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Research Article

SOME INTEGRAL INEQUALITIES FOR THE NEW CONVEX FUNCTIONS

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ABSTRACT

In this study, we obtained the Hermite-Hadamard integral inequality for *MφA-P-* function. Then we gave a new identity for *MφA-P-* function and using these identity, we obtained the theorems and the results.
Keywords: *MφA-P-* function, Hermite-Hadamard type inequality.
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1. INTRODUCTION

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a,b\in I$ with $a < b$. The following inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2} \tag{1.1}$$

holds. This double inequality is known in the literature as Hermite-Hadamard integral inequality for convex functions. Note that some of the classical inequalities for means can be derived from (1.1) for appropriate particular selections of the mapping f. Both inequalities hold in the reversed direction if f is concave. For some results which generalize, improve and extend the inequalities (1.1) we refer the reader to the recent papers (see [1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 13]).

In [7], Varosanec got the new convex class as follow:

Definition 1 [7] Let $f:J\subseteq[0,\infty)\to\mathbb{R}$, be a non-negative function, $h\neq 0$. We say that $f:I\subseteq[0,\infty)\to\mathbb{R}$ is an h-convex function, or that f belongs to the class $SX(h,I)$, if f is non-negative and for all $x, y \in I$, $\alpha\in(0,1)$ we have

$$f(\alpha x+(1-\alpha)y) \le h(\alpha)f(x)+h(1-\alpha)f(y). \tag{1.2}$$

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If inequality (1.2) is reversed, then f is said to be h -concave, i.e. $f \in SV(h,I)$.

Theorem 1 [7] Assume that the function $f : C \subseteq X \to [0,\infty)$ is an h -convex function with $h \in L[0,1]$. Let $y, x \in C$ with $y \neq x$ and assume that the mapping $t \to f[(1-t)x+ty]$, $t \in [0,1]$ is Lebesgue integrable on $[0,1]$. Then

$$\frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{x+y}{2}\right) \leq \int_0^1 f[(1-t)x+ty]dt \leq [f(x)+f(y)]\int_0^1 h(t)dt . \tag{1.3}$$

In [5], Dragomir et.al. gave the new theorem for the Hermite-Hadamard inequality via P -function as follow:

Definition 2 [5] A function $f : I \subseteq [0,\infty) \to \mathbb{R}$ is said to be P -function, if

$$f(tx+(1-t)y) \leq f(x) + f(y) \tag{1.4}$$

for $\forall x, y \in I$, $t \in [0,1]$.

Theorem 2 Let $f \in P(I)$, $a, b \in I$, with $a < b$ and $f \in L_1[a,b]$. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{2}{b-a}\int_a^b f(x)dx \leq 2(f(a)+f(b)) \quad . \tag{1.5}$$

Both inequalities are the best possible.

In [14], Ion, D. A. revealed the new identity for quasi-convex function as follow:

Lemma 1 Assume $a, b \in \mathbb{R}$ with $a < b$ and $f : [a,b] \to \mathbb{R}$ is a differentiable function on (a,b). If $f' \in L^1(a,b)$ then the following equality holds

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx = \frac{b-a}{2}\int_0^1 (1-2t) f'(ta+(1-t)b)dt . \tag{1.6}$$

In this study, we have gotten the generalization of the (1.6) equation for $M_\varphi A - p$ -function. We use the identity the theorems and corollary that is descent from previous study.

2. MAIN RESULTS

Definition 3 Let I be a interval, $\varphi : I \to \mathbb{R}$ be a continuous and strictly monotonic function. $f : I \to \mathbb{R}$ is said to be $M_\varphi A - p$ -function, if

$$f\left(\varphi^{-1}(t\varphi(x)+(1-t)\varphi(y))\right) \leq f(a)+f(b) \tag{2.1}$$

for all $x, y \in I$ ve $t \in [0,1]$.

Lemma 2 Let $f : I \subseteq [0,\infty) \to \mathbb{R}$ be a differentiable function on I^0, $\varphi : I \to \mathbb{R}$ be a continuous and strictly monotonic function and $a, b \in I^0$ with $0 < a < b$. If $f' \in L([a,b])$, then we get

$$\frac{f(a)+f(b)}{2} - \frac{1}{\varphi(b)-\varphi(a)}\int_a^b f(x)\varphi'(x)dx \tag{2.2}$$
$$\frac{\varphi(b)-\varphi(a)}{2}\left[\int_0^1 (1-2t)\left(\varphi^{-1}\right)'\left(t\varphi(a)+(1-t)\varphi(b)\right) f'\left(\varphi^{-1}\left(t\phi(a)+(1-t)\varphi(b)\right)\right)\right].$$

Proof. Firstly we use partial integration method on the right of (2.2) equality as follow

$$\int_0^1 (1-2t)\left(\varphi^{-1}\right)'\left(t\varphi(a)+(1-t)\varphi(b)\right) f'\left(\varphi^{-1}\left(t\phi(a)+(1-t)\varphi(b)\right)\right)$$
$$= \frac{(1-2t)}{\varphi(b)-\varphi(a)} f\left(\varphi^{-1}\left(t\varphi(a)+(1-t)\varphi(b)\right)\right)\Big|_0^1 + \frac{2}{\varphi(b)-\varphi(a)}\int_0^1 f\left(\varphi^{-1}\left(t\varphi(a)+(1-t)\varphi(b)\right)\right)$$
$$= \frac{f(a)+f(b)}{\varphi(b)-\varphi(a)} - \frac{2}{\left(\varphi(b)-\varphi(a)\right)^2}\int_a^b f(x)\varphi'(x)dx .$$

If we compare both sides of the last equality with $\frac{\varphi(b)-\varphi(a)}{2}$, the proof is completed.

Theorem 3 Let $f : I \subseteq [0,\infty) \to \mathbb{R}$ be differentiable on I^0 and $a, b \in I^0$ with $a < b$, $\varphi : I \to \mathbb{R}$ be a continuous and strictly monotonic function such that $\varphi^{-1} : \varphi(I^0) \to (I^0)$ is continuously differentiable, $f' \in L[a,b]$ and f′ is $M_{\varphi}A-p$ - function, we have

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{\varphi(b)-\varphi(a)}\int_a^b f(x)\varphi'(x)dx\right| \tag{2.3}$$
$$\leq \frac{|\varphi(b)-\varphi(a)|}{2}\left[A_1(t)+A_2(t)\right]\left(|f'(a)|+|f'(b)|\right)$$

where

$$A_1(t) = \int_0^{1/2} (1-2t)\left|\left(\varphi^{-1}\right)'\left(t\varphi(a)+(1-t)\varphi(b)\right)\right|dt , \tag{2.4}$$

$$A_2(t) = \int_{1/2}^1 (2t-1)\left|\left(\varphi^{-1}\right)'\left(t\varphi(a)+(1-t)\varphi(b)\right)\right|dt . \tag{2.5}$$

Proof. Firstly we take absolute value on both sides of the equality and then use the f' is $M_{\varphi}A-p$ -function, we get

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{\varphi(b)-\varphi(a)}\int_a^b f(x)\varphi'(x)dx\right| \tag{2.6}$$

$$\leq \frac{|\varphi(b)-\varphi(a)|}{2}\left[\int_0^1 |1-2t|\left|\left(\varphi^{-1}\right)'(t\varphi(a)+(1-t)\varphi(b))\right|\left|f'\left(\varphi^{-1}(t\varphi(a)+(1-t)\varphi(b))\right)\right|dt\right]$$

$$= \frac{|\varphi(b)-\varphi(a)|}{2}\left[\int_0^{1/2} (1-2t)\left|\left(\varphi^{-1}\right)'(t\varphi(a)+(1-t)\varphi(b))\right|\left|f'\left(\varphi^{-1}(t\varphi(a)+(1-t)\varphi(b))\right)\right|dt\right.$$

$$\left.+\int_0^{1/2} (2t-1)\left|\left(\varphi^{-1}\right)'(t\varphi(a)+(1-t)\varphi(b))\right|\left|f'\left(\varphi^{-1}(t\varphi(a)+(1-t)\varphi(b))\right)\right|dt\right]$$

$$\leq \frac{|\varphi(b)-\varphi(a)|}{2}\left[\int_0^{1/2} (1-2t)\left|\left(\varphi^{-1}\right)'(t\varphi(a)+(1-t)\varphi(b))\right|dt+\int_0^{1/2} (2t-1)\left|\left(\varphi^{-1}\right)'(t\varphi(a)+(1-t)\varphi(b))\right|dt\right]\left(|f'(a)|+|f'(b)|\right)$$

This proof is completed.

Corollary 1 ***i***. If we take $\varphi(x) = mx + n$ to (2.3), we get

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{a}\left[|f'(a)|+|f'(b)|\right]. \tag{2.7}$$

ii. If we take $\varphi(x) = \ln x$ to (2.3), we get

$$\left|f\left(\sqrt{ab}\right)-\frac{1}{\ln b-\ln a}\int_a^b f(x)dx\right| \leq \frac{\ln b-\ln a}{2}\left[B_1(t)+B_2(t)\right]\left(|f'(a)|+|f'(b)|\right)$$

where

$$B_1(t) = \int_0^{1/2} (1-2t)a^t b^{1-t}dt,$$

$$B_2(t) = \int_{1/2}^1 (2t-1)a^t b^{1-t}dt.$$

iii. If we take $\varphi(x) = x^{-1}$ to (2.3), we get

$$\left|f\left(\frac{2ab}{a+b}\right)-\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \leq \frac{b-a}{2ab}\left[C_1(t)+C_2(t)\right]\left(|f'(a)|+|f'(b)|\right)$$

where

$$C_1(t) = \int_0^{1/2} (1-2t)\frac{(ab)^2}{(tb+(1-t)a)^2}dt,$$

$$C_2(t) = \int_{1/2}^1 (2t-1)\frac{(ab)^2}{(tb+(1-t)a)^2}dt.$$

Theorem 4 Let $f : I \subseteq [0,\infty) \to \mathbb{R}$ be differentiable on I^0 and $a, b \in I^0$ with $a < b$, $\varphi : I \to \mathbb{R}$ be a continuous and strictly monotonic function such that $\varphi^{-1} : \varphi(I^0) \to (I^0)$ is continuously differentiable functions. If $|f'|^q$, $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ is $M_\varphi A - p$-function on $[a,b]$ then we get

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{\varphi(b)-\varphi(a)}\int_a^b f(x)\varphi'(x)dx\right| \tag{2.8}$$

$$\leq \frac{|\varphi(b)-\varphi(a)|}{2^{1+1/p}(p+1)^{1/p}}\left[D_1^{1/q}(t)+D_2^{1/q}(t)\right]\left(|f'(a)|^q+|f'(b)|^q\right)^{\frac{1}{q}}$$

where

$$D_1 = \int_0^{1/2}\left|\left(\varphi^{-1}\right)'\left(t\varphi(a)+(1-t)\varphi(b)\right)\right|^q dt,$$

$$D_2 = \int_{1/2}^{1}\left|\left(\varphi^{-1}\right)'\left(t\varphi(a)+(1-t)\varphi(b)\right)\right|^q dt.$$

Proof. By using Hölder inequality on (2.6) inequality, we get

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{\varphi(b)-\varphi(a)}\int_a^b f(x)\varphi'(x)dx\right|$$

$$\leq \frac{|\varphi(b)-\varphi(a)|}{2}\left[\left(\int_0^{1/2}(1-2t)^p dt\right)^{\frac{1}{p}}\left(\int_0^{1/2}\left|\left(\varphi^{-1}\right)'\left(t\varphi(a)+(1-t)\varphi(b)\right)\right|^q\left|f'\left(\varphi^{-1}\left(t\varphi(a)+(1-t)\varphi(b)\right)\right)\right|^q dt\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_{1/2}^{1}(2t-1)^p dt\right)^{\frac{1}{p}}\left(\int_{1/2}^{1}\left|\left(\varphi^{-1}\right)'\left(t\varphi(a)+(1-t)\varphi(b)\right)\right|^q\left|f'\left(\varphi^{-1}\left(t\varphi(a)+(1-t)\varphi(b)\right)\right)\right|^q dt\right)^{\frac{1}{q}}\right].$$

Since $|f'|^q$, $q > 1$, is $M_\varphi A - p$-function, we have

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{\varphi(b)-\varphi(a)}\int_a^b f(x)\varphi'(x)dx\right| \tag{2.9}$$

$$\leq \frac{|\varphi(b)-\varphi(a)|}{2^{1+1/p}(p+1)^{1/p}}\left[\left(\int_0^{1/2}\left|\left(\varphi^{-1}\right)'\left(t\varphi(a)+(1-t)\varphi(b)\right)\right|^q dt\right)^{\frac{1}{q}}+\left(\int_{1/2}^{1}\left|\left(\varphi^{-1}\right)'\left(t\varphi(a)+(1-t)\varphi(b)\right)\right|^q dt\right)^{\frac{1}{q}}\right]\left(|f'(a)|^q+|f'(b)|^q\right)^{\frac{1}{q}}.$$

This completed is proof.

Corollary 2 ***i.*** If we take $\varphi(x) = mx + n$ to (2.8), we obtain

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{|b-a|}{4(p+1)^{\frac{1}{q}}}\left[|f'(a)|^q+|f'(b)|^q\right]^{\frac{1}{q}} \tag{2.10}$$

ii. If we take $\varphi(x) = \ln x$ to (2.8), we obtain

$$\left| f\left(\sqrt{ab}\right) - \frac{1}{\ln b - \ln a}\int_a^b f(x)dx \right| \le \frac{\ln b - \ln a}{2^{1+\frac{1}{p}}(p+1)^{\frac{1}{p}}}\left[B_1^{\frac{1}{q}}(t) + B_2^{\frac{1}{q}}(t)\right]\left(\left|f'(a)\right|^q + \left|f'(b)\right|^q\right)^{\frac{1}{q}}$$

where

$$E_1 = \int_0^{\frac{1}{2}} a^{qt} b^{q(1-t)} dt \ ,$$

$$E_2 = \int_{\frac{1}{2}}^{1} a^{qt} b^{q(1-t)} dt \ .$$

iii. If we take $\varphi(x) = x^{-1}$, to (2.8), we obtain

$$\left| f\left(\frac{2ab}{a+b}\right) - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \right| \le \frac{b-a}{2^{1+\frac{1}{p}}(p+1)^{\frac{1}{pab}}}\left[F_1^{\frac{1}{q}}(t) + F_2^{\frac{1}{q}}(t)\right]\left(\left|f'(a)\right|^q + \left|f'(b)\right|^q\right)^{\frac{1}{q}}$$

where

$$F_1(t) = \int_0^{\frac{1}{2}} \frac{(ab)^2}{\left(tb+(1-t)a\right)^{2q}} dt \ ,$$

$$F_2(t) = \int_{\frac{1}{2}}^{1} \frac{(ab)^2}{\left(tb+(1-t)a\right)^{2q}} dt \ .$$

Theorem 5 Let $f : I \subseteq [0,\infty) \to \mathbb{R}$ be differentiable on I^0 and $a, b \in I^0$ with $a < b$, $\varphi : I \to \mathbb{R}$ be a continuous and strictly monotonic function such that $\varphi^{-1} : \varphi(I^0) \to (I^0)$ is continuously differentiable functions. If $|f'|^q$, $q \ge 1$, is $M_\varphi A - p$ -function on $[a,b]$ then we get

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{\varphi(b)-\varphi(a)}\int_a^b f(x)\varphi'(x)dx \right| \tag{2.11}$$

$$\le \frac{|\varphi(b)-\varphi(a)|}{2^{3-\frac{2}{q}}}\left[G_1^{\frac{1}{q}}(t) + G_2^{\frac{1}{q}}(t)\right]\left(\left|f'(a)\right|^q + \left|f'(b)\right|^q\right)^{\frac{1}{q}}$$

where

$$G_1 = \int_0^{\frac{1}{2}} (1-2t)\left|\left(\varphi^{-1}\right)'\left(t\varphi(a)+(1-t)\varphi(b)\right)\right|^q dt \ ,$$

$$G_2 = \int_{\frac{1}{2}}^{1} (2t-1)\left|\left(\varphi^{-1}\right)'\left(t\varphi(a)+(1-t)\varphi(b)\right)\right|^q dt .$$

Proof. We use with the power mean inequality on (2.6) and the $|f'|^q$, $q \geq 1$, is $M_{\varphi}A-p$-function then we get

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{\varphi(b)-\varphi(a)}\int_a^b f(x)\varphi'(x)dx\right|$$

$$\leq \frac{|\varphi(b)-\varphi(a)|}{2}\left[\left(\int_0^{\frac{1}{2}}(1-2t)dt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}}(1-2t)\left|\left(\varphi^{-1}\right)'\left(t\varphi(a)+(1-t)\varphi(b)\right)\right|^q\left|f'\left(\varphi^{-1}\left(t\varphi(a)+(1-t)\varphi(b)\right)\right)\right|^q dt\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_{\frac{1}{2}}^{1}(2t-1)dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^{1}(2t-1)\left|\left(\varphi^{-1}\right)'\left(t\varphi(a)+(1-t)\varphi(b)\right)\right|^q\left|f'\left(\varphi^{-1}\left(t\varphi(a)+(1-t)\varphi(b)\right)\right)\right|^q dt\right)^{\frac{1}{q}}\right]$$

$$\leq \frac{|\varphi(b)-\varphi(a)|}{2^{3-\frac{2}{q}}}\left[\left(\int_0^{\frac{1}{2}}(1-2t)\left|\left(\varphi^{-1}\right)'\left(t\varphi(a)+(1-t)\varphi(b)\right)\right|^q dt\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{2}}^{1}(2t-1)\left|\left(\varphi^{-1}\right)'\left(t\varphi(a)+(1-t)\varphi(b)\right)\right|^q dt\right)^{\frac{1}{q}}\right]\left(|f'(a)|^q+|f'(b)|^q\right)^{\frac{1}{q}}$$

Corollary 3 ***i.*** If we take $\varphi(x) = mx+n$ to (2.11), we obtain

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{2^{3-\frac{1}{q}}}\left[|f'(a)|^q+|f'(b)|^q\right]^{\frac{1}{q}} \tag{2.12}$$

ii. If we take $\varphi(x) = \ln x$ to (2.11), we obtain

$$\left|f\left(\sqrt{ab}\right)-\frac{1}{\ln b-\ln a}\int_a^b f(x)dx\right| \leq \frac{\ln b-\ln a}{2^{3-\frac{2}{p}}}\left[H_1^{\frac{1}{q}}(t)+H_2^{\frac{1}{q}}(t)\right]\left(|f'(a)|^q+|f'(b)|^q\right)^{\frac{1}{q}}$$

where

$$H_1 = \int_0^{\frac{1}{2}} a^{qt}b^{q(1-t)}dt$$

$$H_2 = \int_{\frac{1}{2}}^{1} a^{qt}b^{q(1-t)}dt$$

iii. If we take $\varphi(x) = x^{-1}$, to (2.11), we obtain

$$\left| f\left(\frac{2ab}{a+b}\right) - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \right| \leq \frac{b-a}{2^{3-\frac{2}{p}}ab}\left[K_1(t)+K_2(t)\right]\left(\left|f'(a)\right|^q + \left|f'(b)\right|^q\right)^{\frac{1}{q}}$$

where

$$F_1(t) = \int_0^{\frac{1}{2}} \frac{(ab)^2}{\left(tb+(1-t)a\right)^{2q}}dt$$

$$F_2(t) = \int_{\frac{1}{2}}^{1} \frac{(ab)^2}{\left(tb+(1-t)a\right)^{2q}}dt .$$

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