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Research Article

ON MULTIPLIERS OF HYPER BCC-ALGEBRAS

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ABSTRACT

In this paper, we introduced the notion of multiplier of a hyper BCC-algebra, and investigated some properties of hyper BCC- algebras. And then we introduced notion of kernels and notion of hyper normal ideal of multipliers on hyper BCC-algebras. Also we gave some propositions related with isotone and $Fix_d(H)$.
Keywords: hyper BCC-algebra, multiplier, isotone, $Fix_d(H)$, hyper normal ideal, regular**.**
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1. INTRODUCTION

The study of BCK-algebras was initiated (1966) by Y. Imai and K. Iseki[4] as a generalization of the concept of set-theoretic difference and propositinal calculus. In 1984, was introduced a notion of BCC-algebra which is a generalization of a BCK-algebra by Y. Komori [9].

The derivation of BCC-algebra was introduced C. Prabpayak and U. Lerrawat [1]. In [6] a partial multiplier on a commutative semigroup $(A,.)$ has been introduced as a function F from a nonvoid subset D_F of A into A such that $F(x).y = x.F(y)$ for all x,y∈ D_F.

The hyperstructure theory(called also multialgebras) was introduced (1934) by

F. Marty [2] and hyper BCK-algebras were studied by many authors and were given some related properties. Also hyper BCC-algebras were studied which was a generalization of BCC-algebras and were investigated different types of hyper BCC-ideals and were defined the relationship among them by (2006) R.A. Borzooei. [5]

The notion of multiplier of a BCC-algebra was introduced and some properties of BCC-algebras were invastigated (2013) by K.H.Kim.[3]

In this study, we introduce the notion of multipler of a hyper BCC-algebra and discuss some properties of hyper BCC-algebras. Also we characterize kernel of multipliers on hyper BCC-algebras. Finally we introduced notion of hyper normal ideal of multipliers on hyper BCC-algebras.

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2. PRELIMINARIES

Definition 2.1 [9] *An algebra* $(X, *, 0)$ *of type* $(2,0)$ *is said to be a BCC-algebra if it satisfies the following:*
for all $x, y, z \in X$,

1. $((x * y) * (z * y)) * (x * z) = 0$,
2. $x * 0 = x$,
3. $x * x = 0$,
4. $0 * x = 0$,
5. $x * y = 0$ and $y * x = 0$ imply $x = y$.

Definition 2.2 *[7] By a hyper BCK-algebra, it is meant a nonempty set* H *endowed with a hyper operation* $'' \circ ''$ *and a constant* $''0''$ *satisfying the following axioms:*

1. $(xoz)o(yoz) << xoy$,
2. $(xoy)oz = (xoz)oy$,
3. $xoH << x$,
4. $x << y$ and $y << x$ imply $x = y$,

for all $x, y, z \in H$, where $x << y$ is defined by $0 \in xoy$ and for every $A, B \subseteq H$, $A << B$ is defined by for all $a \in A$, there exists $b \in B$ such that $a << b$. In such case, $'' << ''$ is called the hyperorder in H.

Definition 2.3 *[5] By a hyper BCC-algebra, it is meant a nonempty set* H *endowed with a hyper operation* $'' \circ ''$ *and a constant* 0 *satisfying the following axioms:*

1. $(x \circ z) \circ (y \circ z) << x \circ y$,
2. $0 \circ x = 0$,
3. $x \circ 0 = x$,
4. $x << y$ and $y << x$ imply $x = y$,

for all $x, y, z \in X$.

Theorem 2.4 *[5] Any hyper BCK-algebra is a hyper BCC-algebra.*

Proposition 2.5 *[5] Let* $(H, \circ, 0)$ *be a hyper BCC-algebra.Then for all* $x, y, z \in H$ *and* $A \subseteq H$ *the following conditions hold:*

1. $0 \circ 0 = 0$,
2. $0 << x$,
3. $x << x$,
4. $x \circ y << x$,
5. $A \circ 0 = A$,
6. $0 \circ A = 0$,
7. $x \circ y = 0$ implies $x \circ z << y \circ z$.

Theorem 2.6 *[5] Let* $(H, \circ, 0)$ *be a hyper BCC-algebra. Then* $(H, \circ, 0)$ *is a hyper BCK-algebra if and only if* $(x \circ y) \circ z = (x \circ z) \circ y$ *is satisfied for all* $x, y, z \in H$.

Definition 2.7 *[8]* $(H, \circ, 0)$ *be a hyper BCC-algebra and* I *be a subset of* H *such that* $0 \in I$ *is said to be the following:*

(i) a hyper BCC-ideal of type1, if

$$(x \circ y) \circ z << I, y \in I \Rightarrow x \circ z \subseteq I,$$

(ii) a hyper BCC-ideal of type2, if

$$(x \circ y) \circ z \subseteq I, y \in I \Rightarrow x \circ z \subseteq I,$$

(iii) a hyper BCC-ideal of type3, if

$$(x \circ y) \circ z << I, y \in I \Rightarrow x \circ z << I,$$

(iv) a hyper BCC-ideal of type4, if

$$(x \circ y) \circ z \subseteq I, y \in I \Rightarrow x \circ z << I,$$

Definition 2.8 *[3] Let $(X,*,0)$ be a BCC-algebra and a map $f: X \to X$ is said to be a multiplier if* $f(x * y) = f(x) * y$ for all $x, y \in X$.

3. ON MULTIPLIERS OF HYPER BCC-ALGEBRAS

Definition 3.1 *Let $(H,\circ,0)$ be a hyper BCC-algebra. A map $d: H \to H$ is said to be a multiplier if for all $x, y \in H$* $d(x \circ y) = d(x) \circ y$.

Example 3.1 *Let $H = \{0, a, b\}$ and $(H,\circ,0)$ be a hyper BCC-algebra with Cayley table as follows:*

Table 1.

∘	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{b}	{0,b}

Define a map $d: H \to H$

$$d_1(x) = \begin{cases} a, & x = a \\ 0, & x = 0, b \end{cases}$$

Then it is easily checked that d_1 is a multiplier of hyper BCC-algebra.

Proposition 3.2 *Let d be a multiplier of H. Then it satisfies $d(x \circ d(x)) << 0$ for all $x \in H$.*

Proof 3.1 *Let $x \in H$. Since d is multiplier, we have* $d(x \circ d(x)) = d(x) \circ d(x)$

From Prop.2.5$(iii), (v)$, we find that $0 \in d(x) \circ d(x)$
Therefore we get $d(x) \circ d(x) << 0$
Thus we obtain $d(x \circ d(x)) << 0$.

Definition 3.3 *Let $(H,\circ,0)$ be a hyper BCC-algebra. A self-map d of H is said to be regular if* $d(0) = 0$.

Example 3.2 *d given in $Ex.$ 3.1. is regular.*

Proposition 3.4 *Let $(H,\circ,0)$ be a hyper BCC-algebra and a map $d: H \to H$ is a regular multiplier of H. Then the following hold for all $x, y \in H$: $d(x) << x$,*

Proof 3.2 *Let $x \in H$ and d be a regular multiplier. Then we find that*

$$0 = d(0) \in d(x \circ x) = d(x) \circ x.$$

Hence we can write $0 \in d(x) \circ x$ for all $x \in H$ and we have $d(x) << x$.

Definition 3.5 *Let $(H,\circ,0)$ be a hyper BCC-algebra and a map $d: H \to H$. If $x << y$ imply $d(x) << d(y)$ for all $x, y \in H$, then d is said to be isotone.*

Proposition 3.6 *Let $(H,\circ,0)$ be a hyper BCC-algebra and d be a regular multiplier of H. If $d: H \to H$ is an endomorphism, then d is isotone.*

Proof 3.3 *Let $x, y \in H$ and $x << y$. Then we find $0 \in x \circ y$ and*

$$0 = d(0) \in d(x \circ y) = d(x) \circ d(y).$$

Hence we get $d(x) << d(y)$.

Definition 3.7 *Let* $(H,\circ,0)$ *be a hyper BCC-algebra and two maps* $d_1, d_2: H \to H$. *Then a map* $d_1 \bullet d_2: H \to H$ *is defined by* $(d_1 \bullet d_2)(x) = d_1(d_2(x))$ for all $x \in H$.

Proposition 3.8 *Let* $(H,\circ,0)$ *be a hyper BCC-algebra and two maps* $d_1, d_2: H \to H$ *are multipliers of* H. *Then* $d_1 \bullet d_2$ *is a multiplier of* H.

Proof 3.4 *We find*

$$\begin{aligned}(d_1 \bullet d_2)(a \circ b) &= d_1(d_2(a \circ b)) \\ &= d_1(d_2(a) \circ b) \\ &= d_1(d_2(a)) \circ b \\ &= (d_1 \bullet d_2)(a) \circ b\end{aligned}$$

for all $x \in H$.

Definition 3.9 *Let* $(H_1,\circ,0)$ *and* $(H_2,\circ,0)$ *be two hyper BCC-algebras.Then* $H_1 \times H_2$ *is also a hyper BCC-algebra with respect to the point-wise operation given by*

$$(a,b) \circ (c,d) = (a \circ_1 c, b \circ_2 d)$$

for all $a, c \in H_1$ and $b, d \in H_2$.

Proposition 3.10 *Let* $(H_1,\circ_1,0)$ *and* $(H_2,\circ_2,0)$ *be two hyper BCC-algebras.Define a map* $d: H_1 \times H_2 \to H_1 \times H_2$ *by* $d(x,y) = (x,0)$ *for all* $(x,y) \in H_1 \times H_2$. *Then* d *is a multiplier of* $H_1 \times H_2$ *with respect to the point-wise operation.*

Proof 3.5 *Let* $(x_1,y_1), (x_2,y_2) \in H_1 \times H_2$. *Then we find*

$$\begin{aligned}d((x_1,y_1) \circ (x_2,y_2)) &= d(x_1 \circ_1 x_2, y_1 \circ_2 y_2) \\ &= (x_1 \circ_1 x_2, 0) \\ &= (x_1 \circ_1 x_2, 0 \circ_2 y_2) \\ &= (x_1, 0) \circ (x_2, y_2) \\ &= d(x_1,y_1) \circ (x_2,y_2)\end{aligned}$$

Hence d is a multiplier of the direct product $H_1 \times H_2$.

Definition 3.11 *Let* $(H,\circ,0)$ *be a hyper BCC-algebra,* d *be a multiplier of* H, $A = \{x \in H | d(x) = x\}$ *and* $d(A_i) = A_i$ *for*$A_i \subseteq H$. *Then a set* $Fix_d(H)$ *is defined by* $Fix_d(H) := A \cup (\bigcup_{i \in I} A_i)$ *for all* $i \in I$. $Fix_d(H) := \{x \in H | d(x) = x\}$.

Proposition 3.12 *Let* $(H,\circ,0)$ *be a hyper BCC-algebra and* d *be a multiplier of* H. *If* $x \in Fix_d(H)$ *then* $(d \circ d)(x) = x$.

Proof 3.6 *Let* $x \in Fix_d(H)$. *Then we have*

$$\begin{aligned}(d \circ d)(x) &= d(d(x)) \\ &= d(x) \\ &= x\end{aligned}$$

Proposition 3.13 *Let* $(H,\circ,0)$ *be a hyper BCC-algebra and* d *be a multiplier of* H.*If* $x \in A$, $y \in H$ *then* $x \circ y \subset Fix_d(H)$

Proof 3.7 *Let* $x \in A$. *Then we have* $d(x) = x$. *Therefore we find*

$$d(x \circ y) = d(x) \circ y = x \circ y$$

Then $x \circ y = A_k$ for some $k \in I$.
Therefore we get $x \circ y \subset \bigcup_{i \in I} A_i$. Hence we have $x \circ y \subset Fix_d(H)$.

Proposition 3.14 *Let* $(H,\circ,0)$ *be a hyper BCC-algebra and* d *be a multiplier of* H. *If* $x \in H$ *and* $y \in A$ *then* $x \wedge y = y \circ (y \circ x) \subset Fix_d(H)$.

Proof 3.8 *Let $y \in A$. We have*

$$\begin{aligned} d(x \wedge y) &= d\big(y \circ (y \circ x)\big) \\ &= d(y) \circ (y \circ x) \\ &= y \circ (y \circ x) \\ &= x \wedge y. \end{aligned}$$

Thus $x \wedge y \subset Fix_d(H)$.

Proposition 3.15 *Let $(H,\circ,0)$ be a hyper BCC-algebra and d be a multiplier of H. If $x \in H$ and $y \in A$ then it satisfies $d(x \circ y) = d(x) \circ d(y)$.*

Proof 3.9 *Let $x \in H$ and $y \in Fix_d(H)$. We have*

$$\begin{aligned} d(x \circ y) &= d(x) \circ y \\ &= d(x) \circ d(y) \end{aligned}$$

Definition 3.16 *Let $(H,\circ,0)$ be a hyper BCC-algebra and d be a multiplier of H. We can define a set $Ker_d(H) = K \cup Z$ by*

$$K = \{x \in H | d(x) = 0\} \text{ and } d(Z) = 0 \text{ for } Z \subset H.$$

Proposition 3.17 *Let $(H,\circ,0)$ be a hyper BCC-algebra and d be a multiplier of H. If $x, y \in K$ then $x \circ y \subset Ker_d(H)$.*

Proof 3.10 *Let $x, y \in K$. We get*

$$\begin{aligned} d(x \circ y) &= d(x) \circ y \\ &= 0 \circ y \\ &= \{0\}. \end{aligned}$$

Thus we can write $x \circ y \subseteq Ker_d(H)$.

Definition 3.18 *Let $(H,\circ,0)$ be a hyper BCC-algebra and a non-empty set I of H is said to be hyper normal ideal if it satisfies the following:*

(i) $0 \in I$,
(ii) $x \in I$ and $y \in H$ imply $x \circ$ y $\subseteq$ I.

Theorem 3.19 *Let $(H,\circ,0)$ be a hyper BCC-algebra and d be a regular multiplier of H. Then the following hold:*

(i) *$Fix_d(H)$ is a hyper normal ideal of H.*
(ii) *$Im(d)$ is a hyper normal ideal of H.*

Proof 3.11 (i) *$d(0) = 0$ so we have $0 \in Fix_d(H)$.*

Let $x \in H$ and $a \in Fix_d(H)$. Then $d(a) = a$.
Therefore we get $d(a \circ x) = d(a) \circ x = a \circ x$.

We find $a \circ x \subseteq Fix_d(H)$. Hence $Fix_d(H)$ is a hyper normal ideal of H.

(ii) *d is regular multiplier so $d(0) = 0$. Let $x \in H$ and $a \in Im(d)$. Then*

$$a = d(b) \text{ for some } b \in H.$$

Therefore we can write $a \circ x = d(b) \circ x = d(b \circ x) \subseteq Im(d)$.
Hence $Im(d)$ is a hyper normal ideal of H.

Example 3.3

For the multiplier given in Ex.3.1. and $I = \{0, b\} \subseteq H$. Then H is easily checked that I is a hyper normal ideal of H.

Theorem 3.20 *Let* $(H,\circ,0)$ *be a hyper BCC-algebra and d be a regular multiplier of* H, I be a hyper normal ideal of H. Then $d(I)$ is a hyper normal ideal of H.

Proof 3.12. Let $x \in H$ and $a \in d(I)$. Then $a = d(b)$ for some $b \in I$.

Therefore $a \circ x = d(b) \circ x = d(b \circ x)$. We get that $d(b \circ x) \subseteq d(I)$.
Hence $a \circ x \subseteq d(I)$. Then $d(I)$ is a hyper normal ideal of H.

Theorem 3.21 *Let* $(H,\circ,0)$ *be a hyper BCC-algebra and d be a regular multiplier of H. Then* $Kerd,\ H$ is a hyper normal ideal of H.

Proof 3.13. Let $0 \in Kerd$ and $a \in Kerd$, $x \in H$. Then we get

$$d(a \circ x) = d(a) \circ x = 0 \circ x = \{0\}$$

Hence $a \circ x \subseteq Kerd$. Therefore $Kerd$ is a hyper normal ideal of H.

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