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ON THE RELATION BETWEEN CATEGORIES OF (*m*, *n*)-ARY HYPERMODULES AND (*m*, *n*)-ARY MODULES

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ABSTRACT

We introduce the category of $R_{(m, n)}$-hypermodules over a Krasner *(m, n)*-hyperring *R* and obtain some categorical objects in this category such as product and coproduct. We apply the fundamental relations ε^* and Γ^* on, *M* and, *R* respectively to construct fundamental functor from the category of $R_{(m, n)}$-hypermodules into category of R/Γ^*-modules. In particular we consider the fundamental relation on *(m, n)*-hypermodules, and construct functor from the category of *(m, n)*-hypermodules to the category of *(m, n)*-modules. Then, we find the relations between hom, product, coproduct and fundamental functor.
Keywords: Category, *(m, n)*-hypermodules, product, coproduct, additive category**.**

1. INTRODUCTION

The concept of hypergroup was introduced by F. Marty in [11]. Afterward, because of many applications of this theory in both pure and applied sciences, many authors study in this context. Some review of the hyperstructure theory can be found in [1, 5, 6, 15], respectively.

In 1928, Dörnte introduced the concept of n-ary groups [8] and since then, n-ary systemes have been studied in different contexts.The research about n-ary hyperstructure was initiated by Davvaz and Vougiouklis who introduced these structures in [7].The notation of (m,n)-ary hyperring was defined by Mirvakili and Davvaz in [12]. After that Anvariyeh et al in [3] defined the notion of (m,n)-hypermodules over (m,n)-ary hyperrings. Ameri and Norouzi introduced in [2] the concept of n-ary prime and n-ary primary hyperideales in Krasner (m,n)-hyperring and proved some result in this respect.

Category theory [4] is the mathematical study of (abstract) algebras of functions. just as group theory is the abstraction of the idea of a system of permutations of a set or symmetries of a geometric object, category theory arises from the idea of a system of functions among some objects. In 1945, Eilenberg and Mac Lane's "General theory of natural equivalences" was the original paper, in which the theory was first formulated. Also, R. Ameri in [1] introduced and studied the categories of hypergroups and hypermodules.

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In this paper we introduce and study the various types of categories of (m,n)-ary hypermodules, based on various types of homomorphism. Finally, we use the fundamental relation to constrict the fundamental functor from the categories of $R_{(m,n)}$-ary hypermodules into category of R/Γ^*-modules. In particular we prove that the fundamental functor is not faithful, but it is additive.

2. PRELIMINARIES

In this section we give some definitions and results of n-ary hyperstructures which we need in what follows.

A mapping $f : \underbrace{H \times \cdots \times H}_{n} \to P^*(H)$ is called an n-ary *hyperoperation,* where $P^*(H)$ is the set of all non-empty subsets of H. An algebraic system (H, f), where f is an n-ary hyperoperation defined on H, is called an n*-ary hypergroupoid.*

We shall use the following abbreviated notation:

The sequence $x_i, x_{i+1}, \ldots, x_j$ will be denoted by x_i^j. For $j < i, x_i^j$ is the empty set. Using this notation,

$$f(x_1, \ldots, x_i, y_{i+1}, \ldots, y_j, z_{j+1}, \ldots, z_n)$$

will be written as $f(x_1^i, y_{i+1}^j, z_{j+1}^n)$. In the case when $y_{i+1} = \ldots = y_j = y$ the last expression will be written $f(x_1^i, y_{(j-i)}, z_{j+1}^n)$.

If f is an n-ary hyperoperation and $t = l(n-1)+1$, for some $l \geq 0$, then t-ary hyperoperation f_l is given by

$$f_l(x_1^{l(n-1)+1}) = \underbrace{f(f(\ldots, f(f}_{l}(x_1^n), x_{n+1}^{2n-1}), \ldots,), x_{(l-1)(n-1)+1}^{l(n-1)+1}).$$

For non-empty subsets $A_1, A_2, \ldots, A_n$ of H we define

$$f(A_1^n) = f(A_1, A_2, \ldots, A_n) = \bigcup \{ f(x_1^n) \mid x_i \in A_i, i = 1,2, \ldots, n \}.$$

An n-ary hyperoperation f is called *associative* if

$$f(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n-1}) = f(x_1^{j-1}, f(x_j^{n+j-1}), x_{n+j}^{2n-1}),$$

hold for every $1 \leq i < j \leq n$ and all $x_1, \ldots, x_{n-1} \in H.$ An n-ary hypergroupoid whit the associative n-ary hyperoperation is called an n*-ary semihypergroup*. Let $e \in H,$ such that for every $x \in H,$ $f(x, \underbrace{e, \ldots, e}_{(n-1)}) = x.$ Then e is called scaler identity. A semihypergroup containing the scaler identity is called an n*-ary hypermonoid.*

An n-ary hypergroupoid (H, f) in which the equation $b \in f(a_1^{i-1}, x_i, a_{i+1}^n)$ has a solution, $x_i \in H$ for every $a_1^{i-1}, a_{i+1}^n, b \in H$ and $1 \leq i \leq n$, is called an n*-ary quasihypergroup*. If (H, f) is an n-ary semihypergroup and n-ary quasihypergroup, then (H, f) is called an n*-ary hypergroup*. An n-ary hypergroupoid (H, f) is *commutative* if for all $\sigma \in \mathbb{S}_n$ and for every $a_1^n \in H$ we have $f(a_1, \ldots, a_n) = f(a_{\sigma(1)}, \ldots, a_{\sigma(n)})$. If $a_1^n \in H$ then we denote $(a_{\sigma(1)}, \ldots, a_{\sigma(n)})$ by $a_{\sigma(1)}\sigma(n)$.

Definition 2.1 *[10] Let (H, f) be a commutative n-ary hypergroup. (H, f) is called canonical n-ary hypergroup if*

- there exists unique $e \in H$, such that for every $x \in H$, $f(x, \underbrace{e, \ldots, e}_{(n-1)}) = x$;
- for all $x \in H$ there exists unique $x^{-1} \in H$, such that $e \in f(x, x^{-1} \underbrace{e, \ldots, e}_{(n-2)})$;
- if $x \in f(x_1^n)$, then for all i, we have $x_i \in f(x, x^{-1}, \ldots, x_{i-1}^{-1}, x_{i+1}^{-1}, \ldots, x_n^{-1})$.

we say that e is the scaler identity of (H, f) and x^{-1} is the inverse of x. Notice the inverse of e is e.

Definition 2.2 *[13] A Krasner (m,n)-hyperring is algebraic hyperstructure (R,h,k) which satisfies the following axioms:*

- (R,h) is a canonical m-ary hypergroup;
- (R,k) is an n-ary semigroup;
- the n-ary operation k is distributive to the m-ary hyperoperation h, i.e, for all $a_1^{i-1}, a_{i+1}^n, x_1^m \in R$, and $1 \leq i \leq n$,

$$k(a_1^{i-1}, h(x_1^m), a_{i+1}^n) = h(k(a_1^{i-1}, x_1, a_{i+1}^n), \ldots, k(a_1^{i-1}, x_m, a_{i+1}^n));$$

- 0 is a zero element (absorbing element), of the n-ary operation k, i.e., for $x_2^n \in R$ we have

$$k(0, x_2^n) = k(x_2, 0, x_3^n) = \ldots = k(x_2^n, 0).$$

Definition 2.3 *[3] Let M be a nonempty set. Then (M, f, g) is an (m,n)-hypermodule over an (m,n)-hyperring (R,h,k), if (M, f) is an m-ary hypergroup and the map*

$$g : \underbrace{R \times \ldots \times R}_{n-1} \times M \to P^*(M)$$

satisfies the following conditions:

- $g(r_1^{n-1}, f(x_1^m)) = f(g(r_1^{n-1}, x_1), \ldots, g(r_1^{n-1}, x_m))$;

- $g(r_1^{i-1}, h(s_1^m), r_{i+1}^{n-1}, x) = f(g(r_1^{i-1}, s_1, r_{i+1}^{n-1}, x), \ldots, g(r_1^{i-1}, s_m, r_{i+1}^{n-1}, x);$
- $g(r_1^{i-1}, k(r_i^{i+n-1}), r_{i+m}^{n+m-2}, x) = g(r_1^{n-1}, g(r_m^{n+m-2}, x));$
- $0 \in g(r_1^{i-1}, 0, r_{i+1}^{n-1}, x).$

Definition 2.4 *[3] A Krasner (m,n)-hypermodule (M, f, g) is an (m,n)-hypermodule with a canonical m-ary hypergroup (M, f) over a Krasner (m,n)-hyperring (R,h,k).*

A Krasner (m,n)-hyperring (R,h,k) is *commutative* if (R,k) is a commutative n-ary semigroup. Also, we say that (R,h,k) is with a *scaler identity* if there exists an element 1_R such that $x = k(x, 1_R^{(n-1)})$ for all $x \in R.$ Later on, let (R,h,k) be a commutative Krasner (m,n)-hyperring whit a scaler identity $1_R.$ For all $r_1^{n-1} \in R$ and $x \in M$ we have

$$g(r_1^{n-1}, 0_M) = \{0_M\},\ g(0_R^{n-1}, x) = \{0_M\} \text{ and } g(1_R^{n-1}, x) = \{x\}.$$

Moreover, let $g(r_1^{i-1}, -r_i, r_{i+1}^{n-1}, x) = -g(r_1, \ldots, r_{n-1}, x) = g(r_1^{n-1}, -x).$

Definition 2.5 *[3] Let (M_1, f_1, g_1) and (M_2, f_2, g_2) be two (m,n)-hypermodules over an (m,n)-hyperring (R,h,k). we say that $\phi : M_1 \to M_2$ is a homomorphism of (m,n)-hypermodules if for all x_1^m, x of M_1 and $r_1^{n-1} \in R$:*

$$\phi(f_1(x_1, \ldots, x_m)) \subseteq f_2(\phi(x_1), \ldots, \phi(x_m))\,;$$

$$\phi(g_1(r_1^{n-1}, x)) = g_2(r_1^{n-1}, \phi(x)).$$

If in the above definition we consider a map $\phi : M_1 \to P^*(M_2),$ then we obtain a *multivalued homomorphism,* shortly we write m-homomorphism.

In Definition 2.5, if the equality holds, then ϕ is called a strong (or good) R-homomorphism.

Definition 2.6 *[13] Let (R,h,k) be (m,n)-hyperring.The relation Γ^* is the smallest equivalence relation such that the quotient $(R/\Gamma^*, h/\Gamma^*, k/\Gamma^*)$ be (m,n)-ring. where R/Γ^* is the set of equivalence classes. The Γ^* is called fundamental equivalence relation.*

Definition 2.7 *[3] Let (M, f, g) be an (m,n)-ary hypermodule. We define ε^* as the smallest equivalence relation such that the quotient $(M/\varepsilon^*, f/\varepsilon^*, g/\varepsilon^*)$ is an (m,n)-ary module over an (m,n)-ary hyperring $R,$ where M/ε^* is the set of equivalence classes. The ε^* is called fundamental equivalence relation.*

Theorem 2.8 *[3] The fundamental relation ε^* is the transitive closure of the relation $\varepsilon,$ i.e., $(\varepsilon^* = \hat{\varepsilon}).$*

Theorem 2.9 *[3] Let* (M, f, g) *be an* (m,n)*-ary hypermodule over an* (m,n)*-ary hyperring* (R,h,k)*. Then,* $(M/\varepsilon^*, f/\varepsilon^*)$ *is an* (m,n)*-ary module over on* (m,n)*-ary ring* $(R/\Gamma^*, h/\Gamma^*, k/\Gamma^*)$.

3. VARIOUS CATEGORIES OF (m,n)-HYPERMODULES

Definition 3.1 *The category* $R_{(m,n)} - Hmod$ *of* (m,n)*-ary hypermodules defined as follows:*

- the objects of $R_{(m,n)} - Hmod$ are (m,n)- hypermodules,
- for the objects M and K, the set of all morphisms from M to K is defined as follows:

$$Hom_R(M,K) = \{ f \mid f : M \to P^*(K) \text{ is an m-homomorphism}\},$$

- the composition gf of morphisms $f : M \to P^*(K)$ and $g : K \to P^*(L)$ defined as follows:

$$gf : H \to P^*(K),\ gf(x) = \bigcup_{t \in f(x)} g(t),$$

- for any object H, the morphism $1_H : H \to P^*(H)$, defined by $1_H(x) = \{x\}$, is the identity morphism.

Remark 3.2 *Consider a category whose objects are all* (m,n)*-hypermodules and whose morphisms are all* R*-homomorphisms denoted by* $R_{(m,n)} - hmod$ *The class of all* R*-homomorphisms from* A *into* B *is denoted by* $hom_R(A,B)$*. In addition,* $R_{s(m,n)} - hmod$*, is the category of all* (m,n)*-hypermodules whose morphisms are all strong* R*-homomorphisms. The class of all strong* R*-homomorphisms from* A *into* B *is denoted by* $hom_{RS}(A,B)$*. It is easy to observe that* $R_{s(m,n)} - hmod$ *is a subcategory of* $R_{(m,n)} - hmod$.

Lemma 3.3 *Let* (M_1, f_1, g_1) *and* (M_2, f_2, g_2) *be two* (m,n)*-hypermodules over an* (m,n)*-hyperring* (R,h,k)*.And the map* $\phi : M_1 \to M_2$ *be a strong homomorphism of* (m,n)*-hypermodules then the map* $\phi^*(\varepsilon^*(x)) = \varepsilon^*(\phi(x))$*, for all* $x \in M$ *is a homomorphism from* $M_1/\varepsilon^* \text{to} M_2/\varepsilon^*$.

Proof. First we show that ϕ^* is well-defined. Suppose that for every $a,b \in M, \varepsilon^*(a) = \varepsilon^*(b)$ then there exist $x_1^m \in M$, $f_1^m \in \mathrm{F}_1$ whit $x_1 = a, x_m = b$ such that $\{x_i, x_{i+1}\} \subseteq f_{1i}, i = 1,2,\ldots,m.$ Since ϕ is a homomorphism and $f_1^m \in \mathrm{F}_1$, we get $\phi(f_1^m) \in \mathsf{F}_2$. Therefore $\phi(a)\varepsilon^*\phi(b)$ which implies $\varepsilon^*(\phi(a)) = \varepsilon^*(\phi(b)),$ and so $\phi^*(\varepsilon^*(a)) = \phi^*(\varepsilon^*(b)).$ Thus ϕ^* is well-defined. Now, we have

$$\begin{aligned}\phi^*(f_1/\varepsilon^*(\varepsilon^*(a_1),\ldots,\varepsilon^*(a_m))) &= \phi^*(\{\varepsilon^*(a) \mid a \in f_1(a_1,\ldots,a_m)\}) \\ &= \phi^*(\varepsilon^*(f_1(a_1^m)) = \varepsilon^*(\phi(f_1(a_1^m)) = \varepsilon^*(f_2(\phi(a_1^m)) \\ &= f_2/\varepsilon^*(\varepsilon^*(\phi(a_1)),\ldots,\varepsilon^*(\phi(a_m))) \\ &= f_2/\varepsilon^*(\phi^*(\varepsilon^*(a_1)),\ldots,\phi^*(\varepsilon^*(a_m)))\end{aligned}$$

and

$$\begin{aligned}\phi^*(g_1/\varepsilon^*(\Gamma^*(r_1),\ldots,\Gamma^*(r_{n-1}),\varepsilon^*(x))) &:= \phi^*(g_1(\Gamma^*(r_1),\ldots,\Gamma^*(r_{n-1}),\varepsilon^*(x))) \\ &= \phi^*(\varepsilon^*(g_1(r_1^{n-1},x))) = \varepsilon^*(g_2(r_1^{n-1},\phi(x))) \\ &= g_2/\varepsilon^*(\Gamma^*(r_1),\ldots,\Gamma^*(r_{n-1}),\varepsilon^*(\phi(x))) \\ &= g_2/\varepsilon^*(\Gamma^*(r_1),\ldots,\Gamma^*(r_{n-1}),\phi^*(\varepsilon^*(x))).\end{aligned}$$

Theorem 3.4 $\mathsf{F}: R_{s(m,n)} - hmod \to R_{(m,n)}/\Gamma^* - mod$ *defined by* $\mathsf{F}(M) = M/\varepsilon^*$ *and* $\mathsf{F}(\phi) = \phi^*,$ *is a functor* $\phi: M_1 \to M_2$ and $\phi^*: M_1/\varepsilon^* \to M_2/\varepsilon^*,$ *where* $R_{(m,n)}/\Gamma^* - mod$ *is the category of all* (m,n)*-modules over* $R/\Gamma^*.$

Proof. By Lemma 3.3, F is well-defined. Let $\phi: M_1 \to M_2$ and $\mu: M_2 \to M_3$ be strong homomorphisms. Then

$$\begin{aligned}\mathsf{F}(\mu\circ\phi) &= (\mu\circ\phi)^* \\ (\mu\circ\phi)^*(\varepsilon^*(a)) &= \varepsilon^*((\mu\circ\phi)(a)) \\ &= \varepsilon^*(\mu(\phi(a))) = \mu^*\varepsilon^*(\phi(a)) \\ &= \mu^*\phi^*(\varepsilon^*(a)) = \mathsf{F}(\mu)\mathsf{F}(\phi)(\varepsilon^*(a)),\end{aligned}$$

for all $a \in M.$ Hence $\mathsf{F}(\mu\circ\phi) = \mathsf{F}(\mu)\mathsf{F}(\phi).$ Also

$$\mathsf{F}(1_M) = 1_M^* : M/\varepsilon^* \to M/\varepsilon^*$$

is defined by

$$1_M^*(\varepsilon^*(a)) = \varepsilon^*(a)$$

is the identity morphism. Therefore F is a functor.

Proposition 3.5 *Let* $\mu: M_1 \to M_2$ *be a homomorphism in* $R_{s(m,n)} - hmod.$ *Then the following diagram is commutative:*

$$\begin{array}{ccc} M_1 & \xrightarrow{\mu} & M_2 \\ \phi_{M_1}\downarrow & & \downarrow\phi_{M_2} \\ M_1/\epsilon^* & \xrightarrow{\mu^*} & M_2/\epsilon^* \end{array}$$

where ϕ_{M_1} and ϕ_{M_2} are the canonical projections of M_1 and M_2, respectively.

Proof. Let $a \in M_1$. Then

$$\phi_{M_2}\mu(a) = \varepsilon^*(\mu(a)) = \mu^*(\varepsilon^*(a)) = \mu^*(\phi_{M_1}(a)) = \mu^*\phi_{M_1}(a).$$

Proposition 3.6 F *is not a faithful functor.*

Proof. Suppose F is a faithful functor, (M_1, f_1, g_1) and (M_2, f_2, g_2) be objects in $H_{(m,n)} - Rmod, \mu_1, \mu_2 : M_1 \to M_2$ be parallel arrows of $H_{(m,n)} - Rmod$ and $\mathsf{F}(\mu_1) = \mathsf{F}(\mu_2)$. Then, for any $a \in M_1, \mathsf{F}(\mu_1(a)) = \mathsf{F}(\mu_2(a))$ implies that

$$\mu_1^*(\varepsilon^*(a)) = \mu_2^*(\varepsilon^*(a))$$
$$\varepsilon^*(\mu_1(a)) = \varepsilon^*(\mu_2(a))$$
$$\phi_{M_2}(\mu_1(a)) = \phi_{M_2}(\mu_2(a))$$
$$\mu_1(a) = \mu_2(a).$$

This yields a contradiction to the definition of ϕ, then F is not a faithful functor.

Definition 3.7 *[9] Let* M *and* N *be two* (m,n)*-hypermodules. Define hyperoperation* $\oplus$ *on* $Hom_R(M,N)$, *as follows:*

$$\varphi_1 \oplus \varphi_2 \oplus \ldots \oplus \varphi_m = \{\varphi \in Hom_R(M,N) \mid \varphi(x) \subseteq f_2(\varphi_1(x), \ldots, \varphi_m(x))\},$$

Note that the hyperoperation $\oplus$ on $hom_R(M,N)$, reduced to the following:

$$\varphi_1 \oplus \varphi_2 \oplus \ldots \oplus \varphi_m = \{\varphi \in hom_R(M,N) \mid \varphi(x) \in f_2(\varphi_1(x), \ldots, \varphi_m(x))\}.$$

Remark 3.8 *In the following of this paper we consider the category of all* (m,n)*-hypermodules over a* (m,n)*-hyperring* R, *in the sense of Krasner* (m,n)*-hypermodules over commutative Krasner* (m,n)*-hyperring* R *with identity. We denote this category by* $R_{(m,n)} - KHmod$. *Hence, the objects of* $R_{(m,n)} - KHmod$ *are Krasner* (m,n)*-hypermodules over commutative Krasner* (m,n)*-hyperring with identity and all morphisms are multivalued homomorphisms.*

Proposition 3.9 *[14] Let* M *and* N *be two* (m,n)*-hypermodules.Then*

- $(Hom_R(M,N), \oplus)$ is an n*-ary commutative hypermonoid,*

- $(hom_R(M,N),\oplus)$ is an n*-ary canonical hypergroup.*

Definition 3.10 *[9] Let* M *be a non-empty set. Then* (M,f,g) *is an* (m,n)*-semihypermodule over an* (m,n)*-hyperring* (R,h,k)*, if* (M,f) *is an* m*-ary commutative hypermonoid and the map*

$g:\underbrace{R\times\ldots\times R}_{n-1}\times M\to P^*(M)$ satisfies the definition 2.3.

Proposition 3.11 *[9] Let* R *be a commutative Krasner* (m,n)*-hyperring with identity and* (M_1,f_1,g_1) *and* (M_2,f_2,g_2) *be two* (m,n)*-ary hypermodules over an* (m,n)*-ary hyperring* R*. Then* $Hom_R(M,N)$ *has* (m,n)*-ary semihypermodule construction.*

Theorem 3.12 *[9]* $Hom_{R_S}(A,-):R_{(m,n)}-KHmod\to(m,n)-sHmod$*, defined by* $B\to Hom_{R_S}(A,B)$ *is a functor from the category of Krasner* (m,n)*-hypermodule to the category of* (m,n)*-ary semihypermodule.*

4. CATEGORICAL PROPERTIES OF $R_{(m,n)}-KHmod$

In this section, concepts of direct hyper product and direct hyper coproduct of a Krasner (m,n)-hypermodule are defined. Also we give some properties of the category $R_{(m,n)}-KHmod.$ and

Definition 4.1 *Let* $\{M_i \mid i\in I\}$ *be a family of* (m,n)*-hypermodules. we define a hyperoperation on* $\prod_{i\in I}M_i$ *as follows:*

$$F\{a_{i1}^{im}\}=\left\{\{t_i\}\mid t_i\in f_i(a_{i1}^{im})\ \{a_{i1}^{im}\}\in\prod_{i\in I}M_i\right\}.$$

For $r\in R$ and $a_i\in\prod_{i\in I}M_i$, define $G(r_1^{(n-1)}\{a_i\}_{(i\in I)})=\left\{g_i(r_1^{(n-1)},a_i)\right\}_{(i\in I)}$.

then $\prod_{i\in I}M_i$, together with m-ary hyperoperation F and n-ary operation G is called *direct hyper product* $\{M_i \mid i\in I\}$.

Proposition 4.2 *If* $\{M_i \mid i\in I\}$ *be a family of* (m,n)*-hypermodules, then*

- the direct hyper product $\prod_{i\in I}M_i$ is an (m,n)-hypermodule,
- for each $k\in I$, $\Pi_k:\prod_{i\in I}M_i\to p^*(M_k)$ given by $\Pi_k(\{a_i\}_{i\in I})=\{a_k\}$ is an m - homomorphism.

Theorem 4.3 *Let* $\{M_i \mid i \in I\}$ *be a family of* (m,n)*-hypermodules, and* $\{\phi_i : M \to p^*(M_i) \mid i \in I\}$ *be a family of* m*-homomorphisms. Then there exists a unique* m*-homomorphism* $\left\{\phi : M \to p^*(\prod_{i\in I} M_i)\right\}$

such that, $\Pi_i \phi = \phi_i$ for all $i \in I$, and this property determines $\prod_{i\in I} M_i$ uniquely up to isomorphism. In the other words $\prod_{i\in I} M_i$ is a product in the category of $R_{(m,n)} - KHmod.$

Proof. It is sufficient to put for each $x \in M$:

$$\phi(x) = \{\phi_i(x)\}_{i\in I}.$$

Then for every $r_1^{n-1} \in R, x_1^m \in M$ we have

$$\phi(f_1(x_1^m)) = \{\phi_i(f_1(x_1^m))\}_{i\in I} = \{f_{1i}(\phi_i(x_1^m))\}_{i\in I} = f_2(\phi(x_1^m))$$

and

$$\phi(g_1(r_1^{n-1}, x)) = \{\phi_i(g_1(r_1^{n-1}, x))\}_{i\in I} = \{g_i(r_1^{n-1}, \phi(x_i))\}_{i\in I} = g_2(r_1^{n-1}, \phi(x)).$$

If $\hat{\phi} : M \to p^*(\prod_{i\in I} M_i)$ is an m-homomorphism such that $\Pi_i \hat{\phi} = \phi_i$, for all $i \in I.$ We have $\forall x \in M\ \hat{\phi}(x) \in p^*(\prod_{i\in I} M_i), \hat{\phi}(x) = \{x_i\}_{i\in I}$. Then

$$\{x_k\} = \Pi_k(\{x_i\}_{i\in I}) = \Pi_k(\hat{\phi}(x)) = \phi_i(x).$$

so $\{x_i\}_{i\in I} = \{\phi_i(x)\}_{i\in I}$, then $\hat{\phi}(x) = \phi(x);$ it means $\hat{\phi} = \phi.$ So ϕ is unique. Therefore $\prod_{i\in I} M_i$ is a product in the category $R_{(m,n)} - KHmod.$

Definition 4.4 *The direct hyper sum of the family* $\{M_i \mid i \in I\}$ *of* (m,n)*-hypermodules, denoted by* $\coprod_{i\in I} M_i$ *is the set of all* $\{a_i\}_{i\in I}$, *where* a_i *can be non-zero only for a finite number of indices.*

Proposition 4.5 *If* $\{M_i \mid i \in I\}$ *is a family of* (m,n)*-hypermodules then*

- $\coprod_{i\in I} M_i$ is an (m,n)-hypermodule,
- for each $k \in I$, the map $\ell_k : M_k \to \coprod_{i\in I} M_i$, given by $l_k(a) = \{a_i\}_{i\in I}$ where $a_i = 0$, for $i \neq k$, and $a_k = a$, is an m-homomorphism,

• for each $i \in I, \ell_i(M_i)$ is a subhypermodule of $\coprod_{i\in I} M_i$. The map ℓ_k is called the canonical injection.

Theorem 4.6 *Let* $\{M_i \mid i \in I\}$ *be a family of* (m,n)*-hypermodules.* $\{\phi_i : M_i \to M \mid i \in I\}$ *be a family of* m*-homomorphisms of* (m,n)*-hypermodules. Then there is a unique* m*-homomorphism* $\phi : \coprod_{i\in I} M_i \to M$ *such that* $\phi\ell_i = \phi_i$, *for all* $i \in I$ *and this property determines* $\coprod_{i\in I} M_i$ *uniquely up to isomorphism. In other words* $\coprod_{i\in I} M_i$ *is a coproduct in the category of* $R_{(m,n)} - KHmod.$

Proof. if $0 \neq \{a_i\} \in \coprod_{i\in I} M_i$, then only a finite number of a_i are non-zero say $a_{i_1}, \ldots, a_{i_r}$, We define

$$\phi(\{a_i\}_{i\in I}) = \begin{cases} f(\phi_{i_1}(a_{i_1}),\ldots\phi_{i_r}(a_{i_r}),0^{(m-r)}) & m \geq r \\ f_l((\phi_{i_1}^{i_{l(m-1)+1}})(a_{i_1}^{i_{l(m-1)+1}}) & m < r = l(m-1)+1 \end{cases}$$

it means $\phi(\{a_i\}_{i\in I}) = \coprod_{i\in I_0} \phi_i(a_i)$ where I_0 is the set $\{i_1, i_2, \ldots, i_r\} = \{i \in I \mid a_i \neq 0\}$. ϕ is a homomorphism

$$\phi(f_1\{x_{i1}^{im}\}_{i\in I}) = \phi\{\{t_i\}_{i\in I} \mid \ t_i \in f_i(x_{i1}^{im})\}$$
$$= \{\phi(\{t_i\}_{i\in I}) \mid t_i \in f_i(x_{i1}^{im})\} = \{f(\phi_i(t_i)) \ \mid t_i \in f_i(x_{i1}^{im})\}$$
$$= f(\phi_i(f_i(x_{i1}^{im}))) = f(f_i(\phi_i(x_{i1}^{im}))) = f(f_i(\phi_i(x_{i1}^{im})))$$

and

$$\phi(g_1(r_1^{n-1}, x_i)) = \phi\{g_i(r_1^{n-1}, x)\}_{i\in I} = f(\phi_i(g_i(r_1^{n-1}, (x_i)))) = g_2(r_1^{n-1}, \phi(x_i))$$

and also $\phi\ell_i = \phi_i$ for all $i \in I$. For each $\{a_i\} \in \coprod_{i\in I} M_i, \{a_i\} \in \coprod_{i\in I_0} \ell_i(a_i)$. If $\xi : \coprod_{i\in I} M_i \to M$ is an m-homomorphism such that $\xi\ell_i = \phi_i$, for all $i \in I$, then

$$\xi(\{a_i\}_{i\in I}) = \xi(\coprod_{i\in I_0} \ell_i(a_i)) = \coprod_{i\in I_0} \xi\ell_i(a_i) = \coprod_{i\in I_0} \phi_i(a_i) = \coprod_{i\in I_0} \phi\ell_i(a_i) = \phi(\coprod_{i\in I_0} \ell_i(a_i)) = \phi(\{a_i\}_{i\in I}).$$

hence $\xi = \phi$ and so ϕ is unique. Therefore $\coprod_{i\in I} M_i$ is a coproduct in the category $R_{(m,n)} - KHmod.$

Theorem 4.7 *The category* $R_{(m,n)}-KHmod$ *has zero object, product, coproduct.*

Proof. The proof is an immediate consequence of Theorems 4.3,4.6.

Theorem 4.8 *[3] Let* (M_1, f_1, g_1) *and* (M_2, f_2, g_2) *be two* (m,n)*-ary hypermodules over an* (m,n)*-ary hyperring* R *and let* $\varepsilon_A^*, \varepsilon_B^*$ *and* $\varepsilon_{A\times B}^*$ *be fundamental equivalence relation on* A, B *and* $A\times B$ *respectively. then*

$$\phi : (A\times B)/\varepsilon_{A\times B}^* \cong A/\varepsilon_A^* \times B/\varepsilon_B^*,$$

as (m,n)-ary modules over an (m,n)-ary hyperring R.

Theorem 4.9 *Let* $\{M_i \mid i\in I\}$ *be a family of* (m,n)*-hypermodules over an* (m,n)*-ary hyperring* R *and let* $\varepsilon_{M_i}^*, i\in I$ *and* $\varepsilon^*_{\prod_{i\in I} M_i}$ $(\varepsilon^*_{\coprod_{i\in I} M_i})$ *be fundamental equivalence relation on* M_i *and* $\prod_{i\in I} M_i (\coprod_{i\in I} M_i)$ *respectively. then*

- $\phi_1 : (\prod_{i\in I} M_i)/\varepsilon^*_{\prod_{i\in I} M_i} \cong \prod_{i\in I} M_i/\varepsilon^*_{M_i},$
- $\phi_2 : (\coprod_{i\in I} M_i)/\varepsilon^*_{\coprod_{i\in I} M_i} \cong \coprod_{i\in I} M_i/\varepsilon^*_{M_i}.$

as (m,n)-ary modules over an (m,n)-ary hyperring R.

Proof.

- First we define relation $\hat{\varepsilon}$ on $\prod_{i\in I} M_i$ as follows:

$$\{a_i\}_{i\in I}\ \hat{\varepsilon}\ \{b_i\}_{i\in I} \Leftrightarrow a_i\ \varepsilon_{M_i}^*\ b_i \quad \forall i\in I,$$

$\hat{\varepsilon}$ is an equivalence relation. We define F on $(\prod_{i\in I} M_i)/\hat{\varepsilon}$ as follows: $F((\hat{\varepsilon}(\{a_i\}_{i\in I})_1^m)) = \hat{\varepsilon}(\{b_i\}_{i\in I}),$ for all $b_i \in f_i(\varepsilon_{M_i}^*(a_{i1}),\ldots,\varepsilon_{M_i}^*(a_{im}))$ and

$$G(r_1^{(n-1)}, \hat{\varepsilon}(\{a_i\}_{(i\in I)})) = \hat{\varepsilon}(\{b_i\}_{i\in I})$$

for all $b_i \in g_i(r_1^{(n-1)}, \varepsilon_{M_i}^*(a_i)).$ Since $\{M_i \mid i\in I\}$ be a family of (m,n)-hypermodules, consequently, $(\prod_{i\in I} M_i)/\hat{\varepsilon}$ is an (m,n)-ary hypermodule. Now, let θ be an equivalence relation on $\prod_{i\in I} M_i$ such that $(\prod_{i\in I} M_i)/\theta$ is an (m,n)-ary hypermodule. Similar to the proof of the Theorem 2.8, we get

$$\{a_i\}_{i\in I}\ \hat{\varepsilon}\ \{b_i\}_{i\in I} \Rightarrow \{a_i\}_{i\in I}\ \theta\ \{b_i\}_{i\in I}.$$

Therefore the relation $\hat{\varepsilon}$ is the smallest be an equivalence relation on $\prod_{i\in I} M_i$ such that $(\prod_{i\in I} M_i)/\hat{\varepsilon}$ is an (m,n)-ary hypermodule, i.e., $\hat{\varepsilon} = \varepsilon^*_{\prod_{i\in I} M_i}$. Now, we consider the map

$\phi_1 : \prod_{i\in I} M_i/\varepsilon^*_{M_i} \to (\prod_{i\in I} M_i)/\varepsilon^*_{\prod_{i\in I} M_i}$, by $\phi_1(\{\varepsilon^*_{M_i}(a_i)\}_{i\in I}) = \varepsilon^*_{\prod_{i\in I} M_i}(\{a_i\}_{i\in I})$.

• If $0 \neq \{a_i\} \in \coprod_{i\in I} M_i$, then only a finite number of a_i are non-zero say $a_{i_1}, \ldots, a_{i_r}$, We define relation $\hat{\varepsilon}$ on $\coprod_{i\in I_0} M_i$ as follows:

$$\{a_i\}_{i\in I_0}\ \hat{\varepsilon}\ \{b_i\}_{i\in I_0} \Leftrightarrow a_i\ \varepsilon^*_{M_i}\ b_i \quad \forall i \in I_0,$$

where I_0 is the set $\{i_1, i_2, \ldots, i_r\} = \{i \in I \mid a_i \neq 0\}$.

Rest of the proof is similar to (i).

Theorem 4.10 *Fundamental functor* F *preserves zero object, product and coproduct.*

Proof. We know $\mathsf{F}(M) = M/\varepsilon^*$, then $\mathsf{F}(\{0\}) = \{0\}/\varepsilon^* = \{0\}$.

For the rest we must prove

$$\mathsf{F}(\prod_{i\in I} M_i) \cong \prod_{i\in I} \mathsf{F}(M_i)$$
$$\mathsf{F}(\coprod_{i\in I} M_i) \cong \coprod_{i\in I} \mathsf{F}(M_i).$$

It is an immediate consequence of Theorem 4.9.

Definition 4.11 *An* (m,n)*-hyperadditive (* $h(m,n)$ *-additive) category* $\hbar$ *is a category such that any two objects have a product and the morphism set* $\hbar(A,B)$ *is a commutative* n*-ary hypergroup such that the composition*

$$\hbar(A,B) \times \hbar(B,C) \to \hbar(A,C)$$

is bilinear.

Proposition 4.12 *Let* $\hbar$ *be the category* $R_{(m,n)} - Khmod$. *Then* $\hbar$ *is* $R_{(m,n)} - Khmod$*-additive category.*

Proof. The proof is an immediate consequence of Proposition 3.9 and Theorem 4.7.

Proposition 4.13 *Fundamental functor* F *is an additive functor.*

Proof. The proof is an immediate consequence of Proposition 4.12 and Theorem 4.10.

5. PROPERTIES OF HOM

In the following theorems we show the relation between product and coproduct of the family of (m,n)-hypermodules with hom.

Theorem 5.1 *Let $\{M_i \mid i \in I\}$ be a family of (m,n)-hypermodules over an (m,n)-ary hyperring R and N also is an (m,n)-hypermodule. Then*

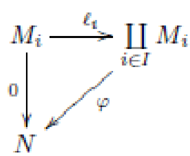
$$hom_R(\coprod_{i\in I} M_i, N) \cong \prod_{i\in I} hom_R(M_i, N)$$

Proof. For each $i \in I$, we consider the map $\ell_i : M_i \to \coprod_{i\in I} M_i$, that introduce in Theorem 4.5. Define the function

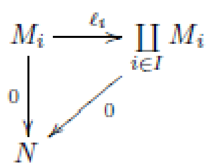
$$\theta : hom_R(\coprod_{i\in I} M_i, N) \to \prod_{i\in I} hom_R(M_i, N)$$

whith $\theta(\varphi) = (\varphi\ell_i)_{i\in I}$. θ is an R - homomorphism. Now we prove that θ is one to one and onto.

Let $\varphi \in Ker\theta$ then $(0)_{i\in I} \in (\varphi\ell_i)_{i\in I} = \theta(\varphi)$. So for each $i \in I, \varphi\ell_i = 0$, therefore φ commutes diagram

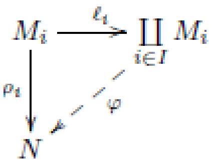


for each $i \in I$. On the other hand, diagram



is also commutative for each $i \in I$, so by Theorem 4.6, $\varphi = 0$. Therefore $Ker\theta = 0$, consequently θ is one to one.

Let $(\rho_i)_{i\in I}$ be an arbitrary member of $\prod_{i\in I} hom_R(M_i, N)$. For Theorem 4.6, there exists a unique homomorphism $\varphi : hom_R(\coprod_{i\in I} M_i, N)$ such that for each $i \in I$ commutes diagram



So $\varphi \in hom_R(\coprod_{i\in I} M_i, N)$ and $\theta(\varphi) = (\varphi\ell_i)_{i\in I} = (\rho_i)_{i\in I}$, consequently θ is onto.

Theorem 5.2 *Let $\{N_i \mid i \in I\}$ be a family of (m,n)-hypermodules over an (m,n)-ary hyperring R and M also is an (m,n)-hypermodule. Then*

$$hom_R(M, \prod_{i\in I} N_i) \cong \prod_{i\in I} hom_R(M, N_i)$$

Proposition 5.3 *Let $\{M_i \mid i \in I\}$ be a family of (m,n)-hypermodules over an (m,n)-ary hyperring R and N also is an (m,n)-hypermodule and F be fundamental functor . Then*

$$\mathsf{F}(hom_R(\coprod_{i\in I} M_i, N)) \cong \prod_{i\in I}(\mathsf{F}(hom_R(M_i, N))).$$

Proof. The proof is an immediate consequence of Theorems 4.9 and 5.2.

Proposition 5.4 *Let $\{N_i \mid i \in I\}$ be a family of (m,n)-hypermodules over an (m,n)-ary hyperring R and M also is an (m,n)-hypermodule and F be fundamental functor. Then*

$$\mathsf{F}(hom_R(M, \prod_{i\in I} N_i)) \cong \prod_{i\in I}(\mathsf{F}(hom_R(M, N_i))).$$

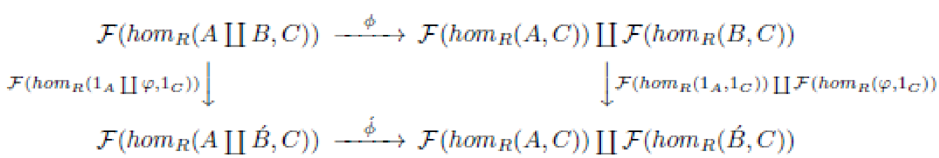
Proof. The proof is an immediate consequence of Theorems 4.10 and 5.2.

Corollary 5.5 *Let A,B,C be (m,n)-hypermodules over an (m,n)-ary hyperring R and and F be fundamental functor . Then isomorphism*

$$\mathsf{F}(hom_R(A\coprod B, C)) \cong \mathsf{F}(hom_R(A,C))\coprod \mathsf{F}(hom_R(B,C))$$

is natural.

Proof. For each homomorphism $\varphi : B \to B'$, the following diagram is commutative:



In this diagram, define a map $1_A \coprod \varphi : A\coprod B \to A\coprod B'$, with $(a,b) \to (a, \varphi(b))$. then, ϕ define an natural isomorphism between two functors S,T that

$$S(B)=\mathrm{F}(hom_R(A\coprod B,C)),T(B)=\mathrm{F}(hom_R(A,C))\coprod \mathrm{F}(hom_R(B,C)).$$

The same prove show that ϕ into A,C also is natural.

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REFERENCES

[1] R. Ameri, On the categories of hypergroups and hypermodules, J. Discrete Math. Sci. cryptogr. 6 (2003) 121-132.

[2] R. Ameri, M.Norouzi, Prime and primary hyperideales in Krasner (m,n)-hyperring, European J. Combin. 34(2013)379-390.

[3] SM. Anvariyeh, S. Mirvakili, B. Davvaz, Fundamental relation on (m,n)-hypermodules over (m,n)-hyperrings. Ars combin. 94(2010)273-288.

[4] S. Awodey, Category theory. vol2, Oxford Press. New York, 2010.

[5] P. Corsini, Prolegemena of Hypergroup Theory, second ed. Aviani, Editor, 1993.

[6] B. Davvaz, V. Leoreanu, Hyperring Theory and Applacations, International Academic Press, 2007, p. 8.

[7] B. Davvaz, T. Vougiouklis , n-ary hypergroups, Iran.J. Sci. Technol. Trans. A. Sci. 30(A2)(2006) 165-174.

[8] W. Dörnte, Untersuchungen Über einen verallgemeinerten Gruppenenbegriff, Math. Z. 29(1928) 1-19.

[9] N. Jafarzadeh, R. Ameri, Hom in the category of (m,n)-ary hypermodules, article in press.

[10] V. Leoreanu, Canonical n-ary hypergroups, Ital.J. Pure Appl. Math. 24(2008).

[11] F. Marty, Sur une generalization de group in: 8^{iem} congres des Mathematiciens Scandinaves, Stockholm. 1934, pp. 45-49.

[12] S. Mirvakili, B. Davvaz, constructions of (m,n)-hyperrings. MATEMAT. 67,1(2015)1-16.

[13] S. Mirvakili, B. Davvaz, Relations on Krasner (m,n)-hyperrings. European J. Combin. 31(2010) 790-802.

[14] H. Shojaei, R. Ameri, Some Results On Categories of Krasner Hypermodules. J FundamAppl Sci. 2016, 8(3S), 2298-2306.

[15] T. Vougiouklis, Hyperstructure and their Representations, Hardonic, Press, Inc, 1994.