Sigma J Eng & Nat Sci 9 (1), 2018, 101-106

Publications Prepared for the 13th Algebraic Hyperstructures and its Applications Conferences AHA 2017

sigma SPECIAL ISSUE

**Research Article**

# AN INTRODUCTION TO ZERO-DIVISOR GRAPHS OF A COMMUTATIVE MULTIPLICATIVE HYPERRING

**Zahra SOLTANI[1], Reza AMERI*[2], Yahya TALEBI-ROSTAMI[3]**

[1]*Department of Mathematics, University of Mazandaran, Babolsar-IRAN*
[2]*School of Mathematics, University of Tehran, Tehran-IRAN;* ORCID:0000-0001-5760-1788
[3]*Department of Mathematics, University of Mazandaran, Babolsar-IRAN*; ORCID:0000-0003-2311-4628

**Received: 01.12.2017 Accepted: 16.04.2018**

**ABSTRACT**

The purpose of this paper is the study of zero-divisor graphs of a commutative multiplicative hyperrings, as a generalization of commutative rings. In this regards we consider a commutative multiplicative hyperring $(R,+,o)$, where $(R,+)$ is an abelian group, $(R,+)$ is a semihypergroup and for all $a,b,c \in R,: a\ o\ (b+c) \subseteq a\ o\ b + a\ o\ c$ and $(a+b)\ o\ c \subseteq a\ o\ b + a\ o\ c$. For $a \in R$ a nonzero element $a \in R$ is said to be a zero-divisor of $a$, if $0 \in a\ o\ b$ and the set of zero-divisors of $R$ is denoted by $Z(R)$. We associative to $R$ a zero-divisor graph $\Gamma(R)$, whose vertices of $\Gamma(R)$ are the elements of $Z(R)^*(= \mathrm{Z(R)}\backslash\{0\})$ and two distinct vertices of $\Gamma(R)$ are adjacent if they were in $Z(R)$. Finally, we obtain some properties of $\Gamma(R)$ and compare some of its properties to the zero-divisor graph of a classical commutative ring and show that almost all properties of zero-divisor graphs of a commutative ring can be extend to $\Gamma(R)$ while $R$ is a strongly distributive multiplicative hyperring.
**Keywords:** Multiplicative hyperring, zero-divisor graph, strongly distributive**.**

## 1. INTRODUCTION

The concept of the zero-divisor graph of a ring was raised by I. Beck when discussing the coloring of a commutative ring in [3] for the first time. Later D. F. Anderson and P. S. Livingston introduced the zero-divisor graph of a unitary commutative ring $R$, denoted by $\Gamma(R)$ in [2]. They considered the set of nonzero zero-divisor of as a vertice of $\Gamma(R)$ and assumed that two distinct vertices $x$ and $y$ are adjacent if and only if $xy = 0$. Subsequently, they proved that if $R$ is a finite ring, then $\Gamma(R)$ is finite and connected and any two vertices can be joined by less than four edges. In particular, they were determined when $\Gamma(R)$ is a complete graph and a star graph.

In this paper we create a connection between the concept of the zero-divisor graph of commutative rings and commutative multiplicative hyperrings and generalize some results and properties of zero-divisor graph of a commutative ring to the strongly distributive multiplicative hyperrings.

In this section we will list some definitions, notions and results about commutative hyperrings from some references.

* Corresponding Author: e-mail: rameri@ut.ac.ir, tel: +989122805056

**Definition 1.1.** Let $H$ be a nonempt y set and $P^*(H)$ denotes the set of all of nonempty subsets of $H$. A hyperoperation $o$ on $H$ is a m apping $o : H \times H \rightarrow P^*(H)$. A nonempty set $H$ together with a family of hyperoperartion is a hyperstructure. A hyperstructure $(H, o)$ is a semihypergroup if for all $a, b, c \in H, (a\ o\ b)\ o\ c = a\ o\ (b\ o\ c)$. (Associativity axiom). A hyperstructure $(H, o)$ is a quasihypergroup if for all $a \in H$, we have $a\ o\ H = H = H\ o\ a$. In the other words for all $a, b, c \in H$ there exist $x, y \in H$ such that $a \in x\ o\ b \cap b\ o\ y$ (Reproduction axiom).

**Definition 1.2.** A hyperstructure $(H, o)$ which is the both semihypergroup and quasihypergroup is called a hypergroup.

**Definition 1.3.** A general hyperring is an algebraic hyperstructure $(R, +, o)$ that satisfies the following axioms:

(1) $(R, +)$ is a hypergroup.
(2) $(R, o)$ is a semihypergroup.
(3) For all $a, b, c \in R, a\ o\ (b + c) = a\ o\ b + a\ o\ c$ and $(a + b)\ o\ c = a\ o\ c + b\ o\ c$.

A hyperring $(R, +, o)$ is commutative, if the both hyperoperations $+$ and $o$ are commutative.The hyperring $R$ is unitary if there exists an element $u \in R$ such that for all $a \in R, a\ o\ u = u\ o\ a = \{a\}$.

**Definition 1.4.** The unitary commutative hyperring $R$ is a hyperfield if for every non-zero element $a \in R$, there exists $b \in R$ such that $u \in a\ o\ b$. where $u$ is an unit element of $R$.

**Definition 1.5.** A commutative hyperring $R$ is a strong hyperdomain if for all $a, b \in R$, if $0 \in a\ o\ b$ with $a \neq 0 (or\ b \neq 0)$, then $b = 0 (or\ a = 0)$. If $a\ o\ b = \{0\}$ implies $a = 0$ or $b = 0$, we will talk about hyperdomain. Obviously, every strong hyperdomain is a hyperdomain and every hyperfield is a strong hyperdomain.

**Definition 1.6.** A nonempty subset $A$ of a hyperring $(R, +, o)$ is subhyperring of $R$ if $(A, +, o)$ is itself a hyperring, under the restriction of hyperoperation $+$ and $o$ to $A$.

**Definition 1.7.** Let $A$ is a subhyperring of a hyperring $R$. We say that $A$ is a left (right) hyperideal of $R$ if for all $r \in R$ and $a \in A$, $r\ o\ a \in A (a\ o\ r \in A)$. $A$ is called a hyperideal if $A$ is both a left and a right hyperideal. A hyperideal $P$ of a commutative hyperring $R$ is said to be prime if $P \neq R$ and for all $a, b \in R$, $a\ o\ b \subseteq P$ implies $a \in P$ or $b \in P$. A hyperideal $P$ of $R$ is said to be strong prime if $a\ o\ b \cap P \neq 0$ implies $a \in P$ or $b \in P$.

**Definition 1.8.** A triple $(R, +, o)$ is multiplicative if $+$ be a classical commutative operation and $o$ be a hyperoperation and following statements hold:

(1) $(R, +)$ is an abelian group.
(2) $(R, o)$ is a semihypergroup.
(3) For all $a, b, c \in R$, $a\ o\ (b + c) \subseteq a\ o\ b + a\ o\ c$ and $(a + b)\ o\ c \subseteq a\ o\ c + b\ o\ c$.
(4) For all $a, b \in R$, $a\ o\ (-b) = (-a)\ o\ b = -(a\ o\ b)$.

If in (3) equality hold, then $R$ is a strongly distributive multiplicative hyperring (briefly, we say that $R$ is a $SDMH$).

**Definition 1.9.** A nonempty subset $S$ of a commutative multiplicative hyperring $(R, +, o)$ is a subhyperring of $R$ if $(S, +, o)$ is a multiplicative hyperring. In other words, $S$ is a subhyperring of $R$ if $(S, +)$ is a subgroup of $(R, +)$ (i.e., $S - S \subseteq S$) and for all $r, s \in S$, $r\ o\ s \subseteq S$.

**Definition 1.10.** A nonempty subset $I$ of a multiplicative hyperring $(R, +, o)$ is a hyperideal if following axioms hold:

(1) $(I, +)$ is a subgroup of $(R, +)$.
(2) $(I\ o\ R) \cup (R\ o\ I) \subseteq I$.

By this definition clearly, every hyperideal is a subhyperring.

Let $(R, +, o)$ be a multiplicative hyperring and $I$ is a hyperideal of $R$. Let $R/I$ be the set of all cosets of $R$ with restrict to $I$, $R/I = \{a + I \mid a \in R\}$ . We define a hyperoperation $*$ on $R/I$ by

$$(a + I) * (b + I) = \{c + I \mid c \in a\ o\ b\}.$$

Then $(R/I\,, +,*)$ is a multiplicative hyperring, moreover if $R$ is a $SDMH$, so is $R/I$.

**Theorem 1.11.** A strongly distributive hyperring $(R, +, o)$ is a ring if and only if there exists $a, b \in R$, such that $|a\ o\ b| = 1$.

**Proof.** Corollary 4.1.6 [5]. □

**Theorem 1.12.** If $I$ is a hyperideal of a commutative multiplicative hyperring $(R, +, o)$, then for every element $a + I \in R/I$, we have $|\,(a + I) * (0 + I)\,| = 1$. In other words, if $R$ is a $SDMH$, then $R/I$ is a ring.

**Proof.** According to Theorem 4.3.5 [5] and Theorem 1.11. □

**Theorem 1.13.** Let $(R, +, o)$ is a $SDMH$, then for all $a, b \in R$, we have:

(1) $0 \in a\ o\ 0$ and $0 \in 0\ o\ a$.
(2) For all $x, y \in a\ o\ 0$, $x - y \in a\ o\ 0$. (i.e., $a\ o\ 0$ is a subgroup of $R$.)
(3) $a\ o\ b$ is a cosets of $0\ o\ 0$.
(4) $0\ o\ 0\ o\ 0 = 0\ o\ 0$.
(5) For all $s \in 0\ o\ 0$ and $r \in R$, $s\ o\ r = 0\ o\ 0$.
(6) If $0 \in a\ o\ b$ then $a\ o\ b = 0\ o\ 0$.

**Proof.** (1) $0\ o\ a = (a - a) o\ a = a\ o\ a - a\ o\ a$. Since $0 \in a\ o\ a - a\ o\ a$, then $0 \in 0\ o\ a$ and similarly $0 \in a\ o\ 0$.

(2) $a\ o\ 0 = a\ o\ (0 - 0) = a\ o\ 0 - a\ o\ 0$. Then for all $x, y \in a\ o\ 0$, $x - y \in a\ o\ 0$.

(3) Let $c \in a\ o\ b$. For all $x \in a\ o\ b$, we have $x - c \in a\ o\ b - a\ o\ b = a\ o\ (b - b) = a\ o\ 0$.

This means that $x + a\ o\ 0 = c + a\ o\ 0$. Thus $a\ o\ b = a\ o\ (b + 0) = a\ o\ b + a\ o\ 0 = \cup_{x \in a\ o\ b}\ x + a\ o\ 0 = c + a\ o\ 0$.Similarly, $a\ o\ b$ is a coset of $a\ o\ b$. Since $a\ o\ 0$ and $0\ o\ b$ are cosets of $0\ o\ 0$, therefore $a\ o\ b$ is a coset of $0\ o\ 0$.

(4) $0\ o\ 0\ o\ 0 = 0\ o\ (0\ o\ 0) = \cup_{a \in 0\ o\ 0}\ 0\ o\ a = \cup_{a \in 0\ o\ 0}\ 0\ o\ 0 = 0\ o\ 0$.

(5) Suppose $s \in 0\ o\ 0$ and $r \in R$, then $s\ o\ r \subseteq 0\ o\ 0\ o\ r = 0\ o\ (0\ o\ r) = 0\ o\ 0$. Since $s\ o\ r$ is a coset of $0\ o\ 0$ then $s\ o\ r = 0\ o\ 0$.

(6) Suppose $0 \in a\ o\ b$, then for $c \in a\ o\ b$,we have $0 \in c + 0\ o\ 0$. Thus there exists $m \in 0\ o\ 0$ such that $0 = c + m$. It follow that $c \in 0\ o\ 0$. Thus $a\ o\ b \subseteq 0\ o\ 0$, and Since $a\ o\ b$ is a coset of $0\ o\ 0$, therefore $a\ o\ b = 0\ o\ 0$. □

**Corollary 1.14.** We denote $0\ o\ 0$ by $\Omega$. then by Theorem 1.12 clearly if $R$ is a $SDMH$, $\Omega$ is a hyperideal of $R$. Moreover, $R/\Omega$ is a ring.

## 2. THE ZERO-DIVISOR GRAPH OF A SDMH WHEN $\boldsymbol{Z(R)^* \cap \Omega = \emptyset}$

In this section, we investigate zero-divisor graph of a strongly distributive multiplicative hyperring and compare their properties with zero-divisor graph of a classical commutative ring.

Let $(R, +, o)$ be a commutative multiplicative hyperring. An element $0 \neq b$ of $R$ is said to be a zero-divisor of $a \in R$, if $0 \in a\ o\ b$. The set of zero-divisors of $R$ denote by $Z(R)$. The zero-divisor graph of $R$ is a graph with elements of $Z(R)^* = Z(R)\backslash\{0\}$ as vertices and two distinct vertices $a, b$ are adjacent if and only if $0 \in a\ o\ b$. This graph denote by $\Gamma(R)$. By definition 1.5, $R$ is a strong hyperdomain if and only if $Z(R) = \{0\}$, and if $R$ is a strong hyperdomain then $\Gamma(R) = \emptyset$. An element $0 \neq a$ of $R$ is regular if $a \notin Z(R)$. The set of regular elements of $R$ denote by $Reg(R)$.

The zero-divisor graph $\Gamma(R)$ is connected if there exists a path between any two distinct vertices. $\Gamma(R)$ is a complete graph if any two distinct vertices of $\Gamma(R)$ are adjacent. $\Gamma(R)$ is a star graph if there exists an unique vertex of $\Gamma(R)$, which is adjacent to every other vertex.

Let $d(a,b)$ be the length of the shortest path from $a$ to $b$ in $\Gamma(R)$. The diameter of $\Gamma(R)$ is denoted by $diam(\Gamma(R))$, is equal to $sup\{d(a,b) \mid a,b\ are\ distinct\ vertices\ of\ \Gamma(R)\}$. The girth of $\Gamma(R)$ is denoted by $gr((R))$, is defined as the length of the shortest cycle in $\Gamma(R)$. ($d(a,b)=\infty$ if there is no such path and $gr(\Gamma(R))=\infty$ if $\Gamma(R)$ contains no cycles).

In the following statements we will generalize some Theorems and results about zero-divisor graph of a commutative ring that were obtained by D. F. Anderson and P. S. Livingstone in [2].

**Theorem 2.1.** Let $(R,+,o)$ be a $SDMH$. Then $\Gamma(R)$ is finite if and only if either $R$ is finite or a strong hyperdomain. In particular, if $1 \le |\Gamma(R)| < \infty$, then $R$ is finite and not a hyperfield.

**Proof.** Suppose that $\Gamma(R)(=Z(R)^*)$ is finite and nonempty. Then there are nonzero $a,b \in R$ such that $0 \in a\ o\ b$. Let $A=\{r \in R \mid 0 \in a\ o\ r\}$. Then $A \subseteq Z(R)$ is finite and for all $r \in R$, $0\ o\ r \subseteq (a\ o\ b)\ o\ r = a\ o\ (b\ o\ r)$. Since $0 \in 0\ o\ r$, therefore $b\ o\ r \subseteq A$. Let $R$ be infinite. Since $A$ is finite, then there are $a_1, a_2, \dots, a_n \in A$ such that $B=\{r \in R \mid b\ o\ r \subseteq \{a_1, a_2, \dots, a_n\}\}$ is infinite. So for all $r,s \in B$, $0 \in b\ o\ (r-s)$. If $C=\{r \in R \mid 0 \in b\ o\ r\}$, then $C \subseteq Z(R)$ is infinite, that is a contradiction. Thus $R$ must be finite. Convers is obviously. □

In this part for determining the zero-divisor graph, we suppose that $(R,+,o)$ is a SDMH, $\Omega = 0\ o\ 0$ and $Z(R)^* \cap \Omega = \emptyset$. According to Corollary 1.14, $R/\Omega$ is a ring. We denoted $R/\Omega$ by $\bar{R}$ and the element $a+\Omega$ of $R/\Omega$ by $\bar{a}$. Here, we state a useful theorem that helps us to determine zero-divisor graph and their properties for a $SDMH$.

**Theorem 2.2.** If $R$ is a $SDMH$ and $Z(R)^* \cap \Omega = \emptyset$. There exists an one-to-one correspondence between the set of zero-divisors of $R$ and the set of zero-divisors of ring $\bar{R}$.

**Proof**. If $a \in Z(R)$, there exists $0 \ne b \in R$ such that $0 \in a\ o\ b$. According to Theorem 1.13(6), $a\ o\ b = \Omega$. Since $Z(R)^* \cap \Omega = \emptyset$, then $\bar{a}, \bar{b} \in \bar{R}$ are nonzero and $\bar{a}\bar{b} = a\ o\ b + \Omega = \Omega$. Therefore $\bar{a} \in Z(\bar{R})$. Conversely, suppose $\bar{a} \in Z(\bar{R})$. There exists $\bar{0} \ne \bar{b} \in \bar{R}$ such that $\bar{a}\bar{b} = (a+\Omega)\ o\ (b+\Omega) = \Omega$. It means that $a\ o\ b + \Omega = \Omega$ and hence $a\ o\ b = \Omega$. Since $0 \in \Omega$, hence $0 \in a\ o\ b$. Then we have $a \in Z(R)$. This complete the proof. □

This results immediately follow from Theorem 2.2:

**Corollary 2.3.** If $R$ is a $SDMH$ and $Z(R)^* \cap \Omega = \emptyset$. There exists an one-to-one correspondence between the set of $Reg(R)$ and the set of $Reg(\bar{R})$.

**Corollary 2.4.** Let $R$ is a $SDMH$ and $Z(R)^* \cap \Omega = \emptyset$. Then $\Gamma(R)$ is isomorphic to $\Gamma(\bar{R})$. In other words, $\bar{a}$ and $\bar{b}$ are adjacent in $\Gamma(\bar{R})$ if and only if $a$ and $b$ are adjacent in $\Gamma(R)$. Hence $\Gamma(\bar{R})$ is connected if and only if $\Gamma(R)$ is so.

**Corollary 2.5.** As another proof of Theorem 2.1, if $R$ is a $SDMH$ and $Z(R)^* \cap \Omega = \emptyset$, $\Gamma(\bar{R})$ is finite if and only if $\Gamma(R)$ is so. According to Theorem 2.2 [2], $\Gamma(\bar{R})$ is finite if and only if $\bar{R}$ is finite or a domain. Also $\bar{R}$ is finite if and only if $R$ is finite. Moreover, since $Z(R)^* \cap \Omega = \emptyset$, $\bar{R}$ is a domain if and only if $R$ is a strong hyperdomain.

**Theorem 2.6.** Let $R$ is a $SDMH$ and $Z(R)^* \cap \Omega = \emptyset$. Then $\Gamma(R)$ is connected and $diam(\Gamma(R)) \le 3$. Moreover, if $\Gamma(R)$ contain a cycle, then $gr(\Gamma(R)) \le 7$.

**Proof.** According to Theorem 2.3 [2], $\Gamma(\bar{R})$ is connected and $diam(\Gamma(\bar{R})) \le 3$, furthermore, if $\Gamma(\bar{R})$ contain a cycle, then $gr(\Gamma(\bar{R})) \le 2diam(\Gamma(\bar{R})) + 1$. Therefore, according to Theorem 2.2, $\Gamma(R)$ is so. □

**Theorem 2.7.** Let $R$ is a finite $SDMH$ and $Z(R)^* \cap \Omega = \emptyset$. If $\Gamma(R)$ contains a cycle then $gr(\Gamma(R)) \le 4$.

**Proof.** Since $R$ is finite if and only if $\bar{R} = R/\Omega$ is finite and $\Gamma(R) \cong \Gamma(\bar{R})$, if $\Gamma(R)$ contains a cycle then $\Gamma(\bar{R})$ is so. By Theorem 2.4 [2], $gr(\Gamma(R)) \leq 4$. □

**Definition 2.8.** A hyperideal $I$ is an annihilator hyperideal if and only if for all $a \in I$ and for all $r \in R$, $0 \in r\ o\ a$ or $0 \in a\ o\ r$.

**Theorem 2.9.** Let $R$ is a $SDMH$ and $Z(R)^* \cap \Omega = \emptyset$. There exists a vertex of $\Gamma(R)$ which is adjacent to every other vertex if and only if either $R/\Omega \cong Z_2 \times A$, where $A$ is an integral domain, or $Z(R)$ is an annihilator hyperideal.

**Proof.** If $\Gamma(R)$ contains a vertex which is adjacent with other vertices, then $\Gamma(\bar{R})$ is so. By Theorem 2.5 [4], we have $\bar{R} = R/\Omega \cong Z_2 \times A$, where A is an integral domain, or $Z(R/\Omega)$ is an annihilator ideal. If $Z(R/\Omega)$ is an annihilator ideal then for all $\bar{a} \in Z(R/\Omega)$ and for all $\bar{r} \in R/\Omega$, $\bar{a}\bar{r} = a\ o\ r + \Omega = \Omega$, Since $Z(R)^* \cap \Omega = \emptyset$, then $a\ o\ r = \Omega$. Since $0 \in \Omega$, then $0 \in a\ o\ r$. Therefore $Z(R)$ is an annihilator hyperideal. □

**Theorem 2.10.** Let $R$ is a $SDMH$ and $Z(R)^* \cap \Omega = \emptyset$. Then $\Gamma(R)$ is a complete graph if and only if $\bar{R} \cong Z_2 \times Z_2$ or $x\ o\ y = \Omega$ for all $x, y \in Z(R)^*$.

**Proof.** Let $\Gamma(R)$ is a complete graph then $\Gamma(\bar{R})$ is so. According to theorem 2.6 [2], $\Gamma(\bar{R})$ is complete graph if and only if $\bar{R} \cong Z_2 \times Z_2$ or $\bar{x}\bar{y} = \Omega$, for all $\bar{x}, \bar{y} \in Z(\bar{R})^*$. If $\bar{x}\bar{y} = \Omega$, according to theorem 2.2, for all $x, y \in Z(R)^*$, $0 \in x\ o\ y$. Then $x\ o\ y = \Omega$. Converse is obviously. □

**Corollary 2.11.** Let $R$ is a $SDMH$ and $Z(R)^* \cap \Omega = \emptyset$. For $x, y \in Z(R)$, define $x \sim y$ if $0 \in x\ o\ y$ or $x = y$. Then relation $\sim$ is an equivalence relation if and only if $\Gamma(R)$ is a complete graph.

## 3. THE ZERO-DIVISOR GRAPH OF A SDMH WHEN $\boldsymbol{Z(R)^* \cap \Omega \neq \emptyset}$

In this section, we suppose that $R$ is a $SDMH$ and $Z(R)^* \cap \Omega \neq \emptyset$. According to Theorem 1.13, for every $a \in Z(R)^* \cap \Omega$, all of elements of $R$ are adjacent to $a$. In this case, $\Gamma(R)$ is connected. But $\Gamma(R)$ and $\Gamma(\bar{R})$ are not isomorphic necessarily.

In the following example we prove that if $R$ is a $SDMH$ and $Z(R)^* \cap \Omega \neq \emptyset$, $\Gamma(R)$ is not isomorphic to $\Gamma(\bar{R})$.

**Example 3.1.** Let $(R, +, .)$ is a ring and $\emptyset \neq P$ be a prime ideal of ring. We define $a\ o_P\ b = ab + P$, for $a, b \in R$. Obviously $(R, +, o_P)$ is a $SDMH$ and $\Omega = 0\ o_P\ 0 = P$. According to Corollary 1.14, $\bar{R} = R/P = \{r + P \mid r \in R\}$ is a ring. Let $a, b \in \Gamma(R)$, are adjacent. Then $0 \in a\ o_P\ b$. Hence $a\ o_P\ b = ab + P = P$ and $ab \in P$. Since $P$ is a prime ideal of $R$, $a \in P$ or $b \in P$. Therefore $\bar{a} \notin Z(\bar{R})^*$ or $\bar{b} \notin Z(\bar{R})^*$.

**Theorem 3.2.** Let $R$ is a $SDMH$ and $Z(R)^* \cap \Omega \neq \emptyset$. Then $\Gamma(R)$ is connected and $diam(\Gamma(R)) \leq 2$. Moreover, if $\Gamma(R)$ contains a cycle, then $gr(\Gamma(R)) \leq 5$.

**Proof.** If $Z(R)^* \cap \Omega \neq \emptyset$, then by theorem 1.13, for all $a \in Z(R)^* \cap \Omega$, and for all $b \in R$, $a\ o\ b = \Omega$. Since $0 \in a\ o\ b$, Then $a$ is adjacent to all of elements of $R$, and $\Gamma(R)(= R^*)$ is connected and $d(a, b) = 1$. Now, we suppose that $a, b \in Z(R)^* \setminus \Omega$. If $0 \in a\ o\ b$, obviously $\Gamma(R)$ is connected and $d(a, b) = 1$. Otherwise, there exist $x \in Z(R)^* \cap \Omega$ such that $0 \in a\ o\ x$ and $0 \in x\ o\ b$. Then $a - x - b$ is a path of length 2 and consequently $\Gamma(R)$ is connected and $diam(\Gamma(\bar{R})) \leq 2$. □

**Theorem 3.3.** Let $R$ is a $SDMH$ and $Z(R)^* \cap \Omega \neq \emptyset$. If $\Gamma(R)$ contains a cycle, then $gr(\Gamma(R)) \leq 3$.

**Proof.** If $\Gamma(R)$ contains a cycle, then there exist $a, b \in Z(R)^* \setminus \Omega$ such that $0 \in a\ o\ b$. On the other hand, for all $x \in Z(R)^* \cap \Omega$,we have $0 \in a\ o\ x$ and $0 \in x\ o\ b$. Then $a - x - b - a$ is a triangle. □

By Theorem 3.3, if $Z(R)^* \cap \Omega \neq \emptyset$, we have seen that $\Gamma(R)$ can be a triangle. But $\Gamma(R)$ cannot be an *n-gon* for any $n \geq 4$.

**Theorem 3.4.** Let $R$ is a $SDMH$ and $Z(R)^* \cap \Omega \neq \emptyset$. Then there is always at least one vertex of $\Gamma(\,R)$ which is adjacent to every other vertex.

**Proof.** According to Theorem 1.13(5). □

**Theorem 3.5.** Let $R$ is a $SDMH$ and $Z(R)^* \cap \Omega \neq \emptyset$. Then $\Gamma(R)$ is complete if and only if for all $x, y \in Z(R)^* \setminus \Omega$, $x\ o\ y = \Omega$.

**Proof.** The proof is obviously. □

**Corollary 3.6.** Let $R$ is a $SDMH$ and $Z(R)^* \cap \Omega = \emptyset$. For $x, y \in Z(R)$, define $x \sim y$ if $0 \in x\ o\ y$ or $x = y$. Then relation $\sim$ is an equivalence relation if and only if $\Gamma(R)$ is a complete graph.

## REFERENCES


[1] D.F. Anderson, A. Frazier, A. Lauve, P.S. Livingston, "The zero-divisor graph of a commutative ring II": *Lecture Notes in Pure and Appl. Math.,* vol. 220, Marcel Dekker, New York, pp. 61-72, 2001.

[2] D.F. Anderson, P.S. Livingston, "The zero-divisor graph of a commutative ring", *J. Algebra*, 217, 434-447, 1999.

[3] I. Beck, "Coloring of commutative rings", *J. Algebra* 116, 208-226, 1988.

[4] P. Corsini, V. Leoreanu, "Applications of hyperstructure theory", *Kluwer academic publications* 2003.

[5] B. Davvaz, V. Leoreanu, "Hyperrings theory and application", *International Academic Press*, 2007.

[6] T. Vougiouklis, "Hyperstructures and their representations", *Hardonic*, press, Inc. 1994.