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DERIVATIONS ON HYPERLATTICES

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ABSTRACT

In this paper, we introduced the notion of derivations on hyperlattices and investigated some related properties. Also, we characterized the $Fix_d(L)$ by derivations.

Keywords: Lattices, hyperlattices, derivation, FixD**.**

1. INTRODUCTION

The theory on hyper algebras called also multi-algebras was introduced by French mathematician Marty [1]. In a classical algebraic structure, the composition of two elements is an element while in an algebraic hyperstructure that is the suitable generalization of classical algebraic structure the composition of two elements is a set. Hyper algebras has many applications to pure and applied mathematics and several papers and books have been written on this topics such as hypergroups [2], hyperrings [3], hyper BCI-algebras [4], and so on.

In algebra, the notion of derivations of rings plays an important role. The notion of derivations in rings were studied by Posner [5] and many results were given on derivations of prime rings. Later, the notion of derivations has been developed by many authors in many different directions like Jordan derivation, generalized derivation in rings and near-rings. Luca Ferrari [9] pursued and completed the problems initiated by Szasz. Then, X. L. Xin, T. Y. Li, and J. H. Lu [10] went on studying the notion of derivation on a lattice and investigated some of its properties. We introduced the notion of symmetric bi-derivations on lattices and derived some related properties [7].

In this paper, the derivation in hyperlattices is introduced and an example is given. Also, some basic properties of derivations are derived.

2. PRELIMINARIES

The composition of two elements is an element in a classical algebraic structure while in an algebraic hyperstructure the composition of two elements is a set. Let H be a non-empty set and

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$\wp^*(H)$ be t he set of all nonempty subsets of H. A binary *hyperoperation* on H is a map $\circ$ from $H \times H$ to $\wp^*(H)$. The couple $(H,\circ)$ is called a *hypergroupoid*. If A and B are nonempty subsets of H, then we denote

$$A \circ B = \bigcup_{a\in A, b\in B} a \circ b, \quad x \circ A = \{x\} \circ A \quad and \quad A \circ x = A \circ \{x\}$$

A hypergroupoid $(H,\circ)$ is called a *semihypergoup* if for all x, y, z of H we have $(x \circ y) \circ z = x \circ (y \circ z)$, which means that

$$\bigcup_{u\in x\circ y} u \circ z = \bigcup_{v\in y\circ z} x \circ v$$

Definition 2.1 *Let* L *be a nonempty set,* $\vee : L\times L \to \wp^*(L)$ *[*$\wedge : L\times L \to \wp^*(L)$*] be a hyperoperation, and* $\wedge : L\times L \to L$ *[*$\vee : L\times L \to L$*] be an operation. Then* $(L,\vee,\wedge)$ *is a (join) meet hyperlattice if for all* $x, y, z \in L$ *the following condions hold:*

$$(hl1) \quad x \in x \vee x \quad and \quad x = x \wedge x \quad (x \in x \wedge x \quad and \quad x = x \vee x);$$
$$(hl2) \quad x \vee (y \vee z) = (x \vee y) \vee z \quad and \quad x \wedge (y \wedge z) = (x \wedge y) \wedge z;$$
$$(hl3) \quad x \vee y = y \vee x \quad and \quad x \wedge y = y \wedge x;$$
$$(hl4) \quad x \in x \wedge (x \vee y) \cap x \vee (x \wedge y);$$

L iscalled a strong join hyperlattice,

$$(hl5) \quad If \quad y \in x \vee y, \quad then \quad x = x \wedge y,$$

where for all nonempty subsets of X and Y of L,

$$X \vee Y = \bigcup\nolimits_{x\in X, y\in Y} x \vee y \quad and \quad X \wedge Y = \{x \wedge y \mid x \in X, y \in Y\}.$$

Similarly, a strong meet hyperlattice can be defined. $(L,\vee,\wedge)$ is called a hyperlattice if L is both join and meet hyperlattices. A hyperlattice $(L,\vee,\wedge)$ is called a superlattice if for all $x, y \in L$, we have

$$x \in x \vee y \quad if \quad and \quad only \quad if \quad y \in x \wedge y.$$

Let L be a join hyperlattice. For each $x, y \in L$, we define two relations on L as follows:

$$(x, y) \in \leq \quad if \quad and \quad only \quad if \quad x = x \wedge y,$$
$$(x, y) \in \prec \quad if \quad and \quad only \quad if \quad y \in x \vee y.$$

Obviously, join and meet (strong) hyperlattice are dual of each other so definitions and properties which have been stated and proved for one of them are correct for one by suitable change.

$(x, y) \in \leq$ is denoted by $x \leq y$ and $(x, y) \in \prec$ is denoted by $x \prec y$. Moreover, for all nonempty subsets X and Y of L, it is defined as $X \leq Y$ if there exists $x \in X$ and $y \in Y$ such that $x \leq y$ and $X \prec Y$ if for each $x \in X$ there exists $y \in Y$ such that $x \leq y$.

A *zero* of a hyperlatti ce (of each kind) L is an element 0 with $0 \leq x$ for all $x \in L$. A *unit*, 1, satisfies $x \leq 1$ for all $x \in L$. $\leq$ is an ordering relation on join strong hyperlattice, so it can be concluded that there are at most one zero and at most one unit. A *bounded hyperlattice* is one that has both 0 and 1. In a bounded join hyperlattice L, y is a *complement* of x if $x \wedge y = 0$ and $1 \in x \vee y$. The set of complement elements of x is denoted by x^c. If $card(x^c) = 1$, then the complement element of x is denoted by x^c too. A *complement hyperlattice* is a bounded hyperlattice in which every element has at least one complement. Also, L is called a *good hyperlattice* if $0 \vee 0 = \{0\}$ for each $x \in L$ and it is called a *s-good hyperlattice* if $x \vee 0 = \{x\}$ for each $x \in L$. Clearly, every s-good hyperlattice is a good hyperlattice.

Proposition 2.2 *[6] Let* $(L, \vee, \wedge)$ *be a join strong hyperlattice. Then for all* $x, y, z \in L$ *and for all nonempty subsets* X, Y *and* Z *of* L *the following hold:*

(i) $\leq = \prec$ and $(L, \leq)$ is a poset. Also, if $\leq = \prec$ then we can replace $hl4$ by $"x \in x \wedge (x \vee y)"$;

(ii) $x \wedge y \leq x$, $y \leq x \vee y$;

(iii) $X \subseteq (X \vee X) \cap (X \wedge Y)$;

(iv) $X \vee (Y \vee Z) = (X \vee Y) \vee Z$ and $X \wedge (Y \wedge Z) = (X \wedge Y) \wedge Z$;

(v) If $x \in x \wedge Y$ then $Y \cap (x \vee Y) \neq \emptyset$;

(vi) If $x \leq y$, then $x \wedge z \leq y \wedge z$;

(vii) If $x, y \in x \vee y$, then $x = y$, so $x \vee y = L$ implies that $x = y$;

(viii) $x \vee y = \{0\}$ implies that $x = y = 0$;

(ix) $A \wedge 0 = \{0\}$;

(x) $0 \in 0 \vee A$ implies that $0 \in A$.

Definition 2.3 *[6] A hyperlattice* $(L, \vee, \wedge)$ *is said to be distributive if for all* $x, y, z \in L$ *we have* $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$. *A distributive hyperlattice* $(L, \vee, \wedge)$ *is said to be s-distributive if and only if* $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

Proposition 2.4 *[6] Let* $(L, \vee, \wedge)$ *be a distributive join strong hyperlattice. Then for each* $x, y, z \in L$ *we have:*

(i) L is a good hyperlattice and $y \in x \vee 0$ implies that $y \vee 0 \subseteq x \vee 0$;

(ii) If L is a distributive and $x \in y \vee y$, then $x \leq y$;

(iii) If $y \leq x$ and $z \leq x$, then $y \vee z \prec x$. In particular, $0 \vee x \prec x$ and $x \vee x \prec x$. Further more, if L is complemented, then

(iv) If $1 \in x \vee z$ where $z \in y^c$, then $y \leq x$.

As usual, a subhyperlattice of a hyperlattice L is a nonempty subset of L which is a strong join hyperlattice, that is, which is closed underhyperoperation $\vee$ and operation $\wedge$ as defined in L.

Example 2.1 *[6] Let* $L := \{0, x_1, x_2, 1\}$ *be a set with the Cayley table;*

$\wedge$	0	x_1	x_2	1
0	0	0	0	0
x_1	0	x_1	0	x_1
x_2	0	0	x_2	x_2
1	0	x_1	x_2	1

$\vee$	0	x_1	x_2	1
0	$\{0\}$	$\{x_1\}$	$\{x_2\}$	$\{1\}$
x_1	$\{x_1\}$	$\{0, x_1\}$	$\{1\}$	$\{x_2, 1\}$
x_2	$\{x_2\}$	$\{1\}$	$\{0, x_2\}$	$\{x_1, 1\}$
1	$\{x_2\}$	$\{x_1\}$	$\{1\}$	L

Then $(L, \vee, \wedge, 0, 1)$ is an s-good complemented distributive join strong hyperlattice.

3. DERIVATIONS ON HYPERLATTICES

Definition 3.1 *Let* L *be a join hyperlattice. A map* d *is called a derivation of* L *if* d *satisfies*

$$d(x \wedge y) \in (d(x) \wedge y) \vee (x \wedge d(y))$$

for all $x, y \in L$.

Example 3.1 *Let* L *be the join hyperlattice given in Example 2.1 and define a function* d *on* L *by*

$$d(x) = \begin{cases} 0, & \text{if} x = 0, x_1, 1 \\ x_2, & \text{if} x = x_2 \end{cases}$$

Then we can see that d is a derivation on L.

Proposition 3.2 *Let* L *be a join hyperlattice with a unit* 1 *and* d *be the derivation on* L. *Then* $d(x) \in d(x) \vee x \wedge d(1)$ *for all* $x \in L$.

Proof. Let x be an element in a join hyperlattice with a unit 1 and d be the derivation on L. Then

$$
\begin{aligned}
d(x) &= d(x \wedge 1) \\
&\in d(x) \wedge 1 \vee x \wedge d(1) \\
&= d(x) \vee x \wedge d(1)
\end{aligned}
$$

So $d(x) \in d(x) \vee x \wedge d(1)$. And also for $x = 1$

$$
\begin{aligned}
d(1) &= d(1 \wedge 1) \\
&\in d(1) \wedge 1 \vee 1 \wedge d(1) \\
&= d(1) \vee d(1)
\end{aligned}
$$

Therefore $d(1) \in d(1) \vee d(1)$.

Proposition 3.3 *Let L be a join hyperlattice with a zero and d be the derivation on L. Then we $d(0) \in 0 \vee 0$.*

Proof. Let 0 be the zero element in a join hyperlattice and d be the derivation on L. Then

$$
\begin{aligned}
d(0) &= d(0 \wedge 0) \\
&\in d(0) \wedge 0 \vee 0 \wedge d(0) \\
&= 0 \vee 0
\end{aligned}
$$

So $d(0) \in 0 \vee 0$.

Proposition 3.4 *Let L be a join hyperlattice with a unit 1 and d be the derivation on L. Then the followings hold for all $x \in L$:*

i) If $x \geq d(1)$ then $d(1) \prec d(x)$

ii) If $x \leq d(1)$ then $x \prec d(x)$.

Proof. Let x be an element in a join hyperlattice with a unit 1 and d be the derivation on L.

i) Let $x \geq d(1)$ for an element $x \in L$.

$x \geq d(1)$ if and only if $x \wedge d(1) = d(1)$. Therefore

$$
\begin{aligned}
d(x) &\in d(x) \vee (x \wedge d(1)) \\
&= d(x) \vee d(1)
\end{aligned}
$$

So $d(x) \in d(x) \vee d(1)$. Therefore $d(1) \prec d(x)$.

ii) Let $x \leq d(1)$ for an element $x \in L$.

$x \leq d(1)$ if and only if $x \wedge d(1) = x$. Therefore

$$
\begin{aligned}
d(x) &\in d(x) \vee (x \wedge d(1)) \\
&= d(x) \vee x
\end{aligned}
$$

So $d(x) \in d(x) \vee x$. Therefore $x \prec d(x)$.

Proposition 3.5 *Let L be a join hyperlattice and d be the derivation on L. Then for any $x, y \in L$*

$$d(x) \in (d(x) \wedge (x \vee y)) \vee (x \wedge d(x \vee y))$$

Proof. Let L be a join hyperlattice and d be the derivation on L. Then we have By $(hl4)$ we have $x \in x \wedge (x \vee y) \cap x \vee (x \wedge y)$ therefore $x \in x \wedge (x \vee y)$. Then

$$\begin{aligned} d(x) &\in d(x \wedge (x \vee y)) \\ &= (d(x) \wedge (x \vee y)) \vee (x \wedge d(x \vee y)) \end{aligned}$$

Therefore $d(x) \in (d(x) \wedge (x \vee y)) \vee (x \wedge d(x \vee y))$.

Proposition 3.6 *Let L be a join strong hyperlattice and d be the derivation on L. Then for any $x \in L$*

$$d(x) \in (d(x) \wedge x) \vee (x \wedge d(x))$$

Proof. Let L be a join strong hyperlattice and d be the derivation on L and $x \in L$. Then

$$\begin{aligned} d(x) &= d(x \wedge x) && \in (d(x) \wedge x) \vee (x \wedge d(x)) \\ &= (d(x) \wedge (d(x) \wedge x) \vee (d(x) \wedge x) \end{aligned}$$

Therefore

$$d(x) \in (d(x) \wedge x) \vee (x \wedge d(x))$$

If L is distributive and $d(x) \leq d(x) \wedge x$ and also $d(x) = d(x) \wedge x$. Therefore $d(x) \leq x$.

Proposition 3.7 *Let L be a join strong hyperlattice and d be the derivation on L. If $y \leq x$ and $d(x) = x$ for any $x, y \in L$ then*

$$d(y) \in y \vee (x \wedge d(y))$$

Proof. Let L be a join hyperlattice and d be the derivation on L and $x, y \in L$ and also assume that $y \leq x$ and $d(x) = x$. Then

$$\begin{aligned} d(y) &= d(x \wedge y) && \in (d(x) \wedge y) \vee (x \wedge d(y)) \\ &= (x \wedge y) \vee (x \wedge d(y) \\ &= y \vee (x \wedge d(y)) \end{aligned}$$

Therefore

$$d(y) \in y \vee (x \wedge d(y))$$

Additionally, if L is distributive

$$\begin{aligned} d(y) &\in y \vee (x \wedge (d(y) \wedge y)) \\ &= y \vee ((x \wedge y) \wedge d(y)) \\ &= y \vee (y \wedge d(y)) \\ &= y \vee d(y) \end{aligned}$$

Therefore

$$d(y) \in y \vee d(y)$$

Definition 3.8 *Let d be a derivation of a join hyperlattice L. We can define a set $Fix_d(L)$ by*

$$Fix_d(L) := \{x \in L \mid d(x) \in \{x\}\}$$

Proposition 3.9 *Let L be a join strong distributive hyperlattice and d be the derivation on L. Define $d^2(x) = d(d(x))$ for all $x \in L$. Then $d^2(x) \in Fix_d(L)$ for all $x \in L$.*

Proof. Let L be a join strong distributive hyperlattice and d be the derivation on L. Then

$$\begin{aligned} d^2(x) &= d(d(x)) \\ &= d(x \wedge d(x)) \\ &= \in (d(x) \wedge d(x)) \vee (x \wedge d^2(x)) \\ &= d(x) \vee (x \wedge d^2(x)) \end{aligned}$$

Since

$$d(d(x)) \le d(x) \le x$$

We get

$$d(x) \vee (x \wedge d^2(x)) = d(x)$$

So $d(d(x)) = d^2(x) \in d(x)$, i.e $d^2(x) \in Fix_d(L)$

Definition 3.10 *If $x \le y$ implies $d(x) \le d(y)$ for all $x, y \in L$ where d is a derivation of the a join hyperlattice L then d is called isotone derivation.*

Proposition 3.11 *Let L be a join strong distributive hyperlattice with a unit 1 and d be the derivation on L. If d is an isotone derivation then the following hold for all $x \in L$:*

i) $d(x) \in x \wedge d(1)$

ii) $d(x) \vee d(y) = d(x \vee y)$

Proof. Let L be a join strong distributive hyperlattice with a unit 1 and d be the derivation on L and also let d be an isotone derivation. Then

i) Since d is isotone we have $d(x) \le d(1)$ and also $d(x) \le x$. Then we $d(x) \le x \wedge d(1)$.We have already known that

$$\begin{aligned} d(x) &\in d(x) \vee x \wedge d(1) \\ &= x \wedge d(1) \end{aligned}$$

Then $d(x) \in x \wedge d(1)$.

ii) Since d is isotone then $d(x) = d(x \vee y)$ and $d(y) = d(x \vee y)$ then $d(x) \vee d(y) = d(x \vee y)$ for all $x, y \in L$.

Proposition 3.12 *Let L be a join hyperlattice and d be the derivation on L. If d is the identity derivation then $d(x \vee y) \in (x \vee d(y)) \wedge (d(x) \vee y)$ for all $x, y \in L$.*

Proof. Let d be an identity derivation on a join hyperlattice L then

$$
\begin{aligned}
d(x \vee y) \quad \in \quad & x \vee y \\
& = (x \vee y) \wedge (x \vee y) \\
& = (x \vee d(y)) \wedge (d(x) \vee y)
\end{aligned}
$$

Therefore, $d(x \vee y) \in (x \vee d(y)) \wedge (d(x) \vee y)$

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