Dergisi

Research Article

ON THE GENERALIZED INTEGRAL INEQUALITIES FOR TWICE DIFFERENTIABLE MAPPINGS

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Received: 14.11.2018 Revised: 27.12.2018 Accepted: 29.12.2018

ABSTRACT

In this paper, an important integral equality is derived. Then, we establish several new inequalities for some twice differentiable mappings that are connected with the celebrated Hermite-Hadamard type and Ostrowski type integral inequalities. Some of the new inequalities are obtained by using Grüss inequality and Chebyshev inequality. The results presented here would provide extensions of those given in earlier works
Keywords: Hermite-Hadamard inequality, Ostrowski inequality, Grüss inequality, Čebyšev inequality, Hölder inequality.

1. INTRODUCTION

In 1938, Ostrowski established the integral inequality which is one of the fundamental inequalities of mathematics as follows (see, [19]):

Let $f:[a,b]\to \mathbb{R}$ be a differentiable mapping on (a,b) whose derivative $f':(a,b)\to \mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty = \sup_{t\in(a,b)}|f'(t)| < \infty$. Then, the inequality holds:

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \le \left[\frac{1}{4} + \frac{(x-\frac{a+b}{2})^2}{(b-a)^2} \right](b-a)\|f'\|_\infty \tag{1.1}$$

for all x in $[a,b]$. The constant $\frac{1}{4}$ is the best possible.

Inequality (1.1) has wide applications in numerical analysis and in the theory of some special means; estimating error bounds for some special means, some mid-point, trapezoid and Simpson rules and quadrature rules, etc. Hence, inequality (1.1) has attracted considerable attention and interest from mathematicians and researchers. Due to this, over the years, the interested reader is also refered to ([1]-[4], [9], [12], [13], [15], [20]-[28]) for integral inequalities in several

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independent variables. In addition, the current approach of obtaining the bounds, for a particular quadrature rule, have depended on the use of Peano kernel. The general approach in the past has involved the assumption of bounded derivatives of degree greater than one.

Definition 1. The function $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$, is said to be convex if the following inequality holds

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y)$$

for all $x, y \in [a,b]$ and $\lambda \in [0,1]$. We say that f is concave if $(-f)$ is convex.

The following inequality is well known in the literature as the Hermite-Hadamard integral inequality (see, [8]):

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2} \tag{1.2}$$

where $f : I \subset \mathbb{R} \to \mathbb{R}$ is a convex function on the interval I of real numbers and $a,b \in I$ with $a<b$.

A largely applied inequality for convex functions, due to its geometrical significance, is Hadamard's inequality, (see [7], [29]-[33]) which has generated a wide range of directions for extension and a rich mathematical literature.

Integration with weight functions is used in countless mathematical problems such as approximation theory, spectral analysis, statistical analysis and the theory of distributions. Grüss developed an integral inequality [14 in 1935. The integral inequality that establishes a connection between the integral of the product of two functions and the product of the integrals is known in the literature as the Grüss inequality. The Grüss inequality is as follows.

Theorem 1. Let $f, g : [a,b] \to \mathbb{R}$ be integrable functions such that $\varphi < f(x) < \Phi$ and $\psi < g(x) < \Psi$ for all $x \in [a,b]$, where $\varphi, \Phi, \psi, \Psi$ are constants. Then

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{b-a}\int_a^b f(x)dx \frac{1}{b-a}\int_a^b g(x)dx\right| \le \frac{1}{4}(\Phi-\varphi)(\Psi-\psi),$$

where the constant $\frac{1}{4}$ is sharp (see, [14]).

In [18, p.40], Čebyšev's inequality is given by the following Theorem:

Theorem 2. Let $f, g : [a,b] \to \mathbb{R}$ be integrable and monotone functions on (a,b) and let p be a positive and integrable function on the same interval. Then

$$\int_a^b p(x) f(x)g(x)dx \int_a^b p(x)dx \ge \int_a^b p(x)g(x)dx \int_a^b p(x)g(x)dx$$

with equality if and only if one of the functions f, g reduces to a constant f and g are monotone in the opposite sense, the reverse inequality holds.

In [5], Bullen proved the following inequality which is known as Bullen's inequality for convex function.

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function on the interval I of real numbers and $a,b \in I$ with $a<b$. The inequality

$$\frac{1}{b-a}\int_a^b f(x)dx \le \frac{1}{2}\left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2}\right].$$

In this study, using twice differentiable functions, we give new inequalities that are connected with the celebrated Hermite-Hadamard type and Ostrowski type integral inequalities. The results presented here would provide extensions of those given in earlier works.

2. MAIN RESULTS

In order to prove our main results we need the following lemma:

Lemma 1. Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be twice differentiable function on I° (the interior of the interval I) such that $f'' \in L[a,b]$ where $a,b \in I^{\circ}$ with $a < b$. Then the following identity holds:

$$\frac{1}{2(b-a)}\int_a^b K_h(x,t)\, f''(t)dt$$

$$= \frac{1}{2}\left[\frac{(b-x)^2-(x-a)^2}{b-a} + h\left(x - \frac{a+b}{2}\right)\right] f'(x) \tag{2.1}$$

$$+\frac{1}{2}\left[(2-h)\, f(x) + h\frac{f(a)+f(b)}{2}\right] - \frac{1}{b-a}\int_a^b f(t)dt$$

$$=: S_h(f)$$

for

$$K_h(x,t) := \begin{cases} (a-t)\left(t-a-h\frac{b-a}{2}\right) & ,a \le t < x \\ (b-t)\left(t-b+h\frac{b-a}{2}\right) & ,x \le t \le b \end{cases}$$

where $h \in [0,1]$ and $a + h\frac{b-a}{2} \le x \le b - h\frac{b-a}{2}$.

Proof. Firstly, we arrange the operations:

$$\int_a^b K_h(x,t)\, f''(t)dt$$

$$= \int_a^x (a-t)\left(t-a-h\frac{b-a}{2}\right) f''(t)dt + \int_x^b (b-t)\left(t-b+h\frac{b-a}{2}\right) f''(t)dt$$

$$= -\int_a^x (t-a)^2 f''(t)dt + h\frac{b-a}{2}\int_a^x (t-a)\, f''(t)dt$$

$$-\int_x^b (t-b)^2 f''(t)dt - h\frac{b-a}{2}\int_x^b (t-b)\, f''(t)dt.$$

By integration by parts twice, we have

$$\begin{aligned}
&\int_a^b K_h(x,t)\, f''(t)dt \\
&= -(x-a)^2 f'(x) + 2(x-a) f(x) - 2\int_a^x f(t)dt \\
&\quad + \frac{h(b-a)}{2}\left[(x-a) f'(x) + f(a) - f(x)\right] + (b-x)^2 f'(x) + 2(b-x) f(x) \\
&\quad -2\int_x^b f(t)dt + \frac{h(b-a)}{2}\left[f(b) - f(x) - (b-x) f'(x)\right].
\end{aligned} \tag{2.2}$$

If we arrange the equality (2.2), then we obtain desired equality (2.1). Hence, the proof is completed.

Remark 1. In Lemma 1, let $h=1$ and $x=\frac{a+b}{2}$. Then, we have the equality

$$\frac{1}{2(b-a)}\int_a^b K_1\left(\frac{a+b}{2},t\right) f''(t)dt$$
$$=\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]-\frac{1}{(b-a)}\int_a^b f(t)dt$$

which is proved by Kirmaci and Dikici in [16] and Minculete et. al in [17].

Theorem 3. Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be twice differentiable function on I°, the interior of the interval I, where $a,b \in I^\circ$ with $a<b$. If $f'' : (a,b) \to \mathbb{R}$ is bounded on $[a,b]$, denote $\|f''\|_\infty = \sup_{t\in[a,b]} |f''(t)| < \infty$, then we have the inequality

$$|S_h(f)| \le \|f''\|_\infty \left\{ \frac{h^3(b-a)^2}{24} + \frac{(x-a)^3+(b-x)^3}{6(b-a)} - h\frac{(b-a)^2}{4}\left[\frac{1}{4}+\frac{(x-\frac{a+b}{2})^2}{(b-a)^2}\right]\right\}$$

for all $h\in[0,1]$ and $a+h\frac{b-a}{2} \le x \le b-h\frac{b-a}{2}$.

Proof. We take absolute value of (2.1). Using bounded of the mapping f'', we find that

$$\begin{aligned}
|S_h(f)| &= \left|\frac{1}{2(b-a)}\int_a^b K_h(x,t)\, f''(t)\,dt\right| \\
&\le \frac{1}{2(b-a)}\int_a^b |K_h(x,t)|\,|f''(t)|\,dt \\
&\le \frac{\|f''\|_\infty}{2(b-a)}\int_a^b |K_h(x,t)|\,dt \\
&= \frac{\|f''\|_\infty}{2(b-a)}\left[\int_a^x |a-t|\left|t-a-h\frac{b-a}{2}\right|dt + \int_x^b |b-t|\left|t-b+h\frac{b-a}{2}\right|dt\right] \\
&= \frac{\|f''\|_\infty}{2(b-a)}L.
\end{aligned}$$

Now, let us observe that

$$\begin{aligned}
\int_p^r |t-p|\,|t-q|\,dt &= \int_p^q (t-p)(q-t)\,dt + \int_q^r (t-p)(t-q)\,dt \\
&= \frac{(q-p)^3}{3} + \frac{(r-p)^3}{3} - \frac{(q-p)(r-p)^2}{2}
\end{aligned}$$

for all r, p, q such that $p \le q \le r.$ Then we get that

$$\int_a^x |t-a|\left|t-a-h\frac{b-a}{2}\right|dt = \frac{h^3(b-a)^3}{24} + \frac{(x-a)^3}{3} - h\frac{(b-a)(x-a)^2}{4}$$

and

$$\int_x^b |t-b|\left|t-b+h\frac{b-a}{2}\right|dt = \frac{h^3(b-a)^3}{24} + \frac{(b-x)^3}{3} - h\frac{(b-a)(b-x)^2}{4}.$$

Then, we have

$$L = \frac{h^3(b-a)^3}{12} + \frac{(x-a)^3+(b-x)^3}{3} - h\frac{b-a}{4}\left[(x-a)^2+(b-x)^2\right]$$

and the proof is thus completed.

Remark 2. If we choose $h=1$ with $x=\frac{a+b}{2}$ in Theorem 3, then we have the following inequality

$$\left|\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \frac{(b-a)^2}{48}\|f''\|_\infty$$

which is given by Dragomir and Sofo in [10].

Remark 3. If we choose $h=0$ and $x=\frac{a+b}{2}$ in Theorem 3, then we have the following midpoint inequality

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)dt\right|\le\frac{(b-a)^2}{24}\|f''\|_\infty$$

which is given by Cerone et al. in [6].

Theorem 4. Let $f: I\subset \mathbb{R}\to\mathbb{R}$ be twice differentiable function on I°, the interior of the interval I, where $a,b\in I^\circ$ with $a<b$. If $|f''|$ is belong to $L_p[a,b]$, $p>1$ and $\frac{1}{p}+\frac{1}{q}=1$, then we have the inequality

$$|S_h(f)|\le\frac{\|f''\|_p}{2(b-a)}\left[2\left(h\frac{b-a}{2}\right)^{2q+1}B(q+1,q+1)+M\left(h\frac{b-a}{2},x-a\right)+M\left(h\frac{b-a}{2},b-x\right)\right]^{\frac{1}{q}} \tag{2.3}$$

for all $h\in[0,1]$ and $a+h\frac{b-a}{2}\le x\le b-h\frac{b-a}{2}$, and where $B(p,q)$ is Euler's Beta function.

Proof. We take absolute value of (2.1). Using Hölder's inequality, we find that

$$\begin{aligned}|S_h(f)|&\le\frac{1}{2(b-a)}\int_a^b|K_h(x,t)||f''(t)|dt\\&\le\frac{1}{2(b-a)}\left(\int_a^b|f''(t)|^p dt\right)^{\frac{1}{p}}\left(\int_a^b|K_h(x,t)|^q dt\right)^{\frac{1}{q}}\\&=\frac{\|f''\|_p}{2(b-a)}\left(\int_a^b|K_h(x,t)|^q dt\right)^{\frac{1}{q}}\\&=\frac{\|f''\|_p}{2(b-a)}K^{\frac{1}{q}}.\end{aligned} \tag{2.4}$$

We need to calculate the integral K to prove the theorem.

$$\begin{aligned}K&=\int_a^b|K_h(x,t)|^q dt\\&=\int_a^x|a-t|^q\left|t-a-h\frac{b-a}{2}\right|^q dt+\int_x^b|b-t|^q\left|t-b+h\frac{b-a}{2}\right|^q dt\end{aligned} \tag{2.5}$$

$$= \int_a^{a+h\frac{b-a}{2}} (t-a)^q \left(a+h\frac{b-a}{2}-t\right)^q dt + \int_{a+h\frac{b-a}{2}}^{x} (t-a)^q \left(t-a-h\frac{b-a}{2}\right)^q dt$$

$$+ \int_x^{b-h\frac{b-a}{2}} (b-t)^q \left(b-h\frac{b-a}{2}-t\right)^q dt + \int_{b-h\frac{b-a}{2}}^{b} (b-t)^q \left(t-b+h\frac{b-a}{2}\right)^q dt$$

$$= I_1 + I_2 + I_3 + I_4$$

Now, we calculate four integrals above . For integral I_1, using the change of variable $t-a = h\frac{b-a}{2}u$ and from $dt = h\frac{b-a}{2}du,$ we write

$$\begin{aligned} I_1 &= \int_a^{a+h\frac{b-a}{2}} (t-a)^q \left(a+h\frac{b-a}{2}-t\right)^q dt \\ &= \left(h\frac{b-a}{2}\right)^{2q+1} \int_0^1 u^q (1-u)^q du \qquad (2.6) \\ &= \left(h\frac{b-a}{2}\right)^{2q+1} B(q+1, q+1) \end{aligned}$$

and, for integral I_4, using the change of variable $t-b+h\frac{b-a}{2} = h\frac{b-a}{2}u$ and from $dt = h\frac{b-a}{2}du,$ we write

$$\begin{aligned} I_4 &= \int_{b-h\frac{b-a}{2}}^{b} (b-t)^q \left(t-b+h\frac{b-a}{2}\right)^q dt = \left(h\frac{b-a}{2}\right)^{2q+1} \int_0^1 (1-u)^q u^q du \qquad (2.7) \\ &= \left(h\frac{b-a}{2}\right)^{2q+1} B(q+1, q+1) \end{aligned}$$

where $B(p,q) = \int_0^1 u^{p-1}(1-u)^{q-1}du, \quad (p,q>0).$

Before calculating the other integrals, let us define that

$$M(c,d)=\int_c^d u^q(u-c)^q du.$$

If we use the change of variable $t-a=u$ for I_2 and $b-t=u$ for I_3, then we get

$$\begin{aligned} I_2 &= \int_{a+h\frac{b-a}{2}}^{x} (t-a)^q \left(t-a-h\frac{b-a}{2}\right)^q dt \\ &= \int_{h\frac{b-a}{2}}^{x-a} u^q \left(u-h\frac{b-a}{2}\right)^q du \qquad (2.8) \\ &= M\left(h\frac{b-a}{2}, x-a\right) \end{aligned}$$

and

$$\begin{aligned} I_3 &= \int_{x}^{b-h\frac{b-a}{2}} (b-t)^q \left(b-t-h\frac{b-a}{2}\right)^q dt \\ &= \int_{h\frac{b-a}{2}}^{b-x} u^q \left(u-h\frac{b-a}{2}\right)^q du \qquad (2.9) \\ &= M\left(h\frac{b-a}{2}, b-x\right). \end{aligned}$$

Substituting (2.6- (2.9) in (2.5), we obtain the equality

$$\begin{aligned} \int_a^b |K_h(x,t)|^q dt &= 2\left(h\frac{b-a}{2}\right)^{2q+1} B(q+1,q+1) \qquad (2.10) \\ &+M\left(h\frac{b-a}{2}, x-a\right) + M\left(h\frac{b-a}{2}, b-x\right). \end{aligned}$$

If we substitute the equality (2.10) in (2.4), then we easily deduce the required inequality (2.3) which completes the proof.

Corollary 1. Let f be as in Theorem 4. If we use the equality

$$M(c,d)=\int_c^d u^q(u-r)^q du = \sum_{k=0}^{q} \frac{q!(-1)^k r^{q-k}}{(q-k)!k!} \frac{d^{q+k+1}-c^{q+k+1}}{q+k+1}.$$

where $q>1$ and $q\in \mathrm{N}$, then we obtain

$$\left|S_h(f)\right| \le \frac{\|f''\|_p}{2(b-a)}\left[2\left(h\frac{b-a}{2}\right)^{2q+1}B(q+1,q+1)\right.$$
$$\left.+\sum_{k=0}^{q}\frac{q!(-1)^k\left(\frac{h(b-a)}{2}\right)^{q-k}}{(q-k)!k!}\frac{(x-a)^{q+k+1}+(b-x)^{q+k+1}-2\left(\frac{h(b-a)}{2}\right)^{q+k+1}}{q+k+1}\right]^{\frac{1}{q}}.$$

Corollary 2. Under the same assumptions of Theorem 4 with $x=\frac{a+b}{2}$ and $h=1,$ then the following inequality holds:

$$\left|\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]-\frac{1}{(b-a)}\int_a^b f(t)dt\right| \le \frac{(b-a)^{1+\frac{1}{q}}\left[B(q+1,q+1)\right]^{\frac{1}{q}}}{8}\|f''\|_p.$$

Remark 4. If we choose $x=\frac{a+b}{2}$ and $h=0$ in Theorem 4, then we have the following midpoint inequality

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)dt\right| \le \frac{(b-a)^{1+\frac{1}{q}}}{8(2q+1)^{\frac{1}{q}}}\|f''\|_p$$

which is proved by Dragomir et al. in [11].

Theorem 5. Let $f: I\subset \mathbb{R}\to\mathbb{R}$ be twice differentiable function on I° (the interior of the interval I) such that $f''\in L[a,b]$ where $a,b\in I^\circ$ with $a<b$. If the mappings

$$\varphi(t)=\begin{cases}(a-t)\left(t-a-h\frac{b-a}{2}\right)f''(t) & ,t\in[a,x)\\ \\ (b-t)\left(t-b+h\frac{b-a}{2}\right)f''(t) & ,t\in[x,b]\end{cases}$$

is convex on $[a,b],$ then we have the inequality

$$\frac{F(x)}{(b-a)} \le S_h(f) \le \frac{F(x)}{2(b-a)}+\frac{1}{8}(b-x)\left(x-b+h\frac{b-a}{2}\right)f''(x) \tag{2.11}$$

where

$$F(x)=\left(\frac{x-a}{2}\right)^2\left(\frac{a-x}{2}+h\frac{b-a}{2}\right)f''\left(\frac{x+a}{2}\right)+\left(\frac{b-x}{2}\right)^2\left(\frac{x-b}{2}+h\frac{b-a}{2}\right)f''\left(\frac{x+b}{2}\right).$$

for all $h\in[0,1]$ and $a+h\frac{b-a}{2}\le x\le b-h\frac{b-a}{2}$.

Proof. If we use left hand side of Hermite-Hadamard's inequality for the mappings φ, then we get

$$\frac{1}{x-a}\int_a^x \varphi(t)dt \ge \varphi\left(\frac{x+a}{2}\right)=\left(\frac{x-a}{2}\right)\left(\frac{a-x}{2}+h\frac{b-a}{2}\right)f''\left(\frac{x+a}{2}\right) \tag{2.12}$$

and

$$\frac{1}{b-x}\int_x^b \varphi(t)dt \geq \varphi\left(\frac{x+b}{2}\right) = \left(\frac{b-x}{2}\right)\left(\frac{x-b}{2}+h\frac{b-a}{2}\right) f''\left(\frac{x+b}{2}\right) \tag{2.13}$$

The inequality (2.12) and (2.13) are multiplied by $x-a$ and $b-x$, respectively. Then, these two inequalities are added, we have

$$2F(x) \leq \int_a^b \varphi(t)dt = 2(b-a)S_h(f) \tag{2.14}$$

Applying the Bullen's inequality for the mappings φ , we get

$$\begin{aligned} \frac{2}{x-a}\int_a^x \varphi(t)dt &\leq \varphi\left(\frac{a+x}{2}\right) + \frac{\varphi(a)+\varphi(x)}{2} \\ &= \left(\frac{x-a}{2}\right)\left(\frac{a-x}{2}+h\frac{b-a}{2}\right) f''\left(\frac{x+a}{2}\right) \\ &\quad + \frac{1}{2}(b-x)\left(x-b+h\frac{b-a}{2}\right) f''(x) \end{aligned} \tag{2.15}$$

and

$$\begin{aligned} \frac{2}{b-x}\int_x^b \varphi(t)dt &\leq \varphi\left(\frac{x+b}{2}\right) + \frac{\varphi(b)+\varphi(x)}{2} \\ &= \left(\frac{x-b}{2}\right)\left(\frac{x-b}{2}+h\frac{b-a}{2}\right) f''\left(\frac{x+b}{2}\right) \\ &\quad + \frac{1}{2}(b-x)\left(x-b+h\frac{b-a}{2}\right) f''(x). \end{aligned} \tag{2.16}$$

The inequality (2.15) and (2.16) are multiplied by $(x-a)/2$ and $(b-x)/2,$ respectively. Then, these two inequalities are added, we have

$$\int_a^b \varphi(t)dt \leq F(x) + \frac{1}{4}(b-a)(b-x)\left(x-b+h\frac{b-a}{2}\right) f''(x) \tag{2.17}$$

Because of (2.14) and (2.17), we easily the deduce required inequality (2.11) which completes the proof.

Remark 5. In Theorem 5, let $h=1$ and $x=\frac{a+b}{2}$. Then we have the inequality

$$\frac{(b-a)^2}{64}\left[f''\left(\frac{3a+b}{4}\right)+f''\left(\frac{a+3b}{4}\right)\right]$$
$$\le \frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]-\frac{1}{(b-a)}\int_a^b f(t)\,dt$$
$$\le \frac{(b-a)^2}{128}\left[f''\left(\frac{3a+b}{4}\right)+f''\left(\frac{a+3b}{4}\right)\right]$$

which is proved by Kirmaci and Dikici in [16] and Minculete et. al in [17].

Corollary 3. Under assumptions of Theorem 5 with $h=0$ and $x=\frac{a+b}{2}$, we have the inequality

$$\frac{(b-a)^2}{128}\left[f''\left(\frac{3a+b}{4}\right)+f''\left(\frac{a+3b}{4}\right)+4f''\left(\frac{a+b}{2}\right)\right]$$
$$\le \frac{1}{b-a}\int_a^b f(t)\,dt-f\left(\frac{a+b}{2}\right)$$
$$\le \frac{(b-a)^2}{64}\left[f''\left(\frac{3a+b}{4}\right)+f''\left(\frac{a+3b}{4}\right)\right].$$

Theorem 6. Let $f : I\subset \mathbb{R}\to\mathbb{R}$ be twice differentiable function on I° such that $\gamma\le f''\le\mu$ on $[a,x)$ and $\gamma'\le f''\le\mu'$ on $[x,b],$ where $a,b\in I^\circ$ with $a<b$. If f'' is integrable on $[a,b]$, then we have the inequality

$$\left|[a+b-2x]\,f'(x)+\left[4-h\frac{(b-a)^2}{2(x-a)(b-x)}\right]f(x)\right.$$
$$h\frac{b-a}{2}\left[\frac{f(a)}{x-a}+\frac{f(b)}{(b-x)}\right]-\frac{2}{x-a}\int_a^x f(t)\,dt-\frac{2}{b-x}\int_x^b f(t)\,dt \tag{2.18}$$

$$-\frac{1}{(x-a)^2}\left[h\frac{(b-a)(x-a)^2}{4}-\frac{(x-a)^3}{3}\right][f'(x)-f'(a)]$$
$$-\frac{1}{(b-x)^2}\left[h\frac{(b-a)(b-x)^2}{4}-\frac{(b-x)^3}{3}\right][f'(b)-f'(x)]\Bigg|$$
$$\leq \frac{1}{4}\left[\left(\frac{h^2(b-a)^2}{16}-(a-x)\left(x-a-h\frac{b-a}{2}\right)\right)(\mu-\gamma)\right.$$
$$\left.+\left(\frac{h^2(b-a)^2}{16}-(b-x)\left(x-b+h\frac{b-a}{2}\right)\right)(\mu'-\gamma')\right]$$

for all $h\in[0,1]$ and $a+h\frac{b-a}{2}\leq x\leq b-h\frac{b-a}{2}$.

Proof. Using Grüss inequality, we find that

$$\left|\frac{1}{x-a}\int_a^x (a-t)\left(t-a-h\frac{b-a}{2}\right)f''(t)dt\right.$$
$$+\frac{1}{b-x}\int_x^b (b-t)\left(t-b+h\frac{b-a}{2}\right)f''(t)dt$$
$$-\frac{1}{(x-a)^2}\int_a^x (a-t)\left(t-a-h\frac{b-a}{2}\right)dt\int_a^x f''(t)dt \tag{2.19}$$
$$\left.-\frac{1}{(b-x)^2}\int_x^b (b-t)\left(t-b+h\frac{b-a}{2}\right)dt\int_x^b f''(t)dt\right|$$
$$\leq \frac{1}{4}\left[(m-n)(\mu-\gamma)+(m'-n')(\mu'-\gamma')\right]$$

Here, we have that

$$\frac{1}{x-a}\int_a^x (a-t)\left(t-a-h\frac{b-a}{2}\right)f''(t)dt$$
$$=-(x-a)f'(x)+2f(x)-\frac{2}{x-a}\int_a^x f(t)dt \tag{2.20}$$
$$+\frac{h(b-a)}{2(x-a)}[(x-a)f'(x)+f(a)-f(x)]$$

and

$$\frac{1}{b-x}\int_x^b (b-t)\left(t-b+h\frac{b-a}{2}\right) f''(t)dt$$
$$= (b-x)f'(x)+2f(x)-\frac{2}{b-x}\int_x^b f(t)dt \tag{2.21}$$
$$+\frac{h(b-a)}{2(b-x)}\left[f(b)-f(x)-(b-x)f'(x)\right].$$

Now, we calculate bounds $m,\ n,\ m^{'}$ and $n^{'}$

$$m=\sup_{t\in[a,x)}\left\{(a-t)\left(t-a-h\frac{b-a}{2}\right)\right\}=\frac{h^2(b-a)^2}{16}, \tag{2.22}$$

$$n=\inf_{t\in[a,x)}\left\{(a-t)\left(t-a-h\frac{b-a}{2}\right)\right\}=(a-x)\left(x-a-h\frac{b-a}{2}\right), \tag{2.23}$$

$$m^{'}=\sup_{t\in[x,b]}\left\{(b-t)\left(t-b+h\frac{b-a}{2}\right)\right\}=\frac{h^2(b-a)^2}{16}, \tag{2.24}$$

$$n^{'}=\inf_{t\in[x,b]}\left\{(b-t)\left(t-b+h\frac{b-a}{2}\right)\right\}=(b-x)\left(x-b+h\frac{b-a}{2}\right). \tag{2.25}$$

On the other hand,

$$\int_a^x (a-t)\left(t-a-h\frac{b-a}{2}\right)dt=h\frac{(b-a)(x-a)^2}{4}-\frac{(x-a)^3}{3}, \tag{2.26}$$

$$\int_x^b (b-t)\left(t-b+h\frac{b-a}{2}\right)dt=h\frac{(b-a)(b-x)^2}{4}-\frac{(b-x)^3}{3}, \tag{2.27}$$

$$\int_a^x f''(t)dt=f'(x)-f'(a),\qquad \int_x^b f''(t)dt=f'(b)-f'(x). \tag{2.28}$$

If we substitute (2.20)-(2.28) in (2.19), then we obtain the inequality (2.18) which completes the proof.

Corollary 4. In Theorem 6, let $h=1$ and $x=\frac{a+b}{2}$. Then we have the inequality

$$\left|\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]-\frac{1}{b-a}\int_a^b f(t)dt-\frac{b-a}{48}\left[f'(b)-f'(a)\right]\right|$$

$$\leq\frac{(b-a)^2}{256}\left[(\mu-\gamma)+\left(\mu^{'}-\gamma^{'}\right)\right]$$

Corollary 5. Under assumption of Theorem 6 with $h=0$ and $x=\frac{a+b}{2}$, we have the inequality

$$\left| f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)dt-\frac{b-a}{24}\left[f'(b)-f'(a)\right]\right|$$

$$\leq \frac{(b-a)^2}{64}\left[(\mu-\gamma)+\left(\mu^{'}-\gamma^{'}\right)\right]$$

“For convenience, we give the following notations used to simplify the details of the next theorem,

$$\alpha_1=\int_a^x (t-a)(f''(t)-k)dt=(x-a)f'(x)-f(x)+f(a)-k\frac{(x-a)^2}{2}, \tag{2.29}$$

$$\begin{aligned}\alpha_2&=\int_a^x\left(a+h\frac{b-a}{2}-t\right)(f''(t)-k)dt\\ &=h\frac{b-a}{2}f'(a)-\left(x-a-h\frac{b-a}{2}\right)f'(x)\\ &\quad+f(x)-f(a)+\frac{k}{2}\left(x-a-h\frac{b-a}{2}\right)^2-\frac{k}{2}\left(h\frac{b-a}{2}\right)^2,\end{aligned} \tag{2.30}$$

$$\alpha_3=\int_a^x (f''(t)-k)dt=f'(x)-f'(a)-k(x-a), \tag{2.31}$$

$$\alpha_4=\int_x^b (b-t)(f''(t)-k)dt=-(b-x)f'(x)+f(b)-f(x)-k\frac{(b-x)^2}{2}, \tag{2.32}$$

$$\begin{aligned}\alpha_5&=\int_x^b\left(t-b+h\frac{b-a}{2}\right)(f''(t)-k)dt\\ &=\left(h\frac{b-a}{2}\right)f'(b)-\left(x-b+h\frac{b-a}{2}\right)f'(x)\\ &\quad+f(x)-f(b)-\frac{k}{2}\left(x-b+h\frac{b-a}{2}\right)^2+\frac{k}{2}\left(h\frac{b-a}{2}\right)^2,\end{aligned} \tag{2.33}$$

$$\alpha_6=\int_x^b (f''(t)-k)dt=f'(b)-f'(x)-k(b-x), \tag{2.34}$$

$$\sigma_1 = \int_a^x (t-a)(K - f''(t))dt = K\frac{(x-a)^2}{2} - (x-a) f'(x) + f(x) - f(a), \tag{2.35}$$

$$\begin{aligned}\sigma_2 &= \int_a^x \left(a + h\frac{b-a}{2} - t\right)(K - f''(t))dt \\ &= \frac{K}{2}\left(h\frac{b-a}{2}\right)^2 - \frac{K}{2}\left(x - a - h\frac{b-a}{2}\right)^2 \\ &\quad + \left(x - a - h\frac{b-a}{2}\right) f'(x) + h\frac{b-a}{2} f'(a) - f(x) + f(a),\end{aligned} \tag{2.36}$$

$$\sigma_3 = \int_a^x (K - f''(t))dt = K(x-a) + f'(a) - f'(x), \tag{2.37}$$

$$\sigma_4 = \int_x^b (b-t)(K - f''(t))dt = -f(b) + f(x) + (b-x) f'(x) + K\frac{(b-x)^2}{2}, \tag{2.38}$$

$$\begin{aligned}\sigma_5 &= \int_x^b \left(t - b + h\frac{b-a}{2}\right)(K - f''(t))dt \\ &= \left(x - b + h\frac{b-a}{2}\right) f'(x) - h\frac{b-a}{2} f'(b) \\ &\quad + f(b) - f(x) + \frac{K}{2}\left(h\frac{b-a}{2}\right)^2 - \frac{K}{2}\left(x - b + h\frac{b-a}{2}\right)^2,\end{aligned} \tag{2.39}$$

$$\sigma_6 = \int_x^b (K - f''(t))dt = K(b-x) + f'(x) - f'(b). \tag{2.40}$$

Theorem 7. Let $f : I \subset \mathrm{R} \to \mathrm{R}$ be twice differentiable function on I°, $a,b \in I^\circ$ with $a < b$. If $f'' \in L[a,b]$ and $k \le f'' \le K$ for all $x \in [a,b]$, then we have the inequality

$$\begin{aligned}&\frac{K}{b-a}\left[\frac{h(b-a)}{8}\left((x-a)^2 + (b-x)^2\right) - \frac{(x-a)^3 + (b-x)^3}{6}\right] - \frac{1}{2(b-a)}\left[\frac{\sigma_1\sigma_2}{\sigma_3} + \frac{\sigma_4\sigma_5}{\sigma_6}\right] \\ &\le S_h(f) \\ &\le \frac{1}{2(b-a)}\left[\frac{\alpha_1\alpha_2}{\alpha_3} + \frac{\alpha_4\alpha_5}{\alpha_6}\right] + \frac{k}{b-a}\left[\frac{h(b-a)}{8}\left((x-a)^2 + (b-x)^2\right) - \frac{(x-a)^3 + (b-x)^3}{6}\right]\end{aligned} \tag{2.41}$$

for all $h \in [0,1]$ and $a + h\frac{b-a}{2} \le x \le b - h\frac{b-a}{2}$.

Proof. From Lemma 1, we have that

$$I_5 + I_6 = 2(b-a)S_h(f) \tag{2.42}$$

where

$$I_5 = \int_a^x (t-a)\left(a + h\frac{b-a}{2} - t\right) f''(t)dt$$

and

$$I_6 = \int_x^b (b-t)\left(t - b + h\frac{b-a}{2}\right) f''(t)dt.$$

By integration by parts, we obtain

$$\int_a^x (t-a)\left(a + h\frac{b-a}{2} - t\right)\left(f''(t) - k\right)dt = I_5 - k\left[\frac{(x-a)^3}{3} - h\frac{(b-a)(x-a)^2}{4}\right] \tag{2.43}$$

and

$$\int_a^x (t-a)\left(a + h\frac{b-a}{2} - t\right)\left(K - f''(t)\right)dt = K\left[\frac{(x-a)^3}{3} - h\frac{(b-a)(x-a)^2}{4}\right] - I_5. \tag{2.44}$$

Also, we have

$$\int_x^b (b-t)\left(t - b + h\frac{b-a}{2}\right)\left(f''(t) - k\right)dt = I_6 - k\left[\frac{(b-x)^3}{3} - h\frac{(b-a)(b-x)^2}{4}\right] \tag{2.45}$$

and

$$\int_x^b (b-t)\left(t - b + h\frac{b-a}{2}\right)\left(K - f''(t)\right)dt = k\left[\frac{(b-x)^3}{3} - h\frac{(b-a)(b-x)^2}{4}\right] - I_6. \tag{2.46}$$

Using Chebychev integral inequality, we find that

$$\begin{aligned}&\int_a^x (t-a)\left(a + h\frac{b-a}{2} - t\right)\left(f''(t) - k\right)dt \\ &\leq \frac{\int_a^x (t-a)\left(f''(t) - k\right)dt \int_a^x \left(a + h\frac{b-a}{2} - t\right)\left(f''(t) - k\right)dt}{\int_a^x \left(f''(t) - k\right)dt},\end{aligned} \tag{2.47}$$

$$\int_a^x (t-a)\left(a+h\frac{b-a}{2}-t\right)(K-f''(t))dt$$
$$\leq \frac{\int_a^x (t-a)(K-f''(t))dt \int_a^x \left(a+h\frac{b-a}{2}-t\right)(K-f''(t))dt}{\int_a^x (K-f''(t))dt}, \quad (2.48)$$

$$\int_x^b (b-t)\left(t-b+h\frac{b-a}{2}\right)(f''(t)-k)dt$$
$$\leq \frac{\int_x^b (b-t)(f''(t)-k)dt \int_x^b \left(t-b+h\frac{b-a}{2}\right)(f''(t)-k)dt}{\int_x^b (f''(t)-k)dt} \quad (2.49)$$

and

$$\int_x^b (b-t)\left(t-b+h\frac{b-a}{2}\right)(K-f''(t))dt$$
$$\leq \frac{\int_x^b (b-t)(K-f''(t))dt \int_x^b \left(t-b+h\frac{b-a}{2}\right)(K-f''(t))dt}{\int_x^b (K-f''(t))dt}. \quad (2.50)$$

Substituting equalities (2.29)-(2.31) in inequality (2.47) and from (2.43), we have

$$I_5 \leq k\left[h\frac{(b-a)(x-a)^2}{4}-\frac{(x-a)^3}{3}\right]+\frac{\alpha_1\alpha_2}{\alpha_3}. \quad (2.51)$$

Substituting equalities (2.32)-(2.34) in inequality (.49) and from (2.45), we have

$$I_6 \leq k\left[h\frac{(b-a)(b-x)^2}{4}-\frac{(b-x)^3}{3}\right]+\frac{\alpha_4\alpha_5}{\alpha_6}. \quad (2.52)$$

Adding (2.51) and (2.52), we obtain

$$I_5+I_6 \leq k\left[\frac{h(b-a)}{4}\left((x-a)^2+(b-x)^2\right)-\frac{(x-a)^3+(b-x)^3}{3}\right]+\frac{\alpha_1\alpha_2}{\alpha_3}+\frac{\alpha_4\alpha_5}{\alpha_6}. \quad (2.53)$$

In a similar way, if we use equalities (45)-(50) and inequalities (32), (34), then we get

$$I_5+I_6 \geq K\left[\frac{h(b-a)}{4}\left((x-a)^2+(b-x)^2\right)-\frac{(x-a)^3+(b-x)^3}{3}\right]-\frac{\sigma_1\sigma_2}{\sigma_3}-\frac{\sigma_4\sigma_5}{\sigma_6}. \quad (2.54)$$

Using the inequalities (2.53), (2.54) and the equality (2.42), we obtain the inequality (2.41) which completes the proof.

Remark 6. If we choose $h=1$ and $x=\frac{a+b}{2}$ in Theorem 7, then Theorem 7 reduces to Theorem 4 in [16].

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