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Research Article

ON SUFFICIENT CONDITION FOR STARLIKENESS OF CONFLUENT HYPERGEOMETRIC FUNCTIONS

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ABSTRACT

In the present paper, firstly some univalent functions are obtained as special cases of hypergeometric functions and then we have discussed the starlikeness of confluent hypergeometric functions.
Keywords: Analytic function, univalent function, starlike functions, classes of starlike functions, hypergometric series, confluent hypergeometric function.

1. INTRODUCTION AND DEFINITIONS

We begin with introducing some of the important functions and their class in this present paper.

Definition 1. A function $f(z)=w$ is called analytic in a neighborhood U of z if it is differentiable everywhere in $U \subset £$.

Corollary 2. If $f(z)=w$ is analytic at point z, then $f(z)=w$ has continuous derivatives of every order at the point Z .

Definition 3. A single-valued function $f(z)=w$ is said to be univalent (schlicht or one-to-one) in domain $U \subset £$ if it never takes the same value twice; that is, if $f(z_1)-f(z_2)\neq 0$ for all z_1 and z_2 points in U with $z_1 \neq z_2$.

For analytic functions $f(z)=w$, the condition $f'(z)\neq 0$ is equivalent to local univalence at the point Z . The condition $f'(z)\neq 0$ is necessary for univalence of $f(z)=w$ but not sufficient. Note that it is a very important property to be univalent for a complex function. Suppose that a function $f(z)=w$ is a univalent function in the open unit disk $\mathcal{U}=\{z:|z|<1\}$.

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Then we say that the function $f(z)=w$ maps the domain $\mathcal{U} \subset £$ into $w-$plane. The domain $f(\mathcal{U})$ is called the image of $\mathcal{U}$ under the mapping $f(z)=w$ and also $\mathcal{U}$ is called the preimage of $f(\mathcal{U})$. If $f(z)=w$ is a univalent mapping of domain $\mathcal{U}$ onto $f(\mathcal{U})$, the derivative $f'(z)$ is nonzero at all finite point of domain $\mathcal{U}$ at which $f(z)=w$ is regular. Therefore, a univalent mapping is also called a conformal mapping of the domain $\mathcal{U}$ onto the domain $f(\mathcal{U})$. In other words, analytic univalent functions are conformal transformations because of the angle preservation.

Definition 4. A domain $f(U)$ in $w-$plane is said to be starlike with respect to a point $w_0 \in f(U)$ if for each point $w \in f(U)$ the line-segment $[w_0 w]$ is entirely contained in $f(U)$. In this case we say that the function $f(z)=w$ is to be starlike function with respect to the $w_0 \in f(U)$.

Note that the theory of functions is dealt with functions $f(z)=w$ which is analytic in the open unit disk $\mathcal{U}$ and normalized to by the condition $f(0)=f'(0)-1=0$. That is we will mostly consider $U = \mathcal{U} = \{z : |z| < 1\}$. Now let $\mathcal{A}$ be the class of analytic and normalized functions in $\mathcal{U}$. That is

$$\mathcal{A} = \{ f(z)=w \,|\, f : \mathcal{U} \to f(\mathcal{U}), f(0)=f'(0)-1=0, |z|<1 \}.$$

Then

$$f(z)=w \in \mathcal{A},\ w = f(z) = z + \sum_{n=2}^{\infty} a_n z^n \ ,$$

that is

$$w = f(z) = f(0) + \frac{f'(0)}{1!} z + \frac{f''(0)}{2!} z^2 + \ldots = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n.$$

This is originally the expansion of Taylor series for $f(z)=w \in \mathcal{A}$. Further, let $\mathcal{S}$ denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathcal{U}$. That is

$$\mathcal{S} = \{ f(z)=w \in \mathcal{A} \,|\, f : \mathcal{U} \to f(\mathcal{U}) \text{ is univalent} \}$$

Definition 5. $f(z)=w \in \mathcal{A}$ function is said to be starlike if the image domain of $f(z)=w$ is starlike with respect to the origin $w_0 = 0$.

Definition 6. The f unctions $f(z)=w \in \mathcal{A}$ starlike are said to be in the class $\mathcal{S}^*$. That is

$$\mathcal{S}^* = \{ f(z)=w \in \mathcal{A} \,|\, f : \mathcal{U} \to f(\mathcal{U}) \text{ is starlike} \}.$$

Definition 7. A series $\sum_{n=0}^{\infty} a_n$ is called hyp ergeometric series if the ratio $\frac{a_{n+1}}{a_n}$ is a rational function of n $\left(n \in \{0,1,2,...\}\right)$.

Definition 8. The hypergeometric function ${}_2F_1\left(a,b;c;z\right)$ is defined by means of a hypergeometric series as

$$
\begin{aligned}
{}_2F_1\left(a,b;c;z\right) &= 1+\frac{a.b}{c}\frac{z}{1!}+\frac{a\left(a+1\right).b\left(b+1\right)}{c\left(c+1\right)}\frac{z^2}{2!}+... \\
&= \sum_{n=0}^{\infty}\frac{\left(a\right)_n\left(b\right)_n}{\left(c\right)_n}\frac{z^n}{n!}. \qquad (1)
\end{aligned}
$$

with $a,b,c \in £$ and $c \neq 0,-1,-2,...$ where $\left(a\right)_n$ denotes the Pochhammer symbol

$$
\left(a\right)_n = \begin{cases} a\left(a+1\right)\left(a+2\right)...\left(a+n-1\right) & \left(n \neq 0\right) \\ 1 & \left(a \neq 0, n=0\right). \end{cases}
$$

Note that,

2 : refers to number of parameters in numerator
1 : refers to number of parameters in denominator.

We consider that ${}_1F_1\left(a,c;z\right)=\varphi\left(z\right)$ is a hypergeometric function. We now list some of simple properties of hypergeometric functions:

(i) The function $\varphi\left(z\right)$ satisfies the following Kummer hypergeometric differential equation

$$z\varphi''\left(z\right)+\left(c-z\right)\varphi'\left(z\right)-a\varphi\left(z\right)=0.$$

(ii) $$\frac{d}{dz}\,{}_1F_1\left(a,c;z\right)=\frac{a}{c}\,{}_1F_1\left(a+1,c+1;z\right).$$

For convenience, we define the general term $A_n=\frac{\left(a\right)_n\left(b\right)_n}{\left(c\right)_n n!}$ so that the hypergeometric series is presented as

$${}_2F_1\left(a,b;c;z\right)=\sum_{n=0}^{\infty}A_n z^n.$$

Otherwise, a hypergeometric function is a polynomial in z. Applying to the standard ratio test, we obtain;

$$\lim_{n\to\infty}\left|\frac{A_{n+1}z^{n+1}}{A_n z^n}\right| = \lim_{n\to\infty}\frac{\left(a+n\right)\left(b+n\right)}{\left(c+n\right)\left(n+1\right)}\left|z\right| = \left|z\right|.$$

It follows that the hypergeometric series is absolutely convergent in the open unit disk $\mathcal{U}$, where it defines an analytic function. All the elementary functions and many special functions including all univalent functions are special cases of the hypergeometric functions. We consider some particularly simple and important examples:

(i) when $a=1$ and $b=c$ in (1)

$$F(z)\equiv {}_2F_1(1,c;c;z)=\sum_{n=0}^{\infty}\frac{(1)_n(c)_n}{(c)_n}\frac{z^n}{n!}$$

$$=\frac{(1)_0.(c)_0}{(c)_0}\frac{z^0}{0!}+\frac{(1)_1.(c)_1}{(c)_1}\frac{z^1}{1!}+\frac{(1)_2.(c)_2}{(c)_2}\frac{z^2}{2!}+\frac{(1)_3.(c)_3}{(c)_3}\frac{z^3}{3!}+...$$

$$=\frac{1.1}{1}\frac{1}{1}+\frac{1.c}{c}\frac{z}{1}+\frac{1.2.c(c+1)}{c(c+1)}\frac{z^2}{2!}+\frac{1.2.3.c(c+1)(c+2)}{c(c+1)(c+2)}\frac{z^3}{3!}+...$$

$$=1+z+z^2+z^3+...$$

$$=\frac{1}{1-z}=(1-z)^{-1}$$

By using the above function, we have

$$z\,{}_2F_1(1,c;c;z)=z\frac{1}{1-z}=z(1-z)^{-1}.$$

Note that the function $w=f(z)=z(1-z)^{-1}$ is one of the leading examples of class $\mathcal{S}$.

(ii) When $a=1$ and $b=c=\frac{3}{2}$ in (1)

$$z\,{}_2F_1\left(1,\frac{3}{2};\frac{3}{2};z^2\right)=z+z^3+z^5+z^7+...$$

$$=z\left(1+z^2+z^4+z^6+...\right)$$

$$=z\frac{1}{1-z^2}=z\left(1-z^2\right)^{-1}.$$

The obtained above function is the most important member of the class $\mathcal{S}^*$.

(iii) when $a=\frac{1}{2}$, $b=1$ and $c=\frac{3}{2}$ in (1)

$$z\,{}_2F_1\left(\frac{1}{2},1;\frac{3}{2};z^2\right)=z\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n(1)_n}{\left(\frac{3}{2}\right)_n}\frac{(z^2)^n}{n!}$$

$$=z\left\{\frac{\left(\frac{1}{2}\right)_0.(1)_0}{\left(\frac{3}{2}\right)_0}\frac{\left(z^2\right)^0}{0!}+\frac{\left(\frac{1}{2}\right)_1.(1)_1}{\left(\frac{3}{2}\right)_1}\frac{\left(z^2\right)^1}{1!}+\frac{\left(\frac{1}{2}\right)_2.(1)_2}{\left(\frac{3}{2}\right)_2}\frac{\left(z^2\right)^2}{2!}+...\right\}$$

$$=z\left\{\frac{1.1}{1}\frac{1}{1}+\frac{\left(\frac{1}{2}\right).(1)}{\left(\frac{3}{2}\right)}\frac{z^2}{1}+\frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}+1\right).(1).(1+1)}{\left(\frac{3}{2}\right)\left(\frac{3}{2}+1\right)}\frac{\left(z^2\right)^2}{2}+...\right\}$$

$$= z\left\{1+\frac{1}{3}z^2+\frac{1}{5}z^4+\ldots\right\}$$

$$= z+\frac{1}{3}z^3+\frac{1}{5}z^5+\ldots$$

$$= \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right).$$

The function $w = f(z) = \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right)$ is a well-known function for the class $\mathcal{S}^*$.

(iv) when $a = c = 1$ and $b = 2$ in (1)

$$z\,{}_2F_1\left(1,2;1;z\right) = z\sum_{n=0}^{\infty}\frac{(1)_n(2)_n}{(1)_n}\frac{z^n}{n!}$$

$$= z\left\{\frac{(1)_0.(2)_0}{(1)_0}\frac{z^0}{0!}+\frac{(1)_1.(2)_1}{(1)_1}\frac{z^1}{1!}+\frac{(1)_2.(2)_2}{(1)_2}\frac{z^2}{2!}+\frac{(1)_3.(2)_3}{(2)_3}\frac{z^3}{3!}+\ldots\right\}$$

$$= z\left\{\frac{1.1}{1}\frac{1}{1}+\frac{1.2}{1}\frac{z}{1}+\frac{1.2.2.3}{1.2}\frac{z^2}{2!}+\frac{1.2.3.2.3.4}{1.2.3}\frac{z^3}{3!}+\ldots\right\}$$

$$= z\left(1+2z+3z^2+4z^3+\ldots\right)$$

$$= z+2z^2+3z^3+4z^4+\ldots$$

$$= z\frac{1}{(1-z)^2} = z(1-z)^{-2}.$$

The obtained above function is the most important member of the class $\mathcal{S}^*$ which is called Koebe function.

Definition 9. The confluent hypergeometric function is defined by ${}_1F_1\left(a,c;z\right) = \sum_{n=0}^{\infty}\frac{(a)_n}{(c)_n}\frac{z^n}{n!}$.

This series defines an entire function of z for with $a,c \in £$ and $c \neq 0,-1,-2,\ldots$.

Note that for a given univalent function in many cases starlikeness is not easy to see. There are a lot of necessary and sufficient conditions of functions to ensure starlikeness. This issue has been investigated by many scientists up until now. They have approached the subject from different points of views:

(i) a function $f(z) = w \in \mathcal{A}$ is said to be starlike in $\mathcal{U}$ if it satisfies

$$\mathcal{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0,\ \ (z \in \mathcal{U}).$$

(ii) a function $f(z) = w \in \mathcal{A}$ is said to be starlike of order α in $\mathcal{U}$ if it satisfies

$$Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \ (z \in \mathcal{U})$$

for some $\alpha, (0 \le \alpha < 1)$.

We recall that the following preliminary lemmas for univalency of the confluent hypergeometric functions are introduced by Miller and Mocanu [2].

Lemma 10. For $a \in \mathrm{R}$ and $c \in \mathrm{R}$ if one of the following conditions

(i) when $0 < a$ and $a \le c$

or

(ii) when $a \le 0$ and $1+\sqrt{1+a^2} \le c$

is satisfied, then

$$\mathrm{Re}\left\{ {}_1F_1(a,c;z) \right\} > 0 \quad (z \in \mathcal{U}).$$

Lemma 11. For $a \in \mathrm{R}$ and $c \in \mathrm{R}$ if one of the following conditions

(i) when $-1 < a$ and $a \le c$

or

(ii) when $-1 \ge a$ and $\sqrt{1+(1+a)^2} \le c$

is satisfied, then

$$\mathrm{Re}\left\{ \frac{c}{a}\, {}_1F_1'(a,c;z) \right\} > 0 \quad (z \in \mathcal{U}).$$

Therefore, ${}_1F_1(a,c;z)$ is univalent in $\mathcal{U}$.

Lemma 12. Let E be a set in the complex plane £ , and let a function $\psi : £^2 \mathrm{x} \mathcal{U} \to £$ satisfy $\psi(is,t;z) \notin E$ for all $z \in \mathcal{U}$ and for all real s and t satisfying $t \le -\frac{(1+s^2)}{2}$. If $\sigma(z)$ is analytic in $\mathcal{U}$ with $\sigma(0)=1$ and $\psi(\sigma(z), z\sigma'(z); z) \in E$ for all $z \in \mathcal{U}$, then $\mathrm{Re}\,\sigma(z) > 0 \ (z \in \mathcal{U})$ ([1]).

2. STARLIKENESS

In section 2, we modified a sufficient condition for starlikeness of confluent hypergeometric function:

$${}_1F_1(a,c;z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} \frac{z^n}{n!}.$$

There are some important studies in the literature that triggering the examination of univalent and starlikeness of hypergeometric functions ([3],[4]).

Theorem 1. Let $a = a_1 + ia_2$ and $c = c_1 + ic_2$ with $c \ne 0, -1, -2, \ldots$, and $\frac{1-2^{-r}}{3-2^{1-r}} c_2 \le a_2$.

Let us ${}_1F_1(a,c;z) \ne 0 \ (z \in \mathcal{U})$. If one of the following conditions

(i) when $a_1 > \dfrac{2\left(1-2^{-r}\right)\left(2-2^{-r}\right)}{\left(3-2^{1-r}\right)} = \dfrac{\left(2-2^{1-r}\right)\left(2-2^{-r}\right)}{\left(3-2^{1-r}\right)}$ and

$$c_2^{\,2} - 2c_2 - \left(3-2^{1-r}\right)\left(2c_1 + 2^{1-r} - 3\right) \le -\frac{2\left(3-2^{1-r}\right)}{\left(1-2^{-r}\right)}\left(a_1 + a_2 + 2^{-r} - 1\right)$$

or

(ii) when $\dfrac{2\left(1-2^{-r}\right)^2}{\left(3-2^{1-r}\right)} < a_1 \le \dfrac{2\left(1-2^{-r}\right)\left(2-2^{-r}\right)}{\left(3-2^{1-r}\right)}$ and

$$\left(1-c_2\right)^2 - \left(3-2^{1-r}\right)\left(2c_1 + 2^{1-r} - 3\right) \le -\frac{\left(3-2^{1-r}\right)}{\left(1-2^{-r}\right)}\left\{\frac{\left(3-2^{1-r}\right)\left(1-2^{-r}-a_1\right)^2}{\left(1-2^{-r}\right)} + 2a_2\right\}$$

or

(iii) when $a_1 \le \dfrac{2\left(1-2^{-r}\right)^2}{\left(3-2^{1-r}\right)}$ and

$$c_2^{\,2} - 2c_2 - \left(3-2^{1-r}\right)\left(2c_1 + 2^{1-r} - 3\right) \le \frac{2\left(3-2^{1-r}\right)}{\left(1-2^{-r}\right)}\left(a_1 - a_2 + 2^{-r} - 1\right)$$

is satisfied for 2^{-r}, where r is a positive integer, then the function $z\,{}_1F_1\left(a,c;z\right)$ is starlike of order 2^{-r} in $\mathcal{U}$. This is an alternative theorem to the theorem in ([5]).

Proof.

It is well-known that the hypergeometric function ${}_1F_1\left(a,c;z\right)$ satisfies the second order differential equation:

$$z^2\,{}_1F_1''\left(a,c;z\right) + \left(c-z\right)z\,{}_1F_1'\left(a,c;z\right) - az\,{}_1F_1\left(a,c;z\right) = 0$$

or equivalently:

$$\frac{z^2\,{}_1F_1''\left(a,c;z\right)}{{}_1F_1\left(a,c;z\right)} + \left(c-z\right)\frac{z\,{}_1F_1'\left(a,c;z\right)}{{}_1F_1\left(a,c;z\right)} - az = 0\,.$$

Let $\phi\left(z\right) = z\,{}_1F_1\left(a,c;z\right)$ and $\sigma(z)$ be defined by

$$\frac{z\phi'\left(z\right)}{\phi\left(z\right)} = 2^{-r} + \left(1-2^{-r}\right)\sigma\left(z\right).$$

Then $\sigma(z)$ is analytic in $\mathcal{U}$ and $\sigma(0) = 1$.

The details are as follows:

$$z^2\,{}_1F_1''\left(a,c;z\right) + \left(c-z\right)z\,{}_1F_1'\left(a,c;z\right) - az\,{}_1F_1\left(a,c;z\right) =$$

$$= \left(1-2^{-r}\right)z\sigma'\left(z\right) + \left(1-2^{-r}\right)^2\left(\sigma\left(z\right)\right)^2 + \sigma\left(z\right)\left(\left(1-2^{-r}\right)\left(c-1-z\right) - 2\left(1-2^{-r}\right)^2\right) + \left(1-2^{-r}\right)^2 -$$

$$\left(1-2^{-r}\right)\left(c-1-z\right) - az = 0$$

$$=\left(1-2^{-r}\right)z\sigma'(z)+\left(1-2^{-r}\right)^2\left(\sigma(z)\right)^2+\sigma(z)\left(1-2^{-r}\right)\left((c-1-z)-2\left(1-2^{-r}\right)\right)+\left(1-2^{-r}-a\right)z+\left(1-2^{-r}\right)\left(2-2^{-r}-c\right)=0$$

$$=\left(1-2^{-r}\right)\left\{z\sigma'(z)+\left(1-2^{-r}\right)\left(\sigma(z)\right)^2+\sigma(z)\left(c-3+2^{1-r}-z\right)+\frac{\left(1-2^{-r}-a\right)}{\left(1-2^{-r}\right)}z+\left(2-2^{-r}-c\right)\right\}=0 \qquad (2)$$

Let us now define $E\equiv\{0\}$ and

$$\psi(w_1,w_2;z)=w_2+\left(1-2^{-r}\right)w_1^2+\left(c-3+2^{1-r}-z\right)w_1+\frac{\left(1-2^{-r}-a\right)}{\left(1-2^{-r}\right)}z+\left(2-2^{-r}-c\right).$$

Then (2) is equivalent to $\psi\left(\sigma(z),z\sigma'(z);z\right)=0\in E$ for all $z\in\mathcal{U}$.

We want to apply Lemma 12 to conclude the proof of the theorem. Therefore, to prove that $\operatorname{Re}\sigma(z)>0$ in $z\in\mathcal{U}$, we must show that the assumptions of our theorem implies that $\psi(is,t;z)$ does not have a zero for $z\in\mathcal{U}$ and $t\le-\frac{\left(1+s^2\right)}{2}$ with all $s\in\mathbb{R}$.

Note that,

$$\psi(is,t;z)=t+\left(1-2^{-r}\right)(is)^2+\left(c_1+ic_2-3+2^{1-r}-x-iy\right)(is)+\frac{\left(1-2^{-r}-a_1-ia_2\right)}{\left(1-2^{-r}\right)}(x+iy)+\left(2-2^{-r}-c_1-ic_2\right)$$

for $z=x+iy$. Then

$$Re\{\psi(is,t;z)\}=t-\left(1-2^{-r}\right)s^2-(c_2-y)s+\frac{\left(1-2^{-r}-a_1\right)}{\left(1-2^{-r}\right)}x+\frac{a_2}{\left(1-2^{-r}\right)}y+\left(2-2^{-r}-c_1\right)\le$$

$$-\frac{1}{2}\left\{\left(3-2^{1-r}\right)s^2+2(c_2-y)s-\frac{2\left(1-2^{-r}-a_1\right)}{\left(1-2^{-r}\right)}x-\frac{2a_2}{\left(1-2^{-r}\right)}y+2c_1+2^{1-r}-3\right\}$$

Let us define the function $\vartheta(s)$ by

$$\vartheta(s)=\left(3-2^{1-r}\right)s^2+2(c_2-y)s-\frac{2\left(1-2^{-r}-a_1\right)}{\left(1-2^{-r}\right)}x-\frac{2a_2}{\left(1-2^{-r}\right)}y+2c_1+2^{1-r}-3.$$

Then the Δ(discrimination) of $\vartheta(s)$ satisfies

$$\Delta=(y-c_2)^2-\left(3-2^{1-r}\right)\left(-\frac{2\left(1-2^{-r}-a_1\right)}{\left(1-2^{-r}\right)}x-\frac{2a_2}{\left(1-2^{-r}\right)}y+2c_1+2^{1-r}-3\right)<$$

$$-x^2+1+c_2^2-\left(3-2^{1-r}\right)\left\{-\frac{2\left(1-2^{-r}-a_1\right)}{\left(1-2^{-r}\right)}x+2\left(\frac{c_2}{\left(3-2^{1-r}\right)}-\frac{a_2}{\left(1-2^{-r}\right)}\right)+2c_1+2^{1-r}-3\right\}.$$

If we define the function $h(x)$ by

$$h(x)=-x^2+1+c_2^2-\left(3-2^{1-r}\right)\left\{-\frac{2\left(1-2^{-r}-a_1\right)}{\left(1-2^{-r}\right)}x+2\left(\frac{c_2}{\left(3-2^{1-r}\right)}-\frac{a_2}{\left(1-2^{-r}\right)}\right)+2c_1+2^{1-r}-3\right\},$$

Then

$$h'(x) = -2(x - x_0),$$

where

$$x_0 = \frac{(3-2^{1-r})(1-2^{-r}-a_1)}{(1-2^{-r})}.$$

Therefore, the conditions (i), (ii) and (iii) of the theorem lead to:

(i) if $x_0 < -1$, equivalently if

$$\frac{(3-2^{1-r})(1-2^{-r}-a_1)}{(1-2^{1-r})} < -1 \Rightarrow (1-2^{-r}-a_1) < \frac{-(1-2^{1-r})}{(3-2^{1-r})}$$

$$\Rightarrow -a_1 < \frac{(1-2^{-r})[-2(2-2^{-r})]}{(3-2^{1-r})}$$

$$\Rightarrow a_1 > \frac{2(1-2^{-r})(2-2^{-r})}{(3-2^{1-r})}$$

then $h(x) < h(-1) \le 0.$

(ii) if $-1 \le x_0 < 1$, equivalently if $\dfrac{2(1-2^{-r})^2}{(3-2^{1-r})} < a_1 \le \dfrac{2(1-2^{-r})(2-2^{-r})}{(3-2^{1-r})}$,

then $h(x) < h(x_0) \le 0.$

(iii) if $x_0 \ge 1$, equivalently if

$$a_1 \le \frac{2(1-2^{-r})^2}{(3-2^{1-r})},$$

then $h(x) < h(1) \le 0.$

Note that the details were omitted at (ii) and (iii).

Thus we have $\Delta < 0$ with the conditions (i), (ii), (iii) of the theorem. This implies that $\vartheta(s) > 0$, that is

$$\operatorname{Re}\{\psi(is,t;z)\} < 0 \quad (z \in \mathcal{U}).$$

Therefore, we conclude that $\psi(is,t;z) \notin E$.

Finally, we obtain $\operatorname{Re}\sigma(z) > 0$ $(z \in \mathcal{U})$ that is equivalent to $z\,{}_1F_1(a;c;z)$ which is starlike of order 2^{-r} in $\mathcal{U}$.

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