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Research Article

THE GENERAL SOLUTION OF A LAYERED MEDIUM RESTING ON A RIGID FOUNDATION

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ABSTRACT

In this study, the general solution of a receding contact problem of layered medium resting on rigid foundation is considered. The layered medium consists of N homogeneous layer with varying material properties and heights. The contact surfaces can be receding or attached. The problem may include M receding contact surfaces. The layered medium is loaded with n concentrated loads transmitted by circular stamps and r distributed loads. The problem is reduced to integral equation system using elasticity theory and Fourier integral transform. The numerical solution of the system is done by using Matlab. A program is written to solve the problem given by the user with any geometry and loading. The program is tested with two studies from the literature and it was shown that the results are compatible.

Keywords: Receding contact, Fourier transform, layered medium.

1. INTRODUCTION

It can be said that the contact mechanics was started with the paper named "On the contact of elastic solids" by Heinrich Hertz in 1882 [1]. Although the expressions of elasticity theory are complex and complicated, it gives better results compared to basic theory. The solution of strain and stress problems of engineering problems using elasticity theory has been increased parallel to the improvements in computer technology and numerical solution methods. Similarly, there has been a significant increase in contact mechanics problems.

In literature, the contact problems of layered mediums are approached generally in two ways. In the first way, the effect of the gravity is considered and the separation occurs in a finite zone. However, the effect of the gravity is neglected in the second way and the separation distance goes to infinite. The second way of the problems are called receding contact in literature and this study is an example of this way. Some of the studies in the literature for receding contact can be summarized as follows.

The contact problem of an infinite layer resting on a half-plane and loaded with a distributed load was investigated by Keer at. al. [2]. Ratwani and Erdoğan [3] studied the frictionless contact problem of a layer lying on a half plane. A compressive load was applied to the layer through a

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frictionless rigid block. Dempsey at. al. [4] in vestigated the contact problem of layer on a Winkler foundation under different loadings. The contact problem of layer rigidly supported at the top and pushed with a rigid block at the bottom was considered by Kahya [5]. Birinci and Erdol [6] was studied the frictionless contact problem of two layers resting on simple support. The contact problem of two layers lying on rigid foundation was investigated by Çömez [7]. The top of the medium was loaded using a rigid block.

The problems in the literature have differences in terms of number of layers, loading, and foundation. Although the solution process in each problem is similar, the analytical solution should be done for each problem separately. In this study, the general solution of contact problem resting on a rigid foundation are investigated and a computer program is developed to solve any contact problem given by the user using obtained general solution.

2. THE DEFINITION OF THE PROBLEM

The general solution of the contact problem of a layered medium resting on rigid foundation is considered. The loading and the geometry of the problem is given in Fig. 1 as representatively.

The layered medium consists of N layers with different material properties and height. The contact surfaces between layers can be attached or not attached. In case of not attached case, a receding contact occurs. In other words, an infinite separation zone occurs and the contact distances changes according to loading. It is assumed that the problem has a total of M contact surfaces with unknown contact distances. m of these contact surfaces are formed between layers and n of these contact surfaces are formed under rigid blocks. The layered medium can be loaded with cases given below.

a) A concentrated load applied by means of a rigid circular block (can be one or more, n loads)
b) Distributed load (can be one or more, r loads)
c) a and b cases act simultaneously

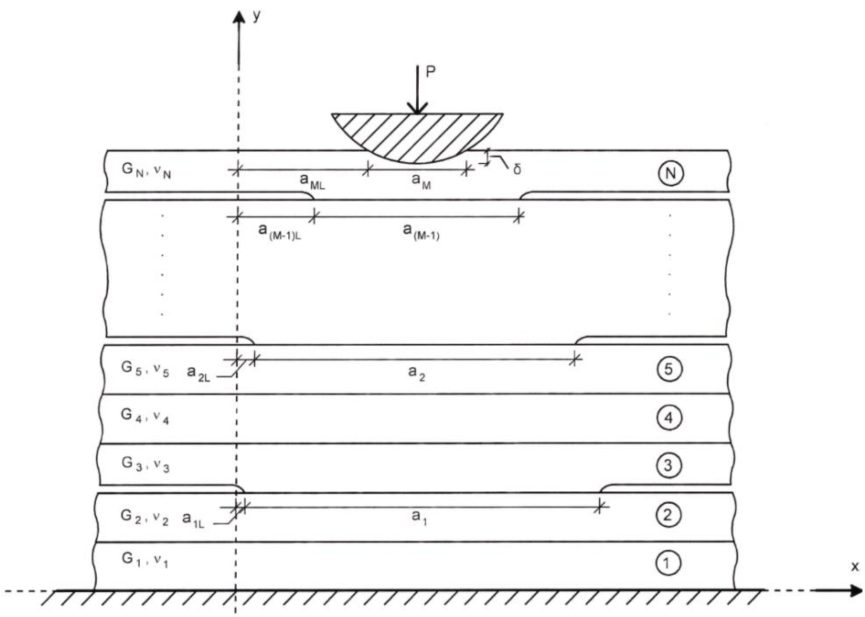

Figure 1. The geometry and the loading of the problem as representatively

It is assumed that the medium is attached to the foundation in the solution. In addition, the effect of the gravity is neglected. The problem is considered as a plain strain problem.

3. THE FORMULATION OF THE PROBLEM

The equilibrium equations in terms of displacements (Navier Equations) according to a local axes passing through under the *i*. layer are obtained as follows.

$$(\lambda_i + G_i)\left[\frac{\partial^2 u_i}{\partial x^2} + \frac{\partial^2 v_i}{\partial x \partial y_i}\right] + G_i\left[\frac{\partial^2 u_i}{\partial x^2} + \frac{\partial^2 u_i}{\partial {y_i}^2}\right] = 0 \tag{1a}$$

$$(\lambda_i + G_i)\left[\frac{\partial^2 u_i}{\partial x \partial y_i} + \frac{\partial^2 v_i}{\partial {y_i}^2}\right] + G\left[\frac{\partial^2 v_i}{\partial x^2} + \frac{\partial^2 v_i}{\partial {y_i}^2}\right] = 0 \tag{1b}$$

If the partially differential equations given by (1) are converted into ordinary differential equations using Fourier integral transform and solved, following stress and displacement expressions are obtained,

$$u_i(x, y_i) = \frac{1}{2\pi}\int_0^\infty \left[(A_{i1} + A_{i2}y_i)e^{-\xi y_i} + (A_{i3} + A_{i4}y_i)e^{\xi y_i}\right]e^{-i\xi x}d\xi \tag{2a}$$

$$v_i(x, y_i) = \frac{1}{2\pi}\int_0^\infty \left\{-i\left[A_{i1} + (\frac{\kappa_i}{\xi} + y_i)A_{i2}\right]e^{-\xi y_i} + i\left[-A_{i3} + (\frac{\kappa_i}{\xi} - y_i)A_{i4}\right]e^{\xi y_i}\right\}e^{-i\xi x}d\xi \tag{2b}$$

$$\sigma_{x_i}(x, y_i) = -\frac{1}{2\pi}\int_0^\infty 2G_i\xi i\left\{\left[\xi(A_{i1} + A_{i2}y_i) - \frac{3-\kappa_i}{2}A_{i2}\right]e^{-\xi y_i} + \left[\xi(A_{i3} + A_{i4}y_i) + \frac{3-\kappa_i}{2}A_{i4}\right]e^{\xi y_i}\right\}e^{-i\xi x}dx \tag{3a}$$

$$\sigma_y(x, y_i) = \frac{1}{2\pi}\int_0^\infty 2G_i\xi i\left\{-\left[\xi(A_{i1} + A_{i2}y_i) + \frac{\kappa_i+1}{2}A_{i2}\right]e^{-\xi y} + \left[-\xi(A_{i3} + A_{i4}y_i) + \frac{\kappa_i+1}{2}A_{i4}\right]e^{\xi y_i}\right\}e^{-i\xi x}dx \tag{3b}$$

$$\tau_{xy}(x, y_i) = -\frac{1}{2\pi}\int_0^\infty 2G_i\xi\left\{-\left[\xi(A_{i1} + A_{i2}y_i) + \frac{\kappa_i-1}{2}A_{i2}\right]e^{-\xi y_i} + \left[\xi(A_{i3} + A_{i4}y_i) - \frac{\kappa_i-1}{2}A_{i4}\right]e^{\xi y_i}\right\}e^{-i\xi x}dx \tag{3c}$$

in which, G_i is the shear modulus of the *i*. layer; κ_i is the material constant of the *i*. layer and becomes $\kappa_i = 3 - 4\upsilon_i$ in case of plain strain; υ_i is the Poisson ratio of *i*. layer; A_{ij} ($i = 1, 2, 3, ..., N$; $j = 1, 2, 3, 4$) are the unknowns coefficient functions of the layers and they will be found using boundary conditions of the problem.

Although the boundary conditions of the problem can differ according to defined problem, they can be defined generally,

i) The boundary conditions (BCs) between the first layer and the rigid foundation (2 BCs)

$$u_1(x,0)=0, \quad v_1(x,0)=0, \qquad (-\infty < x < \infty) \tag{4a,b}$$

ii) The BCs between the *i.* layer and *(i+1).* layer (*i=1,2,3,...,N-1*)
In case of attached contact (4 BCs)

$$u_i(x,h_i)-u_{i+1}(x,0)=0, \quad v_i(x,h_i)-v_{i+1}(x,0)=0, \quad (-\infty < x < \infty) \tag{5a,b}$$

$$\sigma_{y_i}(x,h_i)-\sigma_{y_{i+1}}(x,0)=0, \quad \tau_{xy_i}(x,h_i)-\tau_{xy_{i+1}}(x,0)=0, \quad (-\infty < x < \infty) \tag{5c,d}$$

In case of not attached contact, for *j.* receding contact surface (5 BCs)

$$\sigma_{y_i}(x,h_i)=-p_j(x), \quad \sigma_{y_{i+1}}(x,0)=-p_j(x), \qquad (a_{jL} \le x \le a_{jR}) \tag{6a,b}$$

$$\tau_{xy_i}(x,h_i)=0, \quad \tau_{xy_{i+1}}(x,0)=0, \qquad (-\infty < x < \infty) \tag{6c,d}$$

$$\frac{\partial v_i(x,h_i)}{\partial x}-\frac{\partial v_{i+1}(x,0)}{\partial x}=0, \qquad (a_{jL} \le x \le a_{jR}), \qquad (j=1,2,3,...,m) \tag{7}$$

iii) The BCs on the top of the medium (2+*n* BCs)

$$\tau_{xy_N}(x,h_N)=0, \qquad (-\infty < x < \infty) \tag{8a}$$

$$\sigma_{y_N}(x,h_N)=\begin{cases} q_1(x), & b_{1L} \le x \le b_{1R} \\ \vdots & \vdots \\ q_r(x), & b_{rL} \le x \le b_{rR} \\ p_{m+1}(x), & a_{(m+1)L} \le x \le a_{(m+1)R} \\ \vdots & \vdots \\ p_{m+n}(x) & a_{(m+n)L} \le x \le a_{(m+n)R} \\ 0 & other \end{cases} \tag{8b}$$

$$\frac{\partial v_N(x,h_N)}{\partial x}=f_j, \qquad a_{jL} \le x \le a_{jR}, \qquad (j=m+1, m+2, m+3, ..., m+n) \tag{9}$$

Used expressions in these BCs are given below.

- u_i, v_i : the displacement components of the *i.* layer in x_i ve y_i , respectively (*i=1,2,3,...,N*)
- h_i : the height of *i.* layer (*i=1,2,3,...,N*)
- p_j : the unknown contact pressures for *j.* receding contact surface (*j=1,2,3,...,M*)
- a_{jL}, a_{jR}: the start and end points of *j.* receding contact surface, respectively (*j=1,2,3,...,M*)
- q_j : the load function for *j.* distributed load (*j=1,2,3,...,r*)
- b_{jL}, b_{jR} : the start and end points of *j.* distributed load, respectively (*j=1,2,3,...,r*)

Using (4-9), *4N+M* BCs can be chosen according to the problem. *4N-2* of these BCs are in terms of stresses and *M+2* of these BCs are in terms of displacements. Using BCs obtained from

(4,5,6,8), the unknown coefficient functions A_{ij} ($i = 1..N$, $j = 1..4$) can be found in terms of unknown contact pressures p_j and contact distances a_{jL}, a_{jR} (*j=1..M*). The BCs chosen from (7,9) have not yet been used and can be used to obtain an integral system consisted of *M* singular integral equations in order to find unknown contact pressures p_j. The solution of this integral system can be done using the method suggested in Erdogan and Gupta [9] by the help of a written program.

The program chooses appreciate BCs from (4-9) according to defined problem and solves the problem as given in [9]. After the solution, the stress and displacement in any point can be obtained or wanted graphs can be drawn using program.

4. NUMERICAL RESULTS

In this section, the solutions obtained from the program are compared to the solution in the literature. The graphs in the existing studies are digitalized using a program named “Engauge Digitizer”. A comparison parameter namely “%Diff” is defined to compare the solutions as follows,

$$\% \, Diff = \frac{x_l - x_p}{x_l} * 100 \tag{10}$$

in which, x_l is the result in the literature, x_p is the solution obtained from the program.

The master thesis of Kahya [5] is chosen as the first study to compare the results. The contact problem between a layer supported rigidly and a rigid block was considered in [5]. A concentrated load *P* was applied by the help of a circular rigid block with radius *R*. The problem considered as a plain strain problem. The problem consisted of one layer (*N=1*) and one integral equation (*M=1*). The geometry and the loading of the problem are given in Fig. 2.

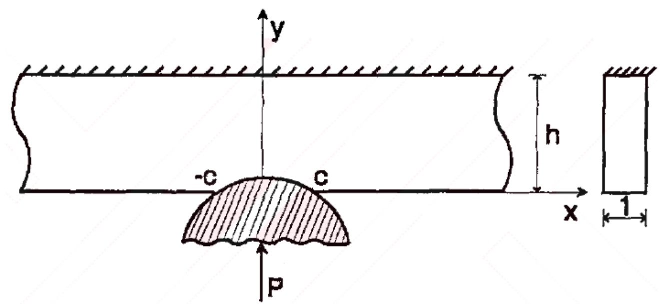

Figure 2. The geometry and the loading of the problem in [5]

Table 1 shows the comparison between the results of [5] and the results obtained from the program for various material properties and loading. The contact distances occurs in *[-c, c]* in [5], whereas the contact distances occurs in [a_{1L} , a_{1R}] in this study. It can be seen from the table, the results are approximately same in the studies. The dimensionless contact pressures under the block for various $\frac{G}{P/h}$ ratios are given in Fig. 3. It can be seen from the figure, the pressures are compatible with each other.

Table 1. The comparison of contact distances for various loading and material properties

			Kahya [5]	This Study (Program)		% Diff	
$\frac{G}{P/h}$	R/h	κ	c/h	a_{1s}	a_{1e}	a_{1s}	a_{1e}
100	10	2	0,215290	-0,21519051	0,21519099	-0,046	0,046
100	100	2	0,622161	-0,62216683	0,62216739	0,001	-0,001
100	1000	2	1,545286	-1,54536214	1,54536244	0,004	-0,005
500	10	2	0,097407	-0,09740715	0,09740715	0,000	-0,000
500	100	2	0,300239	-0,30024043	0,30024056	0,000	-0,000
500	1000	2	0,830313	-0,83032547	0,83032634	0,001	-0,001
1000	10	2	0,068987	-0,06898726	0,06898725	0,000	-0,000
1000	100	2	0,215208	-0,21520847	0,21520900	0,000	-0,000
1000	1000	2	0,622166	-0,62217092	0,62217082	0,001	-0,001
100	100	1,5	0,567560	-0,56756741	0,56756741	0,001	-0,001
100	100	2,5	0,670244	-0,67025250	0,67025328	0,001	-0,001

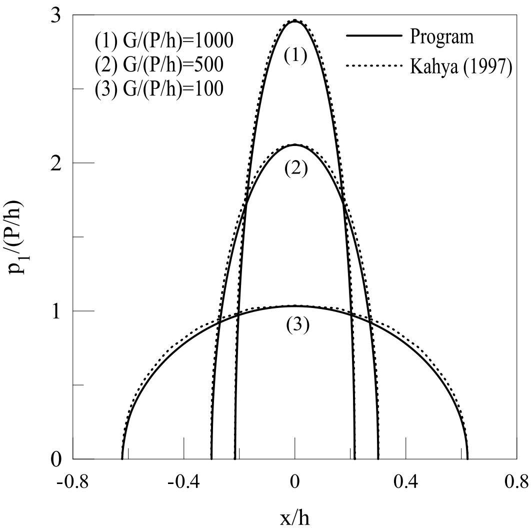

Figure 3. The comparison of the contact pressures for various $\frac{G}{P/h}$ ratios

($\kappa = 2, R/h = 100$)

As the second study, the master thesis done by Çömez [7] is chosen. In his study, the receding contact problem of a layered medium consisted of two layers was investigated. The bottom layer supported rigidly and a concentrated load *P* was applied by the means of a rigid circular block with radius *R* to the top of the medium. The layers were not attached and could have different heights and material properties. The contact distances occur in *[-a, a]* and *[-b, b]* in

[7], whereas the contact distances occurs in $[a_{2L}, a_{2R}]$ and $[a_{1L}, a_{1R}]$ in this study, respectively. The problem considered as a plain strain problem. The problem consisted of two layers (*N=2*) and two integral equations (*M=2*). The geometry and the loading of the problem are given in Fig. 4.

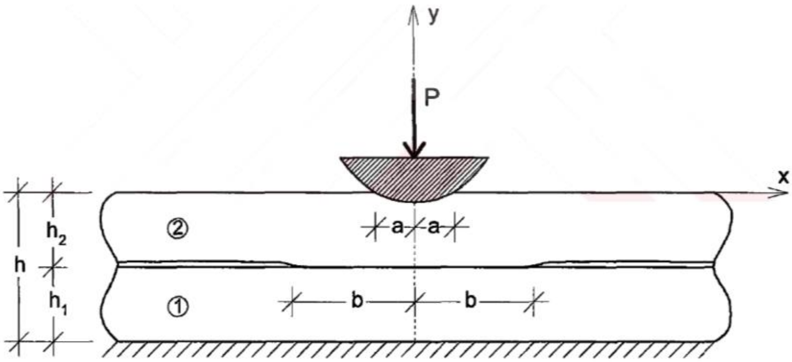

Figure 4. The geometry and the loading of the problem in [7]

The comparison of contact distances between the results given in [7] and obtained in this study for various loadings and material properties is given in Table 2. It can be said that, the obtained results are approximately same compared to existing results. Fig. 5 and 6 show the contact pressures between the layers and between the layer and the block for various R/h ratios, respectively. It can be seen from the figures, the pressures are compatible with each other.

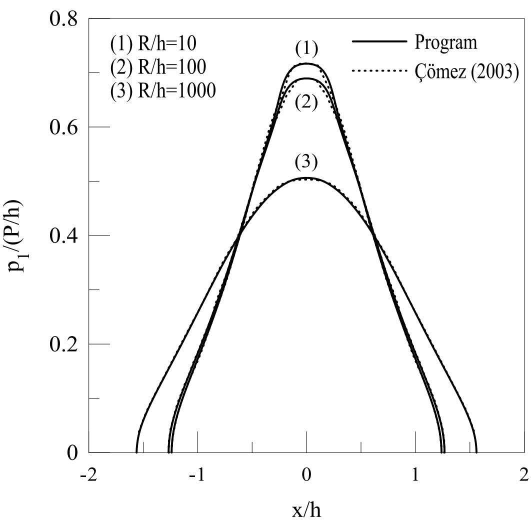

Figure 5. The comparison of the contact pressures between layers for various R/h

$$\left(\frac{G_2}{P/h_2}=500,\ \kappa_1=\kappa_2=2,\ h_1/h_2=2,\ G_1/G_2=1\right)$$

Table 2. The comparison of contact distances for various loading and material properties ($\kappa_1 = \kappa_2 = \kappa$, $h_1 / h_2 = 2$, $G_1 / G_2 = 1$)

Problem Parameters			Çömez (2003)		This Study (Program)				% Diff			
$\frac{G_2}{P/h_2}$	R/h_2	κ	b/h	a/h	a_{1L}	a_{1R}	a_{2L}	a_{2R}	a_{1L}	a_{1R}	a_{2L}	a_{2R}
100	10	2	1,25340	0,22026	-1,251823	1,251823	-0,220269	0,220269	-0,13	0,13	0,00	0,00
100	500	2	1,95470	1,58730	-1,953003	1,953003	-1,587949	1,587949	-0,09	0,09	0,04	-0,04
100	1000	2	2,40810	2,13520	-2,409606	2,409607	-2,134888	2,134888	0,06	-0,06	-0,01	0,01
500	10	2	1,24200	0,09788	-1,239558	1,239575	-0,097885	0,097886	-0,20	0,20	0,00	-0,01
500	500	2	1,40250	0,72430	-1,399719	1,399719	-0,724458	0,724458	-0,20	0,20	0,02	-0,02
500	1000	2	1,56410	1,03120	-1,561074	1,561075	-1,031475	1,031475	-0,19	0,19	0,03	-0,03
1000	10	2	1,23950	0,06916	-1,238065	1,238065	-0,069157	0,069157	-0,12	0,12	0,00	0,00
1000	500	2	1,31720	0,50440	-1,316202	1,316201	-0,504431	0,504431	-0,08	0,08	0,01	-0,01
1000	1000	2	1,40240	0,72430	-1,399719	1,399719	-0,724459	0,724459	-0,19	0,19	0,02	-0,02
500	1000	1,12	1,43510	0,85610	-1,433461	1,433461	-0,856263	0,856263	-0,11	0,11	0,02	-0,02
500	1000	2,6	1,63100	1,13350	-1,632405	1,632407	-1,133304	1,133304	0,09	-0,09	-0,02	0,02

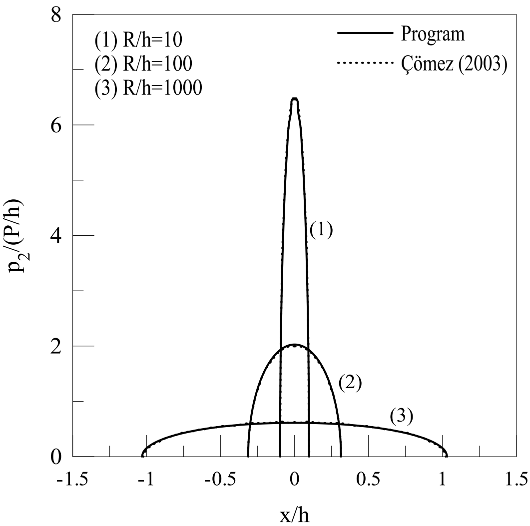

Figure 6. The comparison of the contact pressures between layer and the block for various R/h

$$\left(\frac{G_2}{P/h_2}=500,\ \kappa_1=\kappa_2=2,\ h_1/h_2=2,\ G_1/G_2=1\right)$$

5. CONCLUSION

The program was compared with two problems from the literature. The first problem had one layer with one integral equation, whereas second problem had two layers with two integral equations. The comparison of the tabulated results showed that the results obtained from the program for contact distances were very close to the literature and maximum %Diff values were less than or equal to 0,2. In addition, obtained graphs for contact pressures were almost overlapped from the existing studies. In other words, it can be said that the program produced correct results for different problems. As a result, it is concluded that the written program can be easily used to solve different receding contact problems and the general solution obtained for a layered medium resting on a rigid foundation can be a foundation for computational purposes.

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