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Research Article

ON SUFFICIENT CONDITIONS FOR CLOSE-TO-CONVEXITY OF ORDER 2^{-r}.

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ABSTRACT

The main idea of the present paper is to obtain sufficient conditions for close-to-convexity of order in 2^{-r}, where *r* is a positive integer.
Keywords: Analytic, univalent, starlike, convex and close-to-convex functions.

1. INTRODUCTION AND DEFINITIONS

Let the class A_n be the class of analytic functions in the unit disk $D=\{z: \ |z|<1\}$ and normalized, by the condition $f(0)=0$ and $f'(0)=1$. Then, A_n consisting of functions of the form

$$f(z)=z+\sum_{k=n+1}^{\infty} a_k z^k, \ \ (n\in\{1,2,3,...\}) \tag{1}$$

with $A_1=A$.

Definition 1. A domain D in the w-plane is said to be starlike with respect to a point $u_0 \in D$ if for each point $u\in D$ the line-segment $[u_0u]$ is contained in D [1].

The theory of univalent functions is dealt with functions $f(z)$ which are analytic and univalent in the unit disk D and normalized to by the $f(0)=0$ and $f'(0)=1$.

Definition 2. Let be the function $f(z)$ with $f(0)=0$. We say that the function $f(z)$ is starlike if $f(z)$ is univalent in D and $f(D)$ is a starlike domain with respect to origin [2].

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Let by $S_n^*(2^{-r})$ denote the subclass of A_n consisting of functions which are univalent in the unit disk D. In this case, a function $f(z) \in S_n^*(2^{-r})$ is said to be starlike of order 2^{-r} if and only if it satisfies the condition:

$$Re\left(\frac{zf'(z)}{f(z)}\right) > 2^{-r} \ \ (z \in D) \ \ ,$$

and a function $f(z) \in A_n$ is said to be close-to-convex of order 2^{-r} if and only if it satisfies the condition

$$Re\left(\frac{zf'(z)}{g(z)}\right) > 2^{-r} \ \ (z \in D, g \in S_n^*(0)).$$

We denote by $C_n(2^{-r})$ the class of all such functions. We note that

$$S_n^*(2^{-r}) \subset C_n(2^{-r}) \subset S_n \quad [9].$$

We now turn to an interesting subclass of S which contains S^* and has a simple geometric description. This is the class of close-to-convex functions. A function f analytic in the unit disk is said to be close-to-convex if there is a convex function g such that

$$Re\left(\frac{f'(z)}{g'(z)}\right) > 0 \quad , \quad \text{for all } z \in D.$$

We shall denote by K the class of close-to-convex functions f normalized by the usual conditions $f(0) = 0$ and $f'(0) = 1$. Note that f is not required a prior to be univalent. Note also that the associated function g need not to be normalized. The additional condition that $g \in C$ defines a proper subclass of K which will be denoted by K_0. Every convex function is obviously close-to-convex. More generally, every starlike function is close-to-convex. Indeed, each $f \in S^*$ has the form $f(z) = zg'(z)$ for some $g \in C$, and

$$\text{Re}\left\{\frac{f'(z)}{g'(z)}\right\} = \text{Re}\left\{\frac{z.f'(z)}{f(z)}\right\} > 0.$$

These remarks are summarized by the chain of proper inclusions

$$C \subset S^* \subset K_0 \subset K.$$

A set $E \subset C$ is said to be starlike with respect to a point $w_0 \in E$ if the linear segment joining w_0 to every other point $w \in E$ lies entirely in E. In more picturesque language, the requirement is that every point of E be visible from w_0. The set E is said to be convex if it is starlike with respect to each of its points; that is, if the linear segment joining any two points of E lies entirely in E. A convex function is one which maps the unit disk conformally onto a convex domain. A starlike function is a conformal mapping of the unit disk onto a domain starlike with respect to the origin. The subclass of S consisting of the convex functions is denoted by C

and S^* denotes the subclass of starlike functions. Thus, it is written as $C \subset S^* \subset S$. Closely related to the classes C and S^* is the class P of all functions φ analytic and having positive real part in D, with $\varphi(0) = 1$. Every $\varphi \in P$ can be represented as a Poisson-Stieltjes integral

$$\varphi(z) = \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} d\mu(t),$$

here $d\mu(t) \geq 0$ and $\int d\mu(t) = 1$. The following lemma is often useful:

Lemma 1. If $\varphi \in P$ and

$$\varphi(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$$

Then $|c_n| \leq 2$, $n = 1, 2, 3, \ldots$ .This inequality is sharp for each n [3].

Proof. Since

$$\frac{e^{it} + z}{e^{it} - z} = 1 + 2\sum_{n=1}^{\infty} e^{-\mathrm{int}} z^n ,$$

the representation lemma gives

$$c_n = 2\int_0^{2\pi} e^{-\mathrm{int}} d\mu(t) \ , \quad n = 1, 2, 3, \ldots .$$

Thus $|c_n| \leq 2$ with equality if and only if $e^{-\mathrm{int}}$ has a constant signum on the support of the measure $d\mu$. In particular, equality holds for all n for the function

$$\varphi(z) = \frac{e^{it} + z}{e^{it} - z} = 1 + 2\sum_{n=1}^{\infty} z^n .$$

The following theorem gives an analytic description of starlike functions:

Theorem 1. Let f be analytic in D, with $f(0) = 0$ and $f'(0) = 1$. Then $f \in S^*$ if and only if $zf'(z)/f(z) \in P$ [3].

Proof. Suppose that $f \in S^*$. Then we claim that f maps each subdisk $|z| < \rho < 1$ onto a starlike domain. An equivalent assertion is that $g(z) = f(\rho z)$ is starlike in D. In other words, we must show that for each fixed t $(0 < t < 1)$ and for each $z \in D$, the point $tg(z)$ is in the range of g. But since $f \in S^*$, an application of the lemma gives $tf(z) = f(w(\rho z))$ for some function w analytic in D and satisfying $|w(z)| < |z|$.

Thus

$$tg(z) = tf(\rho z) = f(w(\rho z)) = g(w_1(z))$$

where

$$w_1(z) = w(\rho z)/\rho \text{ and } |w_1(z)| \leq |z|$$

Theorem 2. Let f be analytic in D, with $f(0)=0$ and $f'(0)=1$. Then $f \in C$ if and only if $\left[1+zf''(z)/f'(z)\right] \in P$ [3].

Proof. Suppose that $f \in C$. Then, we claim that f must map each subdisk $|z|<r$ onto a convex domain. To show this, choose points z_1 and z_2 with $|z_1| \le |z_2| < r$. Let $w_1 = f(z_1)$ and $w_2 = f(z_2)$.

Let

$$w_0 = tw + (1-t)w_2, \quad 0<t<1$$

Then, since f is a convex mapping, there is a unique point $z_0 \in D$ for which $f(z_0)=w_0$. We have to show that $|z_0| < r$. But the function

$$g(z) = tf(zz_1/z_2) + (1-t)f(z)$$

is analytic in D, with $g(0)=0$ and $g(z_2)=w_0$. Because $f \in C$, the function $h(z) = f^{-1}(g(z))$ is well defined. Since $h(0)=0$ and $|h(z)| \le 1$ thus it tells us that $|h(z)| \le |z|$.

Thus

$$|z_0| = |h(z_2)| \le |z_2| < r,$$

which was to be shown. Hence f maps each circle $|z| = r < 1$ onto curve C which bounds a convex domain. The convexity implies that the slope of the tangent to C is nondecreasing as the curve is traversed in the positive direction. Analytically, this condition is

$$\frac{\partial}{\partial\theta}\left(\arg\left\{\frac{\partial}{\partial\theta} f(re^{i\theta})\right\}\right) \ge 0,$$

or

$$\operatorname{Im}\left\{\frac{\partial}{\partial\theta}\log\left[ire^{i\theta} f'(re^{i\theta})\right]\right\} \ge 0,$$

which reduces to the condition

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} \ge 0, \qquad |z| = r.$$

By the maximum principle for harmonic functions

$$\left[1+zf''(z)/f'(z)\right] \in P.$$

Conversely, suppose f is a normalized analytic function with

$$\left[1+zf''(z)/f'(z)\right] \in P.$$

The above calculation shows that the slope of the tangent to the curve $\mathcal{C}_r$ increases monotonically. But as a point makes a complete circuit of $\mathcal{C}_r$, the argument of the tangent vector has a net change

$$\int_0^{2\pi} \frac{\partial}{\partial\theta}\left(\arg\left\{\frac{\partial}{\partial\theta} f(re^{i\theta})\right\}\right)d\theta = \int_0^{2\pi} \operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}d\theta$$

$$= \operatorname{Re}\left\{\int_{|z|=r}\left[1+\frac{zf''(z)}{f'(z)}\right]\frac{dz}{iz}\right\} = 2\pi, \quad z = re^{i\theta}.$$

This shows that C_r is a simple closed curve bounding a convex domain. This for arbitrary $r < 1$ implies that f is a univalent function with convex range. Every close-to-convex function is univalent. This can be inferred from the following simple but important criterion for univalence.

Theorem 3. If f is analytic in a convex domain D and $\operatorname{Re}\{f'(z)\} > 0$ there, then f is univalent in D [3].

Proof. Let z_1 and z_2 be distinct points in D. Then f is defined on the linear segment joining z_1 to z_2, and

$$f(z_2) - f(z_1) = \int_{z_1}^{z_2} f'(z)dz$$

$$f(z_2) - f(z_1) = (z_2 - z_1)\int_0^1 f'\left[tz_2 + (1-t)z_1\right]dt \neq 0 ,$$

since $\operatorname{Re}\{f'(z)\} > 0$.

Theorem 4. Every close-to-convex functions is univalent [3].

Proof. If f is close-to-convex, then $\operatorname{Re}\left[f'(z)/g'(z)\right] > 0$ for some convex function g. Let D be the range of g and consider the function

$$h(w) = f\left(g^{-1}(w)\right), w \in D.$$

Then

$$h'(w) = \frac{f'(g'(w))}{g'(g^{-1}(w))} = \frac{f'(z)}{g'(z)}$$

so $\operatorname{Re}\{h'(w)\} > 0$ in D. Thus h is univalent, and so f is univalent.

2. ORDER OF CLOSE-TO-CONVEXITY

The object has been investigated and introduced by many scientists until this time[5],[6],[7],[8]. The following lemmas will be required for our main idea:

Lemma 2. Let the function $f(z)$ defined by (1) be in the class $S_n^*(\alpha)$. Then

$$\operatorname{Re}\left(\frac{f(z)}{z}\right)^{\lambda} > \frac{n}{2\lambda(1-\alpha)+n}, \quad (z \in D)$$

where

$$0 < \lambda \leq \frac{n}{2(1-\alpha)} \text{ and } 0 \leq \alpha < 1 \text{ [9].}$$

Main Theorem. If the function $f(z) \in A_n$ satisfies the inequality the condition

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) > 2^{-r} - \lambda \qquad (z \in D),$$

for $\alpha = 2^{-r}$ (r is a positive integer), $0 < \lambda \le \frac{n(1+\lambda)}{\left[2(1+\lambda)-2^{1-r}\right]}$ and, $\mu = \frac{2^{-r}}{1+\lambda}$ the $f(z)$ belongs to the class $\mathcal{C}_n(\upsilon)$, where $v = \frac{n(1+\lambda)}{(1+\lambda)(n+2\lambda)-2^{1-r}\lambda}$.

Thus, $f(z)$ is close-to-convex of order υ in D. The proof will require by defining a function $g(z)$ by

$$f'(z) = \left(\frac{g(z)}{z}\right)^{1+\lambda} \quad (z \in D)$$

or

$$\frac{zf'(z)}{g(z)} = \left(\frac{g(z)}{z}\right)^{\lambda} \quad (z \in D).$$

Therefore,

$$\frac{zf''(z)}{f'(z)} = \frac{z\left[(1+\lambda)\left(\frac{g(z)}{z}\right)^{\lambda}\left(\frac{g'(z)z-g(z)}{z^2}\right)\right]}{\left(\frac{g(z)}{z}\right)^{1+\lambda}}$$

$$= \frac{z\left[(1+\lambda)\left(\frac{g'(z)z-g(z)}{z^2}\right)\right]}{\left(\frac{g(z)}{z}\right)}$$

$$= (1+\lambda)\left(\frac{zg'(z)}{g(z)}-1\right).$$

That is,

$$(1+\lambda)\left(\frac{zg'(z)}{g(z)}-1\right) = \frac{zf''(z)}{f'(z)} \Rightarrow \frac{zg'(z)}{g(z)} - 1 = \frac{1}{(1+\lambda)}\cdot\frac{zf''(z)}{f'(z)}$$

$$= 1 + \frac{1}{(1+\lambda)}\cdot\frac{zf''(z)}{f'(z)}$$

$$= \frac{1}{(1+\lambda)}\left(1+\lambda+\frac{zf''(z)}{f'(z)}\right).$$

Proof of Main Theorem. Applying Lemma 2 to $g(z)$ we obtain

$$Re\left(1+\frac{zf'(z)}{g(z)}\right)=Re\left(\frac{g(z)}{z}\right)^{\lambda}>\frac{n}{2\lambda\left(1-\frac{2^{-r}}{(1+\lambda)}\right)+n}$$

$$=\frac{n}{2\lambda\left(\frac{1+\lambda-2^{-r}}{(1+\lambda)}\right)+n}$$

$$=\frac{n}{\frac{2\lambda+2\lambda^{2}-\lambda 2^{-r}+n+n\lambda}{(1+\lambda)}}$$

$$=\frac{n(1+\lambda)}{(1+\lambda)(n+2\lambda)-2^{1-r}\lambda}.$$

This completes the proof of main theorem. Letting $r=1$ in the main theorem, we obtain

Corollary 1 If the functions $f(z)$ and $g(z)$ in $\mathcal{A}_n$ satisfy the condition

If the functions $f(z)$ and $g(z)$ in $\mathcal{A}_n$ satisfies the condition

$$Re\left(1+\frac{zf''(z)}{f'(z)}\right)>\frac{1}{2}-\lambda \quad (z\in D)$$

for $0<\lambda\le n(1+\lambda)/\left[2(1+\lambda)-2^{1-r}\right]$ and , $\mu=2^{-r}/(1+\lambda)$ then $f(z)$ belongs to the class $\mathcal{C}_n(\upsilon)$, where

$$\upsilon=\frac{n(1+\lambda)}{n(1+\lambda)+\lambda(1+2\lambda)}.$$

Thus, $f(z)$ is close-to- convex of order υ in D.

Form corollary 1 we obtain

$$Re\left(\frac{1}{2}+\frac{zf''(z)}{f'(z)}\right)>-\lambda \quad (z\in D).$$

By setting $r=1$, $\lambda=\frac{1}{2}$ and $n=1$ in main theorem, we also find that

Corollary 2 If the function $f(z)\in\mathcal{A}_1$ satisfies the condition

$$Re\left(1+\frac{zf''(z)}{f'(z)}\right)>0 \quad (z\in D),$$

then $f(z)$ belongs to the class $\mathcal{C}_1(\frac{3}{5})$.Therefore, if $f(z)$ is convex in D, then $f(z)$ is close-to-convex of order $\frac{3}{5}$ in D.

Proof. Taking $r=1$ and $n=1$ in main theorem, we obtain

$$\upsilon = \frac{(1+\lambda)}{(1+\lambda)(1+2\lambda)-\lambda} = \frac{1+\lambda}{1+2\lambda+2\lambda^2}.$$

Now, setting $\lambda = \frac{1}{2}$

$$\upsilon = \frac{1+\frac{1}{2}}{1+2\left(\frac{1}{2}\right)+2\left(\frac{1}{2}\right)^2} = \frac{3}{5}.$$

It is easy see that $f(z) \in \mathcal{C}_n(\upsilon)$, where $0<\lambda \le 2^{-1}$ and since $\upsilon \ge \frac{3}{5}$. That is close-to- convex of order $\frac{3}{5}$ in D.

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