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Research Article

THE SIGMA INDEX OF GRAPH OPERATIONS

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ABSTRACT

Let G be a graph of order n with vertices labeled as $v_1, v_2, \ldots, v_n$. Let d_i be the degree of the vertex vi, for $i = 1, 2, \ldots, n$. The sigma index, which have been defined very recently, of G is $\sigma(G) = \Sigma_{vivj \in E(G)}(d_i - d_j)^2$. In this paper, the sigma index of the Cartesian product, composition, join and disjunction of graphs are computed. We apply some of our results to compute the sigma index of some special graph classes.

Keywords: Graph operations, sigma index, zagreb index.
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1. INTRODUCTION

All graphs considered in this paper are assumed to be simple connected graphs. Let G be a (molecular) graph with vertex set $V(G) = \{v_1, v_2, ..., v_n\}$ and edge set $E(G)$. The degree of a vertex $v_i \in V(G)$ is the number of vertices adjacent to v and is denoted by d_i. If v_i and v_j are adjacent vertices of G, then the edge connecting them is denoted by v_iv_j. A *walk* from a vertex u to a vertex v is a finite alternating sequence $v_0(= u)e_1v_1e_2 \ldots v_{k1}e_k$ $v_k(= v)$ of vertices and edges such that $e_i = v_{i-1}v_i$

for $i = 1, 2, \ldots, k$. The number k is the *length* of the walk. In particular, if the vertex $v_i, i = 0, 1, ..., k$ in the walk are all distinct then the walk is called a *path*, a path of order n is denoted by P_n. A *closed path* or *cycle*, is obtained from a path $v_1, \ldots, v_k$ (where $k \geqslant 3$) by adding the edge v_1v_k, a cycle of order n is denoted by C_n. A wheel graph W_n is a cycle graph C_n with an additional central vertex adjacent to all the vertices on the cycle graph. A graph is *connected* if each pair of vertices in a graph is joined by a walk. A *bipartite* graph is a graph such that its vertex set can be partitioned into two sets X and Y (called the partite sets)

such that every edge meet both X and Y. A *complete bipartite graph* is a bipartite such that any vertex of a partite set is adjacent to all vertices of the other partite set. A complete bipartite graph with partite set of cardinalities p and q is denoted by $K_{p,q}$. The graph $K_{1,n-1}$ is also called a *star* of order n, denoted by S_n. A simple

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undirected graph in which every pair of distinct vertices is connected by a unique edge, is the *complete graph* and is denoted by K_n. For other graph theory notation and terminology we refer to [12]. The Cartesian product $G \times H$ of graphs G and H has the vertex set $V(G \times H) = V(G) \times V(H)$ and $(a, x)(b, y)$ is an edge of $G \times H$ if $a = b$ and $xy \in E(H)$, or $ab \in E(G)$ and $x = y$. If (a, x) is a vertex of $G \times H$, then $d_{G \times H}((a, x)) = d_G(a) + d_H(x)$. The composition $G[H]$ of graphs G and H with disjoint vertex sets $V(G)$ and $V(H)$ and edge sets $E(G)$ and $E(H)$ is the graph with vertex set $V(G) \times V(H)$ and (a, x) is adjacent to (b, y) whenever b is adjacent to y or $a = b$ and x is adjacent to y. If (a, x) is a vertex of $G[H]$, then $d_{G[H]}((a, x)) = |V(H)|d_G(a) + d_H(x)$. The corona product $G \circ H$ is defined as the graph obtained from G and H by taking one copy of G and $|V(G)|$ copies of H and then by joining with an edge each vertex of the i^{th} copy of H which is named (H, i) with the i^{th} vertex of G for $i = 1, 2...|V(G)|$. If u is a vertex of $G \circ H$, then

$$d_{G \circ H}(u) = \begin{cases} d_G(u) + |V(H)| & \text{if} \quad u \in V(G) \\ d_H(u) + 1 & \text{if} \quad u \in (H, i). \end{cases}$$

The join $G + H$ of graphs G and H is a graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{uv : u \in V(G)$ and $v \in V(H)\}$. The symmetric difference $G \oplus H$ of two graphs G and H is the graph with vertex set $V(G) \oplus V(H)$ and $E(G \oplus H) = \{(u_1, u_2)(v_1, v_2)|u_1v_1 \in E(G)$ or $u_2v_2 \in E(H)$ but not both$\}$. The tensor product $G \otimes H$ of two graphs G and H is the graph with vertex set $V(G) \otimes V(H)$ and $E(G \otimes H) = \{(u_1, u_2)(v_1, v_2) \mid u_1v_1 \in E(G), u_2v_2 \in E(H)\}$. A topological index Top(G) of a graph G, is a number with this property that for every graph H isomorphic to G, Top(H) = Top(G). Usage of topological indices in chemistry began in 1947 when chemist *Harold Wiener* developed the most widely known topological descriptor, the *Wiener* index, and used it to determine physical properties of types of alkanes known as paraffin. In Mathematical Chemistry, there is a large number of topological indices of the form

$$TI = TI(G) = \sum_{v_i, v_j \in E(G)} \mathbb{F}(d_i, d_j).$$

The most popular topological indices of this kind are the:

- First Zagreb index, $M_1(G) = \sum_{u \in V(G)} d_G(u)^2$,
- Second Zagreb index, $M_2(G) = \sum_{uv \in E(G)} \Big(d_G(u)d_G(v)\Big)$,
- Third Zagreb index, $M_3(G) = \sum_{uv \in E(G)} \Big|d_G(u) - d_G(v)\Big|$,
- Hyper-Zagreb Index, $HM(G) = \sum_{uv \in E(G)} \Big(d_G(u) + d_G(v)\Big)^2$

Note that there are several more indices, see ([1], [4], [11]). The *Zagreb* indices are widely studied degree-based topological indices, and were introduced by *Gutman* and *Trinajstić* [3] in 1972. Recently, there was a vast research on comparing *Zagreb* indices see ([6], [8], [9]). A survey on the first *Zagreb* index see [2]. The sigma index can also be expressed as a sum over edges of G, is defined as [5]

$$\sigma(G) = \sum_{v_i v_j \in E(G)} (d_i - d_j)^2.$$

In [5], the authors defined the sigma index and studied inverse problems for the sigma index. We begin with the following basic example.

Example 1. *In this example the sigma index of some well-known graphs are calculated. We first consider the complete graph K_n and let $K_{n,n}$, denote a complete bipartite graph. Then*

$$\sigma(K_n) = \sum_{uv\in E(K_n)} \Big(d_G(u) - d_G(v)\Big)^2 = 0.$$

$$\sigma(K_{n,n}) = \sum_{uv\in E(K_{n,n})} \Big(d_G(u) - d_G(v)\Big)^2 = 0.$$

Let $K_{m,n}(m > n)$, denote a bipartite graph. Then we have:

$$\sigma(K_{m,n}) = \sum_{uv\in E(K_{m,n})} \Big(d_G(u) - d_G(v)\Big)^2 = m(m-n)^2.$$

For a cycle graph with n vertices, we have:

$$\sigma(C_n) = \sum_{uv\in E(C_n)} \Big(d_G(u) - d_G(v)\Big)^2 = 0.$$

For a path with $n > 2$ vertices, we have:

$$\sigma(P_n) = \sum_{uv\in E(P_n)} \Big(d_G(u) - d_G(v)\Big)^2 = 2.$$

For wheel on $n+1$ vertices, we have:

$$\sigma(W_n) = \sum_{uv\in E(W_n)} \Big(d_G(u) - d_G(v)\Big)^2 = (n-1)(n-4)^2.$$

In this paper, the sigma index of the Cartesian product, composition, join and disjunction of graphs are computed. Also, we apply some of our results to compute the sigma index of some special graph classes.

2. SIGMA INDEX OF GRAPH OPERATIONS

We begin this section with standard lemma as follows.

Lemma 1. *Let G and H be two connected graphs, then we have:*

(a)$|V(G\times H)| = |V(G\vee H)| = |V(G[H])| = |V(G\oplus H)| = |V(G)||V(H)|$,
$|E(G\times H)| = |E(G)||V(H)| + |V(G)||E(H)|$,
$|E(G+H)| = |E(G)| + |E(H)| + |V(G)V(H)|$,
$|E(G[H])| = |E(G)||V(H)|^2 + |E(G)||V(H)|$,
$|E(G\vee H)| = |V(G)||V(H)|^2 + |E(G)||V(G)|^2 - 2|E(G)||E(H)|$,
$|E(G\oplus H)| = |E(G)||V(H)|^2 + |E(h)||V(G)|^2 - 42|E(G)||E(H)|$.

(b) $G\times H$ *is connected if and only if* G *and* H *are connected.*

(c) *If* (a,b) *is a vertex of* $G\times H$ *then* $d_{G\times H}(a,b) = d_G(a) + d_H(b)$.

(d) *If* (a,b) *is a vertex of* $G[H]$ *then* $d_{G[H]}(a,b) = |V(G)|d_G(a) + d_H(b)$.

(e) *If* (a,b) *is a vertex of* $G \oplus H$ *or* $G \vee H$, *we have*:

$d_{G \oplus H}(a,b) = |V(G)|d_G(a) + |V(G)|d_H(b) - 2d_G(a)d_H(b)$.

$d_{G \vee H}(a,b) = |V(H)|d_G(a) + |V(G)|d_H(b) - d_G(a)d_H(b)$.

(f)) *If* u *is a vertex of* $G+H$ *then we have*:

$$d_{G+H}(u) = \begin{cases} d_G(u) + |V(H)| & \text{if} \quad u \in V(G) \\ d_H(u) + |V(G)| & \text{if} \quad u \in V(H). \end{cases}$$

Proof. The parts (a) and (b) are consequence of definitions and some famous results of the book of Imrich and Klavzar [7]. For the proof of (c-f) we refer to [10].

Theorem 1. *Let G and H be graphs. Then*

$$\sigma(G \times H) = |V(G)|\sigma(H) + |V(H)|\sigma(G).$$

Proof. From the definition of the Cartesian product of graphs, we have:

$$E(G \times H) = \{(a,x)(b,y) : ab \in E(G), x = y \text{ or } xy \in E(H), a = b\}$$

therefore we can write:

$$\begin{aligned} \sigma(G \times H) &= \sum_{(a,x)(b,y) \in E(G \times H)} [d_{G \times H}((a,x)) - d_{G \times H}((b,y))]^2 \\ &= \sum_{a \in V(G)} \sum_{(x,y) \in E(H)} [d_G(a) + d_H(x) - d_G(a) - d_H(y)]^2 \\ &+ \sum_{x \in V(H)} \sum_{(a,b) \in E(G)} [d_H(x) + d_G(a) - d_H(x) - d_G(b)]^2 \\ &= \sum_{a \in V(G)} \sum_{(x,y) \in E(H)} [d_H(x) - d_H(y)]^2 \\ &+ \sum_{x \in V(H)} \sum_{(a,b) \in E(G)} [d_G(a) - d_G(b)]^2 \\ &= |V(G)|\sigma(H) + |V(H)|\sigma(G). \end{aligned}$$

As an application of Theorem 1, we list explicit formulae for the sigma index of the rectangular grid $P_r \times P_S$, C_4 -nanotube $P_r \times C_q$ and C_4-nanotorus $P_r \times W_S$. The formulae follow from Theorem 1 by using the expressions, $M_1(P_n) = 4n - 6, n > 1; M_1(C_n) = 4n$.

Corollary 1.

1) $\sigma(P_r \times P_S) = 2(r+s), \quad r,s > 3,$
2) $\sigma(P_r \times C_q) = 2q,$
3) $\sigma(P_r \times W_n) = r(n-1)(n-4)^2 + 2n \;\; n > 4.$

Theorem 2. *Let G and H be graphs. Then*

$$\sigma(G[H]) = |V(H)|^4\sigma(G) + 2|E(H)||V(H)|M_1(H) - 8|E(H)|^3 - |V(G)|\sigma(H).$$

Proof. From the definition of the composition $G[H]$ we have:

$$\sigma(G[H]) = \sum_{\substack{(a,x)(b,y)\in E(G[H])\\ ab\in E(G)}} [d_{G[H]}((a,x)) - d_{G[H]H}((b,y))]^2$$

$$- \sum_{\substack{(a,x)(a,y)\in E(G[H])\\ xy\in E(G)}} [d_{G[H]}((a,x)) - d_{G[H]H}((a,y))]^2$$

$$= \sum_{x\in V(H)}\sum_{y\in V(H)}\sum_{ab\in E(G)} \Big(|V(H)|d_G(a) + d_H(x) - |V(H)|d_G(b) - d_H(y)\Big)^2$$

$$- \sum_{a\in V(G)}\sum_{xy\in E(H)} [d_H(x) - d_H(y)]^2$$

$$= \sum_{x\in V(H)}\sum_{y\in V(H)}\sum_{ab\in E(G)} \Big(|V(H)|d_G(a) + d_H(x) - |V(H)|d_G(b) - d_H(y)\Big)^2$$

$$- |V(G)|AI(H)$$

$$= \sum_{x\in V(H)}\sum_{y\in V(H)}\sum_{ab\in E(H)} \Big(|V(H)|^2(d_G(a) - d_G(b))^2 + d_H(x)^2 + d_H(y)^2 -$$

$$2d_H(x)d_H(y) - 2|V(H)|(d_G(a) - d_G(b))(d_H(x) - d_H(y))\Big)$$

$$- |V(G)|\sigma(H)$$

$$= |V(H)|^4\sigma(G) + 2|E(H)||V(H)|M_1(H) - 8|E(H)|^3 - |V(G)|\sigma(H).$$

□

As an application of Theorem 2, we present formulae for the sigma index of the fence graph $C_q[P_r]$ and the closed fence graph $P_r[C_q]$.

Corollary 2. $(C_q[P_r]) = 4r^2 + 12r - 2q + 8, \quad (P_r[C_q]) = 2q^4.$

Theorem 3. *Let G and H be graphs. Then*

$\sigma(G \circ H) = \sigma(G) + |V(G)|\sigma(H) + |V(H)|M_1(G) + |V(G)|M_1(H) - 8|E(G)||E(H)|$
$+ (|V(H)|-1)^2 - 4(|V(H)|)^2|E(H)| + 4|V(G)||V(H)||E(H)| - 4|V(H)||E(G)|$
$+ 4|V(G)||E(H)|.$

Proof. Using the definition of the sigma index, we have

$$\begin{aligned}\sigma(G\circ H) &= \sum_{uv\in(G\circ H)}[d_{(G\circ H)}(u)-d_{(G\circ H)}(v)]^2\\ &= \sum_{uv\in E(G)}[d_{(G}(u)+|V(H)|-d_{(G)}(v)-|V(H)|]^2\\ &+ \sum_{uv\in E(H)}\sum_{i=1}^{|V(G)|}[d_{(H)}(u)+1-d_{(H)}(v)-1]^2\\ &+ \sum_{u\in V(G)}\sum_{v\in V(H)}[d_{(G)}(u)+|V(H)|-d_{(H)}(v)-1]^2\\ &= \sigma(G)+|V(G)|\sigma(H)\\ &+ \sum_{u\in V(G)}\sum_{v\in V(H)}[d_{(G)}(u)^2+d_{(H)}(v)^2-2d_{(G)}(u)d_{(H)}(v)\\ &+ (|V(H)|-1)^2-2(|V(H)|-1)(d_{(G)}(u)-d_{(H)}(v))]\\ &= \sigma(G)+|V(G)|\sigma(H)+|V(H)|M_1(G)+|V(G)|M_1(H)-8|E(G)||E(H)|\\ &+ (|V(H)|-1)^2-4(|V(H)|)^2|E(H)|+4|V(G)||V(H)||E(H)|-4|V(H)||E(G)|\\ &+ 4|V(G)||E(H)|.\end{aligned}$$

□

Corollary 3. $\sigma(P_r \circ C_q) = 4q^3 - q^2(4r-1) + 4q + 3.$

Theorem 4. *Let G and H be graphs. Then we have:*
$\sigma(G+H) = |V(H)|M_1(G) + |V(G)|M_1(H) - 8|E(G)||E(H)| - 4|V(G)||V(H)||E(G)|$
$+ 4(|V(G)|)^2|E(H)| + 4(|V(H)|)^2|E(G)| - 4|V(G)||V(H)||E(H)|$
$+\sigma(G)+\sigma(H)$.

Proof. From the definition we know:
$E(G+H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$.

So, we have:

$$\begin{aligned}\sigma(G+H) &= \sum_{uv\in(G+H)}[d_{(G+H)}(u)-d_{(G+H)}(v)]^2\\ &= \sum_{uv\in E(H)}[d_{(G+H)}(u)-d_{(G+H)}(v)]^2\\ &+ \sum_{uv\in E(G)}[d_{(G+H)}(u)-d_{(G+H)}(v)]^2\\ &+ \sum_{u\in V(G)}\sum_{v\in V(H)}[d_{(G+H)}(u)-d_{(G+H)}(v)]^2.\end{aligned}$$

It is easy to see that:

$$\sum_{uv\in E(G)} [d_{(G+H)}(u) - d_{(G+H)}(v)]^2$$
$$= \sum_{uv\in E(G)} [d_{(G)}(u) - d_{(G)}(v)]^2 = \sigma(G). \tag{1}$$

and similarly we have:

$$\sum_{uv\in E(H)} [d_{(G+H)}(u) - d_{(G+H)}(v)]^2$$
$$= \sum_{uv\in E(H)} [d_{(H)}(u) - d_{(H)}(v)]^2 = \sigma(H). \tag{2}$$

Finally, we can write:

$$\sum_{u\in V(G)} \sum_{v\in V(H)} [d_{(G+H)}(u) - d_{(G+H)}(v)]^2$$
$$= \sum_{u\in V(G)} \sum_{v\in V(H)} [d_{(G)}(u) + |V(H)| - d_{(H)}(v) - |V(G)]^2$$
$$= \sum_{u\in V(G)} \sum_{v\in V(H)} \Big[d_{(G)}(u)^2 + d_{(H)}(v)^2 - 2d_{(G)}(u)d_{(H)}(v) - (|V(H)| - |V(G)|)^2$$
$$- 2\Big(|V(H)| - |V(G)|)(G)(d_{(G)}(u) - d_{(H)}(v)\Big)\Big]$$
$$= |V(H)|M_1(G) + |V(G)|M_1(H) - 8|E(G)||E(H)| - 4|V(G)||V(H)||E(G)|$$
$$+ 4(|V(G)|)^2|E(H)| + 4(|V(H)|)^2|E(G)| - 4|V(G)||V(H)||E(H)|. \tag{3}$$

□

Combining these three equations (1), (2), (3) will complete the proof.

Corollary 4. $\sigma(P_r + C_q) = 4q^2 - qr - 2q - 2$.

3. CONCLUDING

In this paper, the sigma index of the Cartesian product, composition, join and disjunction of graphs are computed. Also, we apply some of our results to compute the sigma index of some special graph classes.

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