Sigma J Eng & Nat Sci 9 (1), 2018, 133-141

Publications Prepared for the 13th Algebraic Hyperstructures and its Applications Conferences AHA 2017

sigma SPECIAL ISSUE

Research Article

VARIOUS KINDS OF QUOTIENT OF A CANONICAL HYPERGROUP

Hossein SHOJAEIJESHVAGHANI[1], Reza AMERI*[2]

[1]*School of Mathematics, University of Tehran, Tehran-IRAN*
[2]*School of Mathematics, University of Tehran, Tehran-IRAN*; ORCID:0000-0001-5760-1788

Received: 05.11.2017 Revised: 20.02.2018 Accepted: 11.06.2018

ABSTRACT

We introduce and study some equivalence relations on a canonical hypergroup to construct a quotient of such hyperstructures. In this regard, we study the relationships among these relations and obtain some conditions such that the extracted quotient structures are equal. Finally, the relationship between the heart of a canonical hypergroup and its quotient via an equivalence relation is studied and some related basic results are obtained.
Keywords: Hypergroup, Canonical hypergroup, Quotient, Heart.
MSC number: 20N20.

1. INTRODUCTION

The concept of hyperstructure, especially hypergroup, was introduced by Marty in 1934 [16]. Hyperstructures have many applications to other areas of various sciences. Many books and papers have been published related to the applications of hyperstructures in the fields of geometry, hypergraphs, binary relations, lattices, fuzzy sets and rough sets, automata, cryptography, combinatorics, codes, artificial intelligence, probabilistic, etc, for example, see [1-6, 8, 11, 17].

Canonical hypergroup as a special kind of hypergroups is indeed a natural generalization of the concept of abelian group. This kind of hypergroup is a basic addidive hyperstructure of many hyperstructures, e.g., Krasner hypermodules [19]. As it is well known, after introducing an algebraic structure, defining and studying its quotient by some substructure is a natural matter. In this regard we chiefly study the concept of quotient of a canonical hypergroup in detail.

This paper is organized as follows. In Section 2, we state some basic and fundamental concepts of hyperstructures theory. In Section 3, we study some relations on a canonical hypergroup, the related quotients and the relationship among them.

2. PRELIMINARIES

Here, we state some requirements. Let $P(X)$ denote the set of all subsets of X, $P(X)^* = P(X) \setminus \{\emptyset\}$ and let H be a non-empty set. Then H together with the map

* Corresponding Author: e-mail: rameri@ut.ac.ir, tel: +989122805056

$$\cdot : H \times H \longrightarrow P(H)^*$$
$$(a, b) \longmapsto a \cdot b$$

denoted by $(H,\cdot)$ is called a *hypergroupoid* and $\cdot$ is called a *hyperproduct* or *hyperoperation* on H. Let $A, B \subseteq H$. The hyperproduct $A \cdot B$ is defined as $A \cdot B = \cup_{(a,b)\in A\times B}\, a \cdot b$. If there is no confusion, then for simplicity $\{a\}$, $A \cdot \{b\}$ and $\{a\} \cdot B$ are denoted by a, $A \cdot b$ and $a \cdot B$, respectively. Also, we use ab instead of $a \cdot b$ for $a, b \in H$.

Definition 2.1. A non-empty set S together with the hyperoperation $\cdot$, denoted by $(H,\cdot)$ is called a *semihypergroup* if for all $x, y, z \in S$, $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.

Definition 2.2. A semihypergroup $(H,\cdot)$ satisfying $x \cdot H = H \cdot x = H$ for every $x \in H$, is called a *hypergroup*.

Let H be a semihypergroup and $\mathcal{U}(H)$ denote the set of all finite hyperproducts of elements of H. Let β be the relation $\cup_{n\geq 1}\beta_n$, where β_1 is the diagonal relation and for every integer $n > 1$, β_n is the relation:

$$x\beta_n y \Leftrightarrow \exists (x_1, x_2, \dots, x_n) \in H^n\!: \ x, y \in \prod_{i=1}^{n} x_i.$$

By $\hat{\beta}$ we mean the *transitive closure* of β.

Remark 2.3. For $x, y \in H$, $x\hat{\beta}y$ if and only if there exist $n \in \mathbb{N}$, $x_1, x_2, \dots, x_n \in H$ and $u_1, u_2, \dots, u_{n-1} \in \mathcal{U}(H)$ such that $x = x_1, x_n = y$ and $x_i, x_{i+1} \in u_i$ for $1 \leq i \leq n-1$, that is $x = x_1\beta x_2\beta x_3 \dots x_{n-1}\beta x_n = y$.

In other words, $x\hat{\beta}y$ if and only if there are $n \in \mathbb{N}$, $x_1, x_2, \dots, x_n \in H$ and $i_1, i_2, \dots, i_{n-1} \in \mathbb{N}$ such that $x = x_1\beta_{i_1}x_2\beta_{i_2}x_3 \dots x_{n-1}\beta_{i_{n-1}}x_n = y$.

The relation β was introduced by Koskas [15] and studied mainly by Corsini [7]. Referring to [10, 21], there is a relation denoted by β^* and called the *fundamental relation* of (semi)hypergroup H, as the smallest equivalence relation such that $(H/\beta^*,\otimes)$ is a (semi)group, where

$$\beta^*(x) \otimes \beta^*(y) = \beta^*(z) \ \ \forall x, y \in H, \ \forall z \in xy.$$

The quotient $(H/\beta^*,\otimes)$ is called the *fundamental (semi)group* of the (semi)hypergroup $(H,\cdot)$. It is shown that the fundamental relation of a hypergroup β^* is indeed $\hat{\beta}$, i.e., $\hat{\beta} = \beta^*$ (see [21]). As a very important result, Freni [14] proved β is transitive on hypergroups, i.e, $\beta = \hat{\beta}$.

Remark 2.4. For every two distinct hypergroups H and K with K⊆H, we use β_K^* and β_H^* to distinguish their fundamental relations. Note that $\beta_K^* \subseteq \beta_H^*$ for all $x \in K$.

Definition 2.5. [10, Definition 2.5.18] Let $(H,\cdot)$ is a hypergroup and consider the canonical projection $\varphi_H\!: H \longrightarrow H/\beta^*$ with $\varphi_H(x) = \beta^*(x)$. The heart of H is the set $\omega_H = \{x \in H \mid \varphi_H(x) = 1\}$, where 1 is the identity of the group H/β^*.

Definition 2.6. Let e be an element of the semihypergroup $(H, +)$ such that $e + x = x$ for all $x \in H$. Then e is called a left scalar identity.

Similarly, a right scalar identity is defined. An element x of the semihypergroup $(H, +)$ is called a scalar identity if it is a left and right scalar identity. Every scalar identity is unique. We denote the scalar identity of H by 0_H.

Let 0_H be the scalar identity of hypergroup $(H, +)$ and $x \in H$. An element $x' \in H$ is called an *inverse* of x in $(H, +)$ if $0_H \in x + x' \cap x' + x$.

A semihypergroup with a scalar identity is called a hypermonoid.

Definition 2.7. A non-empty set M together with the hyperoperation $+$ is called a *canonical hypergroup* if the following axioms hold:

1. $(M, +)$ is a semihypergroup (associativity);

2. $(M,+)$ is commutative (commutativity);
3. there is a scalar identity 0_M (existence of scalar identity);
4. for every $x \in M$, there is a unique element denoted by $-x$ called inverse of x such that $0_M \in x+(-x)$, which for simplicity, we write $0_M \in x-x$ (existence of inverse);
5. $\forall x,y,z \in M$: $x \in y+z \Longrightarrow y \in x-z$ (reversibility).

Sometimes, for simplicity, we use M instead of $(M,+)$.

Definition 2.8. A non-empty subset N of M is called a canonical subhypergroup of M, denoted by $N \leq M$ if it is a canonical hypergroup itself.

It is easy to verify that $N \leq M$ if and only if $N \neq \emptyset$ and $x-y \subseteq N$ for all $x,y \in N$. Clearly, always $0_M \in N$.

For more basic concepts and definitions about hyperstructures, we refer the reader to the books [7-10] and the papers [12-14, 18].

3. MAIN RESULTS

3.1. Structure of M/N

Here, we plan to study (the structure of) a quotient of a canonical hypergroup by some its canonical subhypergroup. Let $(M,+)$ be a canonical hypergroup, N be an arbitrary canonical subhypergroup of M and set $M/N := \{x+N \mid x \in M\}$. Consider the hyperaddition $+'$ on M/N defined as

$$(x+N)+'(y+N) = \{t+N \mid t \in x+y\}. \tag{3.1}$$

In order to be more ready and familiar with the hyperoperation $+'$ (or $-'$ later on) defined on M/N in the next subsections, first we intentionally study the structure of $(M/N,+')$ in detail.
In the sequel, $-x$ denotes the inverse of x in M and we write $x-y$ instead of $x+(-y)$. Also, for convenience, we use $\bar{x}$ instead of $x+N$.

Lemma 3.1. $\bar{x} \cap \overline{x'} \neq \emptyset$ implies $\bar{x} = \overline{x'}$.

Proof. Let $t \in \bar{x} \cap \overline{x'}$. Clearly $t \in x+n'$ and $t \in x'+n''$ for some $n', n'' \in N$. Since $x' \in t-n''$, we have $x' \in t-n'' \subseteq x+n'-n'' \subseteq x+N$. So $x'+N \subseteq x+N+N = x+N$ and thus $\bar{x} \subseteq \overline{x'}$. Similarly, $\overline{x'} \subseteq \bar{x}$. ■

We prove $(M/N,+')$ is a canonical hypergroup. Indeed,

Proposition 3.2. For every canonical hypergroup M, if $N \leq M$, then $(M/N,+')$ is a canonical hypergroup.

Proof. We first show that $+'$ is well defined, i.e., if $\overline{x_1} = \overline{x_2}$ and $\overline{y_1} = \overline{y_2}$, then $\overline{x_1} +' \overline{y_1} = \overline{x_2} +' \overline{y_2}$.

Let $\bar{z} \in \overline{x_1} +' \overline{y_1}$. Then there exists some $t \in x_1+y_1$ such that $\bar{z} = \bar{t}$ or $z+N = t+N$. So $z \in t+n$ for some $n \in \mathbb{N}$. Thus $z \in t+n \subseteq x_1+y_1+n$. On the other hand, $\overline{x_1} = \overline{x_2}$ and $\overline{y_1} = \overline{y_2}$ imply that $x_1 \in x_2+n_1$ and $y_1 \in y_2+n_2$ for some $n_1,n_2 \in \mathbb{N}$. Thus $z \in x_1+y_1+n \subseteq x_2+y_2+n_1+n_2+n$ from commutativity. So $z \in t'+n_3$ for some t'$\in x_2+y_2$ and some $n_3 \in \mathbb{N}$, and thus $z \in \overline{t'} \cap \bar{z}$. Hence from Lemma 3.1, $\overline{t'} = \bar{z}$ which $\overline{t'} \in \overline{x_2} +' \overline{y_2}$. Thus $\overline{x_1} +' \overline{y_1} \subseteq \overline{x_2} +' \overline{y_2}$. Similarly, $\overline{x_2} +' \overline{y_2} \subseteq \overline{x_1} +' \overline{y_1}$. Consequently, $\overline{x_1} +' \overline{y_1} = \overline{x_2} +' \overline{y_2}$.
Associative axiom:

$$\begin{aligned}(\bar{x} +' \bar{y}) +' \bar{z} &= \Big(\bigcup_{w\in x+y}\{\bar{w}\}\Big) +' \bar{z} \\ &= \Big(\bigcup_{w\in x+y}(\bar{w} +' \bar{z})\Big) = \bigcup_{\substack{w\in x+y\\ t\in w+z}}\{\bar{t}\} \\ &= \bigcup_{w\in (x+y)+z}\{\bar{t}\} = \bigcup_{w\in x+(y+z)}\{\bar{t}\} \\ &= \bigcup_{\substack{v\in y+z\\ u\in x+v}}\{\bar{t}\} = \Big(\bigcup_{v\in y+z}(\bar{x} +' \bar{v})\Big) \\ &= \bar{x} +' (\cup_{v\in y+z}\{\bar{v}\}) = \bar{x} +' (\bar{y} +' \bar{z})\end{aligned}$$

Commutativity axiom:

Clearly, $\bar{x} + '\bar{y} = \{\bar{t} |\ t \in x + y\} = \{\bar{t} |\ t \in y + x\} = \bar{y} + '\bar{x}$.

Existence of (the unique) scalar identity:

$\overline{0_M} = N$ is the identity of $(M/N, +')$. In fact,

$\overline{0_M} +' \bar{x} = \{\bar{t} \mid t \in 0_M + x\} = \{\bar{t} |\ t \in \{x\}\} = \{\bar{x}\}$. Similarly, $\bar{x} + '\overline{0\_M} = \{\bar{x}\}$.

Existence of inverse of an element:

$\bar{x}$ has an inverse and that is $\overline{-x}$. Indeed,

$$\bar{x} + '(\overline{-x}) = \{\ \bar{t} |\ t \in x + (-x)\} = \{\bar{t} |\ t \in x - x\} \ni \overline{0_M},$$

since $0_M \in x - x$. (Also $\overline{0_M} \in (\overline{-x}) + '\bar{x}$ from commutativity.) So $\overline{-x}$ is the inverse of $\bar{x}$. In order to show the uniqueness of inverse, let $\bar{y} \in M/N$ such that $\overline{0_M} \in \bar{x} + '\bar{y}$. Then $\overline{0_M} \in \{\bar{t} |\ t \in x + y\} = \{\bar{t} |\ t \in x + y\}$. So there exists some $z \in x + y$ such that $\overline{0_M} = \bar{z}$ or $N = z + N$. So $z = n$ for some $n \in \mathbb{N}$. From the reversibility axiom we have $y \in -x + n \subseteq -x + N$. Consequently $y \in -x + N$. Now since $y \in \bar{y} \cap \overline{-x}$, we get $\bar{y} = \overline{-x}$ by Lemma 3.1.

Sometimes we may use $0_{\mathrm{M/N}}$ and $-'\bar{x}$ to denote 0_M and $\overline{-x}$ (the inverse of $\bar{x}$), respectively. So we can speak of hyperoperation $-'$ on M/N and write $\bar{x} - '\bar{y}$ instead of $\bar{x} + '(-'\bar{y})$.

Reversibility axiom:

$$\begin{aligned}\bar{x} \in \bar{y} +' \bar{z} &\Longrightarrow \exists t \in y + z, \quad \bar{x} = \bar{t} \\ &\Longrightarrow y \in t - z, \quad \bar{x} = \bar{t} \\ &\Longrightarrow \bar{y} \in \bar{t} -' \bar{z},\ \bar{x} = \bar{t}\end{aligned}$$

So $\bar{y} \in \bar{x} + '(\overline{-z})$.

Hence $(M/N, +')$ is a canonical hypergroup.■

The following result states another way to present the hyperoperation $+'$ in (3.1).

Proposition 3.3. The hyperoperation $+'$ on M/N is the same as $+''$ defined by:

$$\bar{x} +'' \bar{y} := \{\bar{t} |\ t \in \bar{x} + \bar{y}\}.$$

Proof. Let $A = \{t + N |\ t \in\ x + y\}$ and $B = \{t + N |\ t \in x + N + y + N\}$. We show $A = B$. Let $t + N \in A$. Then clearly, $t \in x + y \subseteq x + N + y + N$ implies $t + N \in B$. So $A \subseteq B$. Conversely, suppose $t + N \in B$. So there exists $p \in x + y$ and $n \in N + N \subseteq N$ such that $t \in p + n$. Clearly, $t \in \bar{t} \cap \bar{p}$. Hence $\bar{t} = \bar{p}$ by Lemma 3.1. On the other hand, $p \in x + y$ implies $\bar{p} \in A$. So $\bar{t} \in A$. Consequently, $B \subseteq A$.■

3.2. Equivalent quotients to M/N

Let $(M,+)$ be a canonical hypergroup and $N \leq M$. In this section, we investigate some relations ρ on a canonical hypergroup M and conditions such that M/ρ is the same as M/N.

We begin with the following result:

Lemma 3.4. In any canonical hypergroup A for all $x, y \in A$, $x = y$ if and only if $0_A \in x - y$.

Proof. Let $x = y \in A$. Then we have $x - y = y - y \ni 0_A$. Conversely, let $0_A \in x - y = x + (-y)$ and thus $0_A \in (-y) + x$. On the other hand, it is clear that always $0_A \in y + (-y) = (-y) + y$. So x and y are inverses of $-y$. For the uniqueness of inverse (of $-y$), we have $x = y$. ■

Remark 3.5. Note that by reversibility axiom, we can also prove that $0_A \in x - y$ implies $x = y$. If we apply Lemma 3.4 for M/N, then we have

$$\begin{aligned} \bar{x} = \bar{y} &\Rightarrow N = 0_{\frac{M}{N}} \in \bar{x} -' \bar{y} = \bar{x} +' \overline{-y} \\ &\Rightarrow \exists t \in x - y, \quad 0_{M/N} = N = \bar{t} \\ &\Rightarrow \exists t \in x - y, \quad N = t + N \\ &\Rightarrow \exists t \in x - y, \quad t \in N \\ &\Rightarrow t \in x - y \cap N \neq \emptyset \end{aligned}$$

On the other hand,

$$\begin{aligned} t \in x - y \cap N \neq \emptyset &\Rightarrow t \in x + (-y), \quad t \in N \\ &\Rightarrow t + N \in \bar{x} +' (\overline{-y}) = \bar{x} -' \bar{y}, \quad t \in N \\ &\Rightarrow 0_{M/N} = N = t + N \in \bar{x} -' \bar{y} \end{aligned}$$

Hence from Lemma 3.4, we have $\bar{x} = \bar{y}$. Thus

Proposition 3.6. $\bar{x} = \bar{y}$ if and only if $x - y \cap N \neq \emptyset$.

Proposition 3.7. $\{N\} = \bar{x} -' \bar{y}$ if and only if $x - y \subseteq N$.

Proof.

$$\begin{aligned} \{N\} = \bar{x} -' \bar{y} &\Leftrightarrow \{N\} = \bar{x} +' \overline{-y} = \{t + N \mid t \in x + (-y)\} \\ &\Leftrightarrow N = t + N, \ \forall t \in x - y \\ &\Leftrightarrow \forall n \in N, \ \forall t \in x - y\colon \ t + n \subseteq N \\ &\Leftrightarrow \{t\} = t + 0_N \subseteq N, \ \forall t \in x - y \\ &\Leftrightarrow x - y \subseteq N. \blacksquare \end{aligned}$$

Lemma 3.8. Let N be a canonical subhypergroup of M such that $\bar{x} -' \bar{y}$ is a singleton. Then

$$\begin{aligned} x - y \cap N \neq \emptyset &\Leftrightarrow \bar{x} = \bar{y} \\ &\Leftrightarrow N \in \bar{x} -' \bar{y} \\ &\Leftrightarrow \bar{x} -' \bar{y} = \{N\} \\ &\Leftrightarrow x - y \subseteq N. \end{aligned}$$

Proof. From Proposition 3.6, $x - y \cap N \neq \emptyset$ if and only if $\bar{x} = \bar{y}$. By the assumption and applying Lemma 3.4 for $A = M/N$, we have $\bar{x} = \bar{y}$ if and only if $\{N\} = \bar{x} -' \bar{y}$. Finally, use Proposition 3.7.■

The next result is just for clarifying the likelihood of confusion may be caused by $+$ or $+'$ in the study of the quotient M/N.

Proposition 3.9. $\bar{x} + \bar{y} = N$ if and only if $x + y \subseteq N$, i.e.,

$$x + N + y + N = N \Leftrightarrow x + y \subseteq N.$$

Proof. From commutativity and since $N + N = N$, we have

$$x + N + y + N = N \Longrightarrow x + y + N = N.$$

So for every $t \in x + y$, we have $t + N \subseteq N$. Thus for every $t \in x + y$, and for all $n \in \mathbb{N}$, $t + n \subseteq N$. In particular $\{t\} = t + 0_N \subseteq N$ and consequently, $x + y \subseteq N$. Conversely, suppose $x + y \subseteq N$. So for every $t \in x + y$, from the reproductivity axiom $t + N = N$. Thus $x + y + N = N$. Now from commutativity and since $N + N = N$ we have $N = x + y + N = x + y + N + N = x + N + y + N$.■

Now let M be a canonical hypergroup and N be a canonical subhypergroup of M. Consider the following relations on M:

for all $x, y \in M$,

$$x\rho_1 y \Leftrightarrow x + N = y + N$$
$$x\rho_2 y \Leftrightarrow x \in y + N$$
$$x\rho_3 y \Leftrightarrow x - y \subseteq N$$
$$x\rho_4 y \Leftrightarrow x - y \cap N \neq \emptyset.$$

In the sequel, our aim is to study the relationship among these relations.

Proposition 3.10. Let N be a canonical subhypergroup of M. Then

1. $x\rho_1 y \Leftrightarrow x\rho_2 y$,
2. $x\rho_2 y \Leftrightarrow x\rho_4 y$.

Proof. 1. Let $\rho_1 y$, i.e., $\bar{x} = \bar{y}$. So $x + N = y + N$ implies that $x \in y + N$. Thus $x\rho_2 y$. Conversely, let $x\rho_2 y$, i.e, $x \in \bar{y}$. Since $x \in \bar{x} \cap \bar{y}$, we have $\bar{x} = \bar{y}$ from Lemma 3.1. Thus the result is true.

2. Let $\rho_2 y$. So $x \in y + N$ and thus $x \in y + n$ for some $n \in \mathbb{N}$. The reversibility axiom implies $n \in -y + x \cap N \neq \emptyset$. From the commutativity axiom $x - y \cap N \neq \emptyset$. Thus $x\rho_4 y$. If $x\rho_4 y$, then $x - y \cap N \neq \emptyset$ and thus there exists some $n \in x - y \cap N$. So $x \in n + y$ or $x \in y + n$ by the commutativity axiom. Therefore $x \in y + N$. This implies $x\rho_2 y$. Hence $x\rho_2 y$ if and only if $x\rho_4 y$.■

Remark 3.11. Clearly, ρ_1 is an equivalence relation on M.

Theorem 3.12. Let N be a canonical subhypergroup of M. Then $\rho_1 = \rho_2 = \rho_4$ as equivalence relations and $\rho_1(x) = \rho_2(x) = \rho_4$ (x)=x+N.

Proof. From Proposition 3.10 and Remark 3.11, $\rho_1 = \rho_2 = \rho_4$. The latter statement is followed from ρ_1 (x)=x+N.■

In this section, we give a condition that implies $\rho_1 = \rho_2 = \rho_3 = \rho_4$ and then obtain some results.

Lemma 3.13. Let N be a canonical subhypergroup of M such that $\bar{x} -' \bar{y}$ is a singleton. Then

$$x\rho_4 y \Leftrightarrow x\rho_1 y \Leftrightarrow x\rho_3 y.$$

Proof. According to Lemma 3.8, it is clear.■

Proposition 3.14. Let N be a canonical subhypergroup of M such that $\bar{x} -' \bar{y}$ is a singleton for all $x, y \in M$. Then $\rho_1 = \rho_2 = \rho_3 = \rho_4$.

Proof. From Proposition 3.10 and Lemma 3.13, $\rho_1 = \rho_2 = \rho_3 = \rho_4$.■

Corollary 3.15. Let N be a canonical subhypergroup of M such that $(M/N, +')$ is a (commutative) group. Then $\rho_1 = \rho_2 = \rho_3 = \rho_4$. as equivalence relations.

Proof. From Proposition 3.14, the result is clear.■

Proposition 3.16. Let N be a canonical subhypergroup of M such that $(M/N, +')$ is a (commutative) group. Then for every $x \in M$,

$$\beta^*(x) \subseteq \rho_1(x) = \rho_2(x) = \rho_3(x) = \rho_4(\mathrm{x}).$$

Proof. Since β^* is the smallest equivalence such that M/β^* is a group, the result is clear.■

In the sequel, we need another definition of heart as follows:

Definition 3.17. [20, Definition 2.5] The heart of a canonical hypergroup M is

$$\omega_M = \{t \in \sum_{i=1}^{n}(x_i - x_i) \mid n \in \mathbb{N},\ x_i \in M\}.$$

Remark 3.18. By definition of $\otimes$ and since $x + 0_M = x = 0_M + x$, it is easily seen that $\beta^*(x) \otimes \beta^*(0_M) = \beta^*(x) = \beta^*(0_M) \otimes \beta^*(x)$. So $\beta^*(0_M)$ is the identity of the group M/β^*. By Definition 2.5,

$$\omega_M = \{x \in M \mid \varphi_M(x) = \beta^*(0_M)\} = \{x \in M \mid \beta^*(x) = \beta^*(0_M)\}.$$

So $\omega_M = \beta^*(0_M)$.

Proposition 3.19. Let N be a canonical subhypergroup of M such that $\bar{x} -' \bar{x}$ is a singleton for each $x \in M$. Then $\omega_M \subseteq N$.

Proof. Since $0_M \in x - x \cap N$, from Lemma 3.8, we have $x - x \subseteq N$. So $\omega_M \subseteq N$.■

When M/N is a (commutative) group, we have

$$\omega_M = \beta^*(0_M) \subseteq \rho_1(0_M) = \rho_2(0_M) = \rho_3(0_M) = \rho_4(0_M) = 0_M + N,$$

i.e., $\omega_M \subseteq N$ (see Proposition 3.19).

Now let $\mathcal{S}$ denote the set of all canonical subhypergroups N of M such that $(M/N, +')$ is a (commutative) group. Then we have the following result:

Corollary 3.20. For every canonical hypergroup M, $\omega_M \subseteq \cap_{N \in \mathcal{S}} N$.

In the following, for completeness, we state a direct proof of being an equivalence relation for ρ_2.

Proposition 3.21. Let N be a canonical subhypergroup of M. Then ρ_2 is an equivalence relation on M.

Proof. Let $x \in M$. Since $\{x\} = x + 0_M \subseteq x + N$. So the relation ρ_2 is reflexive. Let $x, y \in M$. If $x \in y + N$, then $x \in y + n$ for some $n \in \mathbb{N}$. That is, $y \in x - n \subseteq x + N$. So, ρ_2 is a symmetric relation. Suppose that $x, y, z \in M$ such that $x\rho_2 y$ and $y\rho_2 z$, then $x \in y + N$ and $y \in z + N$. Therefore, $x \in$

$y + n$, and $y \in z + n'$, for some $n, n' \in \mathbb{N}$. So, $x \in y + n \subseteq (z + n) + n' = z + (n + n') \subseteq z + N$. Hence $x\rho_2 z$. Therefore, the relation ρ_2 is transitive.

3.3. Quotient by normal canonical subhypergroup

A canonical subhypergroup N of M is said to be *normal* if for all $x \in M$, $x + N - x \subseteq N$.

Proposition 3.22. If N is a normal canonical subhypergroup of M, then

$$x\rho_1 y \Leftrightarrow x\rho_2 y \Leftrightarrow x\rho_3 y \Leftrightarrow x\rho_4 y.$$

Proof.

$$\begin{gathered} x\rho_1 y \Longrightarrow \bar{x} = \bar{y} \Longrightarrow x \in \bar{y} \Longrightarrow x\rho_2 y \\ \Longrightarrow x \in y + N \Longrightarrow x - y \subseteq y + N - y \subseteq N \Longrightarrow x\rho_3 y \\ \Longrightarrow x - y \cap N \Longrightarrow x\rho_4 y \\ \Longrightarrow \exists t \in x - y\colon\ t + N = N \Longrightarrow \bar{t} = N = 0_{\frac{M}{N}} \in \bar{x} -' \bar{y} \\ \Longrightarrow \bar{x} = \bar{y} \Longrightarrow x\rho_1 y. \end{gathered}$$

So

$$x\rho_1 y \Longrightarrow x\rho_2 y \Longrightarrow x\rho_3 y \Longrightarrow x\rho_4 y \Longrightarrow x\rho_1 y$$

Now by Theorem 3.12, the result is followed.■

According to Proposition 3.22, ρ_3 is an equivalence relation for every canonical subhypergroup N of M, since ρ_1 is an equivalence relation. Moreover,

Theorem 3.23. If N is a normal canonical subhypergroup. Then $\rho_1 = \rho_2 = \rho_3 = \rho_4$ as equivalence relations.

Proof. Since ρ_1 is an equivalence relation, from Proposition 3.22, the result is followed.■
Although we proved ρ_3 is an equivalence relation by ρ_1 when N is normal, we can prove this fact independent of ρ_1 as follows:

Proposition 3.24. ρ_3 is an equivalence relation if N is a normal canonical subhypergroup of M.

Proof. Clearly, $x \in x + 0_M$ implies $x - x \subseteq x + 0_M - x \subseteq x + N - x \subseteq N$. So ρ_3 is reflexive.

Also, $x - y \subseteq N$ if and only if $y - x \subseteq N$. So ρ_3 is symmetric. For transitivity, let $x - y \subseteq N$ and $y - z \subseteq N$. So by normality of N, we have

$$x - y + y - z = x - y + 0_M + y - z \subseteq x + N + -z \subseteq x - z + N \subseteq N.■$$

Remark 3.25. According to the proof of Proposition 3.22, if N is a normal canonical subhypergroup, then $\bar{x} = \bar{y}$ if and only if $\bar{x} -' \bar{y} = \{\bar{t} | \, t \in x - y\} = \{N\} = \{0_{M/N}\}$, i.e., $\bar{x} -' \bar{y}$ is the singleton $\{0_{M/N}\}$.

Lemma 3.26. Let N be a normal canonical subhypergroup of a canonical hypergroup M. Then $\omega_M \subseteq N$.

Proof. Clearly for every $x \in M$, $0_M \in x - x \cap N \neq \emptyset$. So by the proof of Proposition 3.22, $x - x \subseteq N$. Since N is a canonical subhypergroup of M, we have $\sum_{i=1}^{n}(x_i - x_i) \subseteq N$ for all $n \in \mathbb{N}$ (which $x_i \in M$). Thus $\omega_M \subseteq N$.■

Let $\mathcal{N}$ denote the set of all normal canonical subhypergroups of M. Then

Theorem 3.27. For every canonical hypergroup M, $\omega_M \subseteq \cap_{N \in \mathcal{S}} N \subseteq \cap_{N \in \mathcal{N}} N$.

Proof. According to Remark 3.25, $\mathcal{S} \supseteq \mathcal{N}$. Hence the result is followed by Corollary 3.20.■

Corollary 3.28. Let M be a canonical hypergroup. If $N = \{0_M\}$ is a normal canonical subhypergroup of M, then $\omega_M = \{0_M\}$.

Proof. From Lemma 3.26, it is clear.■

We say a canonical hypergroup M has the trivial fundamental group if $M = \omega_M$ (see [18]).

Proposition 3.29. Let M be a canonical hypergroup with the trivial fundamental (commutative) group. Then the canonical subhypergroup $\{0_M\}$ is normal if and only if $M = \{0_M\}$.

Proof. According to Lemma 3.26, since $M = \omega_M$, normality of $\{0_M\}$ implies $M = \{0_M\}$. The converse is clear.■

Acknowledgement

The authors are extremely grateful to the anonymous referees for their valuable comments to improve the paper.

REFERENCES

[1] Ameri R., Amiri-Bideshki M., Hoskova-Mayerova S. and Saeid A. B., (2017) Distributive and Dual Distributive Elements in Hyperlattices, Analele Stiintifice ale Univ. Ovidius Constanta, Ser. Matematica 25(3), 25-36.

[2] Ameri R., Amiri-Bideshki, M., Saeid A. B. and Hoskova-Mayerova S., (2016) Prime filters of hyperlattices, An. Stiint. Univ. Ovidius Constanta Ser. Mat. 24(2), 15-26.

[3] Ameri R., Kordi A. and Hoskova-Mayerova S., (2017) Multiplicative hyperring of fractions and coprime hyperideals, An. Stiint. Univ. Ovidius Constanta Ser. Mat. 25(1), 5-23.
[4] Chvalina J. and Hoskova-Mayerova S., (2014) On certain proximities and preorderings on the transposition hypergroups of linear first-order partial differential operators, An. Stiint. Univ. Ovidius Constanta Ser. Mat. 22(1), 85-103.
[5] Chvalina J., Hoskova-Mayerova S. and Deghan Nezhad, A., (2013) General actions of hypergroups and some applications, An. Stiint. Univ. Ovidius Constanta Ser. Mat. 21(1), 59-82.
[6] Cristea I. and Hoskova-Mayerova S., (2009) Fuzzy topological hypergroupoids, Iranian Journal of Fuzzy Systems 6(4), 11-19.
[7] Corsini P., (1993) Prolegomena of Hypergroup Theory. Second edition, Aviani editore, Tricesimo.
[8] Corsini P. and Leoreanu, V., (2003) Applications of Hyperstructures Theory. Advanced in Mathematics, Kluwer Academic Publisher, Dordrecht.
[9] Davvaz B., (2013) Polygroup Theory and Related Systems, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ.
[10] Davvaz B. and Leoreanu, V., (2007) Hyperring Theory and Applications. International Academic Press, USA.
[11] Hoskova-Mayerova S., (2012) Topological hypergroupoids, Computers and Mathematics with Applications 64(9), 2845-2849.
[12] Freni D., (2002) A new characterization of the derived hypergroup via strongly regular equivalences, Comm. Algebra 30, 3977-3989.
[13] Freni D., (2004) Strongly transitive geometric spaces: applications to hypergroups and semigroups theory, Comm. Algebra 32, 969-988.
[14] Freni D., (1991) Une note sur le cur d'un hypergroupe et sur la cloture transitive β^* de β, Riv. Mat. Pura Appl. 8, 153-156.
[15] Koskas M., (1970) Groupoides, demi-hypergroupes et hypergroupes, J. Math. Pures Appl. 49, 155-192.
[16] Marty F., (1934) Sur une generalization de la notion de groupe, 8th Congress Math. Scandenaves, Stockholm, 45-49.
[17] Saeid A. B., Flaut C., Hoskova-Mayerova S., Afshar M., Cristea R. L. and Rafsanjani Kuchaki M., (2018) Some connections between BCK-algebras and n-ary block codes. Soft Computing 22, 41–46.
[18] Shojaei H. and Ameri R., (2015) On hypergroups with trivial fundamental group, 46th Annual Iranian Mathematics Conference (AIMC46), 25-28 August 2015, Yazd University, Yazd, Iran, 238-241.
[19] Shojaei H. and Ameri R., (2016) Some results on categories of Krasner Hypermodules, Journal of Fundamental and Applied Sciences 8(3S), 2298-2306.
[20] Velrajan M. and Arjunan A., (2010) Note on isomorphism theorems of hyperrings, Int. J. Math. And Math. Sci., Article ID 376985.
[21] Vougiouklis T., (1994) Hyperstructures and their Representations, Hadronic Press, Inc., Palm Harber, USA.