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Research Article

KRASNER HYPERMODULES OF GENERALIZED FRACTIONS

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ABSTRACT

In this paper, we define the Krasner hypermodule of generalized fractions of a hypermodule M over a Krasner hyperring R. If M is an R-hypermodule, then we construct the Krasner hypermodule of generalized fractions $U^{-n}M$ consisting of all fractions $\frac{m}{(u_1,\dots,u_n)}$ with $m \in M$ and $(u_1,\dots,u_n) \in U$.

We show that $U^{-n}M$ is a Krasner hypermodule. Then, we consider the category of Krasner hypermodules and prove that the direct limit always exists.

We consider the fundamental equivalence relation ε_u^* and prove some results about the connections between the Krasner hypermodule of fractions and the fundamental Krasner modules, direct systems and direct limits.
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1. INTRODUCTION

Hyperstructure theory, introduced in 1934 by Marty [16], is studied from the theoretical point of view and for its applications to many subjects of pure and applied mathematics (see [4]) like geometry, topology, cryptography and code theory, graphs and hypergraphs, probability theory, binary relations, theory of fuzzy and rough sets, automata theory, economy, etc. A hypergroup is an algebraic structure similar to a group, but the composition of two elements is a non-empty set. The hypergroup theory have applications to several domains and some books have been written on this topic, for example see [5, 7, 8, 23]. Mittas [17] introduced the notion of canonical hypergroups. Several kinds of hyperrings are introduced and analyzed. One of the important class of hyperrings is Krasner hyperrings [13]. Davvaz and Leoreanu studied hyperrings in more details in [9], also see [11, 22, 23]. Moreover, one of the class of algebraic hyperstructures satisfying the module-like axioms, as a generalization of module, is the class of hypermodules [1, 2, 3, 4, 9,12, 21]. There are various types of hypermodules as a generalization of modules. In [9], Davvaz

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studied the concept of H_v -modules of fractions. In [20], Sharp and Zakeri introduced the concept of modules of generalized fractions.

The purpose of this paper is as stated in the abstract. We continue this section with some basic and fundamental concepts. For details and definitions refer to [5, 7, 8, 10, 21]. Let R be a non-empty set. Then, a mapping $+ : R\times R \to \mathcal{P}^*(R)$ is called a hyperoperation on R , where $\mathcal{P}^*(R)$ is the family of all non-empty subsets of R . The couple $(R,+)$ is called a hyperoperation. For non-empty subsets A,B of R and $x \in R$, by $A+B$ we mean the set

$$\bigcup_{\substack{a\in A\\ b\in B}} a+b$$

And $A+x = A+\{x\}$, $B+x = B+\{x\}$ and also $-A = \{-a : a\in A\}$. A non-empty set R together with a hyperoperation $"+"$ and a multiplication $"."$ is called a Krasner hyperring [13] if it satisfies the following:

(1) $(R,+)$ is a canonical hypergroup[17], i.e.,

i. for every x , y , $z \in R$, $x+(y+z) = (x+y)+z$,

ii. for every $x,y \in R$, $x+y = y+x$,

iii. there exists $0 \in R$ such that $0+x = x$ for all $x \in R$

iv. for every $x \in R$ there exists an unique element denoted by $-x \in R$ such that $0 \in x+(-x)$,

v. for every $x,y,z \in R$, $z \in x+y$ implies $y \in -x+z$ and $x \in z-y$.

(2) $(R,.)$ is a semigroup having 0 as a bilaterally absorbing element, i.e.,

i. for every $x,y,z \in R$, $(x.y).z = x.(y.z)$,

ii. $x.0 = 0.x = 0$ for all $x \in R$.

(3) The multiplication $"."$ is distributive with respect to the hyperoperation $"+"$, i.e., for every $x,y,z \in R$, $x.(y+z) = x.y + x.z$ and $(x+y).z = x.z + y.z$.

The following elementary facts in a hyperring easily follow from the axioms:

(1) $-(-a) = a$ for every $a\in R$;(2) 0 is the unique element such that for every $a\in R$, there is an element $-a\in R$ with the property $0\in a+(-a)$ and $-0=0$; (3) $-(a+b) = -a-b$ for all $a,b\in R$; (4) $-(ab) = (-a)b = a(-b)$ for all $a,b \in R$. In a Krasner hyperring R , if there exists an element $1\in R$ such that $1a = a1 = a$ for every $a\in R$,then the element 1 is called the identity element of the hyperring R . In fact, the element 1 is unique. Further, if $ab = ba$ for every $a,b \in R$ then the hyperring R is called a commutative hyperring.

Let $(R,+,.)$ be a Krasner hyperr ing. The canonical hypergroup $(M,+)$ along with the map $*: R\times M \to M$ is called a Krasner hypermodule over R if for all $r_1, r_2 \in R$ and $m_1, m_2 \in M$ the following axioms hold:

1) $r_1 *(m_1 + m_2) = r_1 * m_1 + r_1 * m_2$;
2) $(r_1 + r_2) * m_1 = r_1 * m_1 + r_2 * m_1$;
3) $(r_1.r_2) * m_1 = r_1.(r_2 * m_1)$;
4) $0_R * m_1 = 0_M$

Throughout the paper, for convenience, by hyperring R , we mean a Krasner hyperring with 1_R and by hypermodule M we mean a unitary Krasner hypermodule over R , unless otherwise stated. In every unitary hypermodule M , for every $m \in M$, we have $(-1_R) * m = -m$. Let $(M,+)$ and $(N,+')$ be two hypermodules over a hyperring R . A function $f : M \to N$ that satisfies the conditions:

1) $f(x+y) \subseteq f(x) +' f(y)$;
2) $f(r*y) \subseteq r*f(x)$

for all $r \in R$ and all $x, y \in M$, is called t an (inclusion) homomorphism from M into N . If in (1) the equality holds, then f is called a homomorphism or strong (good) homomorphism.

2. HYPERMODULES OF GENERALIZED FRACTIONS

The first time, the concept of modules of generalized fractions on an arbitrary commutative ring was introduced by Sharp and Zakeri in [20]. Our goal in this section is to introduce a hyperstructure for these modules.

Definition 2.1. [20] Let R be a ring and n be a positive integer. A subset U of $R^n = R \times R \times ... \times R$ (n factors) is said to be triangular if

1) U is non-empty;
2) whenever $(u_1,...,u_n) \in U$, then $(u_1^{\alpha_1},...,u_n^{\alpha_n}) \in U$ for all choices of positive integers $\alpha_1,...,\alpha_n$; and
3) whenever $(u_1,...,u_n)$ and $(v_1,...,v_n) \in U$, then there exists $(w_1,...,w_n) \in U$ such that for all $i = 1,\ldots,n$, $w_i \in \underline{u}_i \cap \underline{v}_i$ which $\underline{u}_i = \sum_{j=1}^{i} R u_j$ and $\underline{v}_i = \sum_{j=1}^{i} R v_j$.

We consider Definition 2.1 for an arbitrary hyperring R . Then, there are lower triangular matrices $\mathbf{H},\mathbf{K} \in D_n(R)$ such that $w^T \in \mathbf{H}u^T \cap \mathbf{K}v^T$ where as usual, $w^T = [w_1,...,w_n]^T$, etc.

In this paper we assume R is a hyperring and U is a triangular subset of R^n, where n is a positive integer. The most of our notations are similar to the notions in [20]. For $\mathbf{H} \in D_n(R)$, we have $\operatorname{adj}(\mathbf{H}) = |\mathbf{H}| I_n$. For a sequenc $(s_1,...,s_n) \in U$, we shall denote the matrix $[s_1,...,s_n]^T$ by s^T .

Lemma 2.2. Let $u,v \in U$ and suppose that there exists $\mathbf{H} = [h_{ij}] \in D_n(R)$ such that $v \in \mathbf{H}u$ Then for all $1 \le i \le n$, $|\mathbf{H}_i| u_i \in \underline{v}_i$, where in $\mathbf{H}_i$ exists a submatrix $i \times i$ of $\mathbf{H}$ which is located at the left top.

Proof. We prove the lemma by induction on i . If $i = 1$, then $v \in |\mathbf{H}_1| u_1 = h_{11}u_1$. Let for each $i = 1,2,...,k$, $|\mathbf{H}_i| u_i \in \underline{v}_i$. According to the assumption $v_{k+1} \in h_{k+1}u_1 + ... + h_{k+1\,k+1}u_{k+1}$. Thus, we obtain

$$|\mathbf{H}_k| v_{k+1} \in |\mathbf{H}_k| \left(h_{k+11}u_1, \ldots, h_{k+1\,k}u_k\right) + |\mathbf{H}_k| h_{k+1\,k+1}u_{k+1}.$$

Therefore, we have

$$|\mathbf{H}_{k+1}| u_{k+1} \in -|\mathbf{H}_k| \left(h_{k+11}u_1, \ldots, h_{k+1\,k}u_k\right) + |\mathbf{H}_k| v_{k+1}.$$

Now, according to the assumption $|\mathbf{H}_{k+1}| u_{k+1} \in \underline{v}_{k+1}$. □

Lemma 2.3 Let $v,u \in U$ and suppose that there exists $\mathbf{H},\mathbf{K} \in D_n(R)$ such that $v^T \in \mathbf{H}u^T \cap \mathbf{K}u^T$. Then $|\mathbf{DH}| - |\mathbf{DK}| \in Rv_1^2 + ... + Rv_{n-1}^2$, where $\mathbf{D}$ is the diagonal matrix $\operatorname{diag}(v_1, \ldots, v_n)$.

Proof. The proof is similar to the proof of Lemma 2.3 in [20]. □

Proposition 2.4. Let M be a hypermodule over a hyperring R . Consider therelation ⊔ on $M \times U$ defined as follows: for $b,c \in M$ and $(u_1,...,u_n),(v_1,...,v_n) \in U$, We write $(b,(u_1,...,u_n)) \sqcup (c,(v_1,...,v_n))$ precisely when there exist $(w_1,...,w_n) \in U$ and lower triangularmatrices $\mathbf{H},\mathbf{K} \in D_n(R)$ such that $w^T \in \mathbf{H}u^T \cap \mathbf{K}v^T$ and $|\mathbf{H}| b - |\mathbf{K}| c \cap \underline{w}_{n-1}M \neq \emptyset$. Therelation ⊔ is an equivalence relation on $M \times U$.

Proof. The proof is similar to the proof of Proposition 2.4 in [20]. □

By using the notations of Proposition 2.4, for $b \in M$ and $u \in U$, define the formal symbol $\frac{b}{u}$ to be the equivalence classes of $\sqcup$ containing $[b,u]$. Let $U^{-n}M$ denote the set of all equivalence classes of $\sqcup$. Our immediate intention is to furnish $U^{-n}M$ with the hyperstructure of a hypermodule.

Lemma 2.5 Let M be a hypermodule over a hyperring R. Let $a,b \in M$ and $(s_1,\dots,s_n),(t_1,\dots,t_n) \in U.$ Suppose that $(u_1,\dots,u_n),(v_1,\dots,v_n) \in U$ and $\mathbf{H},\mathbf{K},\mathbf{P},\mathbf{Q} \in D_n(R)$ aresuch that

$$u^T \in \mathbf{H}s^T \cap \mathbf{K}t^T \quad \text{and} \quad v^T \in \mathbf{P}s^T \cap \mathbf{Q}t^T.$$

Then in $U^{-n}M$ for all $\mathbf{H},\mathbf{K},\mathbf{P},\mathbf{Q} \in D_n(R)$ which the condition 3 of Definition2.1 is true, the following sets

$$A = \left\{ \frac{c}{u} \mid u^T \in \mathbf{H}s^T \cap \mathbf{K}t^T,\ c \in |\mathbf{H}|a + |\mathbf{K}|b \right\},$$

$$B = \left\{ \frac{d}{v} \mid v^T \in \mathbf{P}s^T \cap \mathbf{Q}t^T,\ d \in |\mathbf{P}|a + |\mathbf{Q}|b \right\}$$

are equal. We show that the sets A and B by $\frac{|\mathbf{H}|a + |\mathbf{K}|b}{u}$ and $\frac{|\mathbf{P}|a + |\mathbf{Q}|b}{v}$,respectively.

Proof. The proof is similar to the proof of Lemma 2.6 in [20]. □

Lemma 2.6. Suppose that $a,a',b,b' \in M$ and $(s_1,\dots,s_n),(s'_1,\dots,s'_n),(t_1,\dots,t_n),$ $(t'_1,\dots,t'_n) \in U$ are such that, in $U^{-n}M$, $\frac{a}{s} = \frac{a'}{s'}$ and $\frac{b}{t} = \frac{b'}{t'}$. Suppose that $(u_1,\dots,u_n),(u'_1,\dots,u'_n) \in U$ and $\mathbf{H},\mathbf{K},\mathbf{H}',\mathbf{K}' \in D_n(R)$ are such that $u^T \in \mathbf{H}s^T \cap \mathbf{K}t^T$. and $u'^T \in \mathbf{H}'s'^T \cap \mathbf{K}'t'^T$. Then in $U^{-n}M$, for all $\mathbf{H},\mathbf{K},\mathbf{H}',\mathbf{K}' \in D_n(R)$ which the condition 3 of Definition 2.1 is true. Then, the following sets

$$A = \left\{ \frac{c}{u} \mid u^T \in \mathbf{H}s^T \cap \mathbf{K}t^T,\ c \in \mathbf{H}a + \mathbf{K}b \right\},$$

$$B = \left\{ \frac{d}{u'} \mid u'^T \in \mathbf{H}'s'^T \cap \mathbf{K}'t'^T,\ d \in \mathbf{H}'a + \mathbf{K}'b \right\}$$

are equal.

Proof. The proof is similar to the proof of Lemma 2.7 in [20]. □

Now, it is a straightforward matter to show that $U^{-n}M$ may be given a natural hypermodule structure. The addition is such that, if $a,b \in M$ and $(s_1,...,s_n),(t_1,...,t_n) \in U$, then

$$\frac{a}{(s_1,...,s_n)} + \frac{b}{(t_1,...,t_n)} = \frac{|\mathbf{H}|a + |\mathbf{K}|b}{(u_1,...,u_n)},$$

for any choice of $(v_1,...,v_n) \in U$ and $\mathbf{H},\mathbf{K} \in D_n(R)$ such that $u^t \in \mathbf{H}s^T \cap \mathbf{K}t^T$ where as usual $s^T = [s_1,...,s_n]^T$, et cetera. The zero of this hypermodule is

$$\frac{0}{(v_1,...,v_n)} ,$$

for any choice of $(u_1,...,u_n) \in U$. The scalar multiplication is

$$r\left(\frac{a}{(s_1,...,s_n)}\right) = \frac{ra}{(s_1,...,s_n)} ,$$

for $r \in R$.

Lemma 2.7. $U^{-n}M$ together with the above hyperoperation (+), is a canonical hypergroup.

Proof.Let $\frac{a}{u},\frac{b}{v},\frac{c}{s} \in U^{-1}M$, $w = (w_1,...,w_n), t = (t_1,...,t_n) \in U$ and $\mathbf{H},\mathbf{K},\mathbf{P},\mathbf{Q} \in D_n(R)$ such that $t^T \in \mathbf{H}u^T \cap \mathbf{K}v^T$ and $w^T \in \mathbf{P}s^T \cap \mathbf{Q}t^T$. Then

$$\left(\frac{a}{u}+\frac{b}{v}\right) + \frac{c}{s} = \frac{|\mathbf{H}|a + |\mathbf{K}|b}{t} + \frac{c}{s} = \frac{|\mathbf{Q}||\mathbf{H}|a + |\mathbf{Q}||\mathbf{K}|b + |\mathbf{P}|c}{w}.$$

On the other hand $w^T \in \mathbf{P}s^T \cap \mathbf{QK}v^T$ and $w^T \in \mathbf{I}w^T \cap \mathbf{QH}u^T$. Therefore

$$\frac{a}{u} + \left(\frac{b}{v}+\frac{c}{s}\right) = \frac{a}{u} + \frac{|\mathbf{Q}||\mathbf{K}|b + |\mathbf{P}|c}{w} = \frac{|\mathbf{Q}||\mathbf{H}|a + |\mathbf{QK}|b + |\mathbf{P}|c}{w}.$$

Hence

$$\left(\frac{a}{u}+\frac{b}{v}\right) + \frac{c}{s} = \frac{a}{u} + \left(\frac{b}{v}+\frac{c}{s}\right).$$

Then hyperoperation is associative. It is easy to see that foreach $\frac{a}{u} \in U^{-n}M$,

$$\frac{0}{v} + \frac{a}{u} = \frac{a}{u} = \frac{a}{u} + \frac{0}{v}.$$

Let $\frac{a}{u}\in U^{-n}M$ and we consider $\frac{-a}{u}\in U^{-n}M$ as theinverse $\frac{a}{u}\in U^{-n}M$. If $\mathbf{H}=\mathbf{K}=\mathbf{I}\in D_n(R)$, then $u^T\in\mathbf{H}u^T\cap\mathbf{K}u^T$. and $0\in|\mathbf{H}|a-|\mathbf{K}|a\cap \underline{u}_{n-1}$, therefore $\frac{0}{u}\in\frac{a}{u}+\frac{-a}{u}$. It is easy to see that for each $\frac{a}{u},\frac{b}{v}\in U^{-n}M$, $\frac{a}{u}+\frac{b}{v}=\frac{b}{v}+\frac{a}{u}$. Since M is a canonical hypergroup, by Lemmas 2.2 and 2.5, if $\frac{a}{u}\in\frac{b}{v}+\frac{c}{s}$, then $\frac{c}{s}\in-\frac{a}{u}+\frac{b}{v}$ and $\frac{b}{v}\in\frac{a}{u}-\frac{c}{s}$. □

Proposition 2.8. $(U^{-n}M,+,.)$ is a hypermodule. $U^{-n}M$ is called the hypermodule of generalized fractions.

Proof. It is straightforward. □

Proposition 2.9. Let M_1 and M_2 be two hypermodules and $f: M_1\to M_2$ be a homomorphism. Then, the map $U^{-n}f: U^{-n}M_1\to U^{-n}M_2$ by $U^{-n}f\left(\frac{a}{s}\right)=\frac{f(a)}{s}$ is a homomorphism and $U^{-n}M$ becomes a functor from the category of hypermodules to itself, and it is easy to see that this functor is covariant.

Proof. Suppose that $\frac{a}{s},\frac{b}{t}\in U^{-n}M_1$ and $r\in R$. First we show that $U^{-n}f$ is well-defined. Let $\frac{a}{s}=\frac{b}{t}$, $(r_1,\dots,r_n)\in U$ and $\mathbf{H},\mathbf{K}\in D_n(R)$ such that

$$r^t\in\mathbf{H}s^T\cap\mathbf{K}t^T \qquad \text{and} \qquad |\mathbf{H}|a-|\mathbf{K}|b\cap\left(\sum_{i=1}^{n-1}R\,r_i\right)M_1\neq\varnothing.$$

Then, for some $m_1,\dots,m_n\in M_1$ we have

$$\varnothing\neq|\mathbf{H}|a-|\mathbf{K}|b\cap\sum_{i=1}^{n-1}r_i m_i.$$

Since f is a homomorphism, it follows that

$$\varnothing\neq|\mathbf{H}|f(a)-|\mathbf{K}|f(b)\cap\sum_{i=1}^{n-1}r_i f(m_i).$$

Thus, we obtain

$$U^{-n}f\left(\frac{a}{s}\right)=\frac{f(a)}{s}=\frac{f(b)}{t}=U^{-n}f\left(\frac{b}{t}\right).$$

Therefore, $U^{-n}f$ is well-defined. Moreover, $U^{-n}f$ is a homomorphism. Let $\frac{a}{s},\frac{b}{t}\in U^{-n}M_1$, $r\in R$, $(u_1,...,u_n)\in U$ and $\mathbf{H},\mathbf{K}\in D_n(R)$ such that $u^T\in \mathbf{H}s^T\cap \mathbf{K}t^T$. Therefore, we have

$$U^{-n}f\left(\frac{a}{s}+\frac{b}{t}\right)=U^{-n}f\left(\frac{|\mathbf{H}|a+|\mathbf{K}|b}{u}\right)=\frac{f\left(|\mathbf{H}|a+|\mathbf{K}|b\right)}{u}=\frac{|\mathbf{H}|f(a)+|\mathbf{K}|f(b)}{u}$$

$$=\frac{f(a)}{s}+\frac{f(b)}{t}=U^{-n}f\left(\frac{a}{s}\right)+U^{-n}f\left(\frac{b}{t}\right).$$

Similarly, we obtain

$$U^{-n}f\left(\frac{r.a}{s}\right)=r.U^{-n}f\left(\frac{a}{s}\right).$$

This proves that $U^{-n}f$ is a homomorphism. Let $\iota : M\to M$ is identity map, it is easy to see that $U^{-n}\iota : U^{-n}M\to U^{-n}M$ is identity. Let $f : M\to N$, $g : N\to P$ are homomoephisms of hypermodeles, $\frac{a}{s}\in U^{-n}M$ and $r\in R$. Then

$$U^{-n}(fog)\left(\frac{a}{s}\right)=\frac{gof(a)}{s}=U^{-n}go\left(\frac{f(a)}{s}\right)=U^{-n}goU^{-n}f\left(\frac{a}{s}\right).$$

Therefore, U^{-n} is a covariant functor. If $f : M\to N$ and $g : M\to N$ are two homomorphisms of hypermodules and $\frac{m}{u}\in U^{-n}M$, we have

$$U^{-n}(f+g)\left(\frac{m}{u}\right)=\frac{(f+g)(m)}{u}=\frac{(f)(m)+(g)(m)}{u}=U^{-n}f\left(\frac{m}{u}\right)+U^{-n}g\left(\frac{m}{u}\right),$$

$$U^{-n}(rf)\left(\frac{m}{u}\right)=\frac{rf(m)}{u}=r\frac{f(m)}{u}=rU^{-n}f\left(\frac{m}{u}\right).$$

So, the functor U^{-n} is additive. □

Proposition 2.10. The functor U^{-n} is right exact.

Proof. Suppose that

$$M'\xrightarrow{f}M\xrightarrow{g}M''\longrightarrow 0$$

is an exact sequence of hypermodules. We prove that $U^{-n}M'\xrightarrow{U^{-n}f}U^{-n}M\xrightarrow{U^{-n}g}U^{-n}M''\longrightarrow 0$ is exact. We show that $U^{-n}g$ is an epimorphism. Let $\frac{m''}{u}\in U^{-n}M''$. Since g is an epimorphism, it follows that there exists $m\in M$ such that $g(m)=m''$. Thus, we have

$$U^{-n}g\left(\frac{\mathrm{m}}{\mathrm{u}}\right)=\frac{g(\mathrm{m})}{\mathrm{u}}=\frac{\mathrm{m}''}{\mathrm{u}}.$$

Therefore, $U^{-n}g$ is an epimorphism. Now, we have $gof=0$. By Proposition 2.9, $U^{-n}goU^{-n}f=U^{-n}(gof)=U^{-n}(0)=0$, and $\mathrm{Im}\,U^{-n}f\subset\ker U^{-n}g$. Let $\frac{\mathrm{m}}{\mathrm{u}}$, where $m\in M$ and $u\in U$, belong to $\ker U^{-n}g$. This means that there exist $(t_1,...,t_n)\in U$ and $\mathbf{H}\in D_n(R)$ such that $t^T\in\mathbf{H}s^T$ and $|\mathbf{H}|g(m)\in\left(\sum_{i=1}^{n-1}Rt_i\right)M''$. Since g is an epimorphism, it follows that there exist $m_1,\dots,m_{n-1}\in M$ such that $g(|\mathbf{H}|m)\in\sum_{i=1}^{n-1}t_ig(m_i)$. Hence,

$$|\mathbf{H}|m-\sum_{i=1}^{n-1}t_im_i\subseteq\ker g=\mathrm{Im}f.$$

Thus, for some $m'\in M'$,

$$\varnothing\neq|\mathbf{H}|b-f(m')\cap\left(\sum_{i=1}^{n-1}Rt_i\right)M.$$

Thus, in $U^{-n}M$, $\frac{\mathrm{b}}{\mathrm{s}}=\frac{f(\mathrm{m}')}{\mathrm{t}}$. Therefore $\ker U^{-n}g\subseteq\mathrm{Im}U^{-n}f$. □

Let R be a hyperring. The triangular subset U' of R^n is expanded [20] if, whenever $(u_1,...,u_n)\in U'$ and i is an integer such that $0\le i\le n$, it is the case that $(u_1,...,u_i,1,\dots,1)\in U'$ also. In particular, this means that U' contains $(1,\dots,1)$. For a general triangular subset U of R^n, we let $\overline{U}$ be the set of all sequences $(v_1,...,v_n)\in R^n$ for which there exist $i\in\mathbb{N}$ with $0\le i\le n$ and $(u_1,...,u_n)\in U$ such that

$$v_j=\begin{cases}u_j & \text{for} \quad j=1,\dots,i,\\ 1 & \text{for} \quad j=i+1,\dots,n.\end{cases}$$

Lemma 2.11. $\overline{U}$ is the smallest expanded triangular subset of R^n which contains U.

Proof. First we show that $\overline{U}$ is a triangular subset of R^n. For every $(u_1,...,u_n)\in U$, with choice $i=n$, we define $v_j=u_j$ for $j\le n$. We have $(u_1,...,u_n)=(v_1,...,v_n)\in\overline{U}$. Then, $U\subseteq\overline{U}$, so $\overline{U}\neq\varnothing$. Now, let $(v_1,...,v_n)\in\overline{U}$. Thus, for $0\le i\le n$ and

$(u_1,...,u_n)\in U$ we have $(v_1,...,v_n)=(u_1,...,u_i,1,\dots,1)$ and let $\alpha_1,\dots,\alpha_n\in$. Then, $(u_1^{\alpha_1},...,u_n^{\alpha_n})\in U$ and $(v_1^{\alpha_1},...,v_n^{\alpha_n})=(u_1^{\alpha_1},...,u_i^{\alpha_i},1,\dots,1)$. So, according to the definition of $\bar{U}$, $(v_1^{\alpha_1},...,v_n^{\alpha_n})\in\bar{U}$. On the other hand, let $(v_1,...,v_n)\in\bar{U}$ and $(w_1,...,w_n)\in\bar{U}$. Thus, for some $j\le n$, $0\le i$ and $(u_1,...,u_n),(u'_1,...,u'_n)\in U$, we have

$$(v_1,...,v_n)=(u_1,...,u_i,1,\dots,1) \text{ and } (w_1,...,w_n)=(u'_1,...,u'_j,1,\dots,1).$$

Since U is triangular subset, it follows that there exists element $(s_1,...,s_n)\in U$ such that for $1\le k\le n$, $s_k\in \underline{u}_k\cap\underline{u}'_k$. Let $i\le j$ and set $(t_1,...,t_n)=(s_1,...,s_j,1,\dots 1)$, it is easy to see that $(t_1,...,t_n)\in\bar{U}$ and for $1\le r\le n$, $t_k\in\underline{w}_r\cap\underline{v}_r$. Therefore, $\bar{U}$ is a triangular subset of R^n. Now, let U' is expanded triangular subset of R^n which contains U . By definition, it is easy to see that $\bar{U}\in U'$. So, $\bar{U}$ is an expanded triangular subset of R^n .□

Proposition 2.12. Let M be a hypermodule over a hyperring R and assume that U is expanded. Let $m\in M$; $u,v\in U$ and $\mathbf{H}\in D_n(R)$ such that $v^T\in\mathbf{H}u^T$. Then

1) $\frac{\mathrm{m}}{\mathrm{u}}=\frac{|\mathbf{H}|\mathrm{m}}{\mathrm{v}}$.
2) $\frac{\mathrm{u_n m}}{\mathrm{u}}=\frac{\mathrm{m}}{(\mathrm{u_1,\dots,u_n},1)}$.
3) If $m\in\left(\sum_{i=1}^{n}Ru_i\right)M$, then $\frac{m}{u}=0$.
4) If $\frac{u_n m}{u}=0$, then $\frac{m}{u}=0$.

Proof. (1) According to assumption, we have $v^T\in\mathbf{H}u^T\cap\mathbf{I_n}v^T$ and $0\in|\mathbf{H}|m-|\mathbf{I_n}||\mathbf{H}|m$, Then by defined $\frac{\mathrm{m}}{\mathrm{u}}=\frac{|\mathbf{H}|\mathrm{m}}{\mathrm{v}}$.

The proofs of (2) and (3) are obvious and similar to the proof of (1).

(4) Since $\frac{u_n m}{u}=0$, it follows that there is $w=(w_1,...,w_n)\in U$ and $\mathbf{H}=[h_{ij}]\in D_n(R)$ such that $w^T\in\mathbf{H}u^T$ and $|\mathbf{H}|u_n m\in\left(\sum_{i=1}^{n}Ru_i\right)M$, but

$w_n \in \sum_{i=1}^{n} h_{ni}u_i = h_{nn}u_n + \sum_{i=1}^{n-1} h_{ni}u_i$, thus $h_{nn}u_n \in w_n - \sum_{i=1}^{n-1} h_{ni}u_i$, therefore $|\mathbf{H}|u_n m \in \prod_{i=1}^{n-1} h_{ii}\left(w_n - \sum_{i=1}^{n-1} h_{ni}u_i\right)\mathrm{m}$. On the other hand, we have

$$\prod_{i=1}^{n-1} h_{ii}\left(\sum_{i=1}^{n-1} h_{ni}u_i\right) \cap \sum_{i=1}^{n-1} Rw_i \neq \varnothing.$$

So, we obtain

$$\prod_{i=1}^{n-1} h_{ii}w_n m \cap \left(\sum_{i=1}^{n-1} Aw_i\right)M \neq \varnothing.$$

By using (3), we obtain

$$\frac{\prod_{i=1}^{n-1} h_{ii}w_n m}{\left(w_1,\ldots,w_{n-1},w_n^2\right)} = 0.$$

Thus, $\frac{|\mathbf{H}|w_n m}{\left(w_1,\ldots,w_{n-1},w_n^2\right)} = 0$ and according to (2), $\frac{|\mathbf{H}|m}{w} = 0$. Since $v^T \in \mathbf{H}u^T$, it follows that $\frac{|\mathbf{H}|m}{w} = \frac{m}{w} = 0$. □

EXAMPLE 1. Let M be a hypermodule over a hyperring R. Let $f_1,\ldots,f_n$ be fixed elements of R, and set $(f_1,\ldots,f_n) \in R^n$. Suppose that

$$U_f = \left\{\left(f_1^{\alpha_1},\ldots,f_n^{\alpha_n}\right) : \alpha_i \in \mathbb{N} \text{ for all } i = 1,\ldots,n\right\}.$$

It is easy to see that $U_f \neq \varnothing$ and for $(g_1,\ldots,g_n) \in U_f$ and $\beta_1,\ldots,\beta_n \in \mathbb{N}$ we have $\left(g_1^{\beta_1},\ldots,g_n^{\beta_n}\right) \in U_f$ .Let $\left(f_1^{\beta_1},\ldots,f_n^{\beta_n}\right), \left(f_1^{\alpha_1},\ldots,f_n^{\alpha_n}\right) \in U_f$. if for $1 \le i \le n$, set $\gamma_i := \max\{\alpha_i, \beta_i\}$, $\mathbf{H} = \mathrm{diag}\left(f_1^{\gamma_1-\alpha_1},\ldots,f_n^{\gamma_n-\alpha_n}\right)$ and $\mathbf{K} = \mathrm{diag}\left(f_1^{\gamma_1-\beta_1},\ldots,f_n^{\gamma_n-\beta_n}\right)$, then $\mathbf{H},\mathbf{K} \in D_n(R)$, $\left(f_1^{\gamma_1},\ldots,f_n^{\gamma_n}\right) \in U$ and

$$\left[f_1^{\gamma_1}\ldots f_n^{\gamma_n}\right]^T \in \mathbf{H}\left[f_1^{\alpha_1}\ldots f_n^{\alpha_n}\right]^T \cap \mathbf{K}\left[f_1^{\beta_1}\ldots f_n^{\beta_n}\right]^T.$$

Then U_f is a triangular subset of R^n. We denote $U_f^{-n}M$ by M_f .

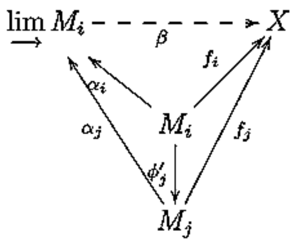
3. DIRECT SYSTEM AND DIRECT LIMIT OF HYPERMODULES

The construction of the direct system and direct limit is similar to the usual module theory (see [12, 15, 18]). In [19], Ghadiri and Davvaz considered the category of H_v -modules and prove that the direct limit always exists in this category. Direct limits are defined by a universal property, and so are unique. Also, already Leoreanu [14, 15] and Romeo [19] studied the notions of direct limits of hyperstructures.

A partially ordered set I is said to be a direct set, if foreach $i, j \in I$ there exists $k \in I$ such that $i \leq k$ and $j \leq k$. Let I be a direct set and ν the category of hypermodules. Let $(M_i)_{i\in I}$ be a family of hypermodules indexed by I. For each pair $i, j \in I$ such that $i \leq j$, let $\phi_j^i : M_i \to M_j$ be a homomorphism and suppose that the following axioms are satisfied:

1) ϕ_i^i is the identity for all $i \in I$,
2) $\phi_k^i = \phi_k^j \phi_j^i$ whenever $i \leq j \leq k$.

Then the hypermodules M_i and homomorphism ϕ_j^i are said to be a direct system $M = (M_i, \phi_j^i)$ over the direct set I. Let $M = (M_i, \phi_j^i)$ be a direct system in ν. The direct limit of this system, denoted by $\varinjlim M_i$,is a hypermodules and a family ofhomomorphism $\alpha_i : M_i \to \varinjlim M_i$, with $\alpha_i = \alpha_j \phi_j^i$ whenever $i \leq j$ satisfying the following universal mapping property:



for every hypermodules X and every family of homomorphism $f_i : M_i \to X$ whit $f_i = f_j \phi_j^i$, whenever $i \leq j$, there is a unique strong homomorphism $\beta : \varinjlim M_i \to X$ making the above diagram commute.

First of all, we define a relation $\leq$ on U as follows:

For $(u_1,...,u_n),(v_1,...,v_n) \in U$, we set $u \leq v$ if and only if there exists $\mathbf{H} \in D_n(R)$ such that $v^T \in \mathbf{H}u^T$.

We show that $\leq$ is a quasi-order on U. With choice $\mathbf{H} = \mathbf{I_n}$, for every $u \in U$, $u^T \in \mathbf{H}u^T$. Then $\leq$ is reflexive. Let $u,v,w \in U$, $v \leq w$ and $u \leq v$ also $\mathbf{H},\mathbf{K} \in D_n(R)$ such that $w^T \in \mathbf{K}v^T$ and $v^T \in \mathbf{H}u^T$. It is clear that $\mathbf{HK} \in D_n(R)$ and $w^T \in \mathbf{KH}u^T$. Therefore, the relation $\leq$ is a quasi-order on U.

Let $u, v \in U$. Then, there exist $w \in U$ and $\mathbf{H},\mathbf{K} \in D_n(R)$ such that $w^T \in \mathbf{K}v^T \cap \mathbf{H}u^T$ .Hence, $u \leq w$ and $v \leq w$ and U is a direct set.

REMARK 1. Let $g = (g_1,...,g_n), f = (f_1,...,f_n) \in U$ and $f \leq g$. By Example 1, $U_f \subseteq U$ and we can build the hypermodule M_f . For every $\frac{m}{f_1^{\alpha_1},...,f_n^{\alpha_n}}$, we define the homomorphism

$$\rho_f : M_f \to U^{-n}M \quad by \quad \rho_f\left(\frac{m}{\left(f_1^{\alpha_1},...,f_n^{\alpha_n}\right)}\right) = \frac{m}{\left(f_1^{\alpha_1},...,f_n^{\alpha_n}\right)}.$$

Moreover, for $\alpha_1,\ldots,\alpha_n \in \square$, we show that there exist $\beta_1,\ldots,\beta_n \in \square$ and $\mathbf{K} \in D_n(R)$ such that $\left[g_1^{\beta_1}\ldots g_n^{\beta_n}\right]^T \in \mathbf{K}\left[f_1^{\alpha_1}\ldots f_n^{\alpha_n}\right]^T$. Since $f \leq g$, by definition there is $\mathbf{H} = \left[h_{ij}\right] \in D_n(R)$ such that $\left[g_1 \ldots g_n\right]^T \in \mathbf{H}\left[f_1 \ldots f_n\right]^T$. Therefore, for $1 \leq i \leq n$, $g_i \in h_{i1} + \ldots, h_{ii} f_i$. Set $\beta = \alpha_1 + \ldots + \alpha_n$, for some k_{ij} which $1 \leq i \ , j \leq n$. We can write $g_i^{\beta} \in k_{i1} f_i^{\alpha_1},\ldots,k_{ii} f_i^{\alpha_i}$ and if for $1 \leq i \leq j \leq n$, $k_{ij} = 0$, then $\mathbf{K}\left[k_{ij}\right] \in D_n R$ and $\left[g_1^{\beta_1},\ldots,g_n^{\beta_n}\right]^T \in \mathbf{K}\left[f_1^{\alpha_1},\ldots,f_n^{\alpha_n}\right]^T$.

Lemma 3.1. Let $(U, \leq)$ be a quasi-order triangular subset of R^n ; $f, g \in U$ and $f \leq g$, then for $m \in M$, $\left(f_1^{\alpha_1},\ldots,f_n^{\alpha_n}\right) \in U_f$, $\left(g_1^{\beta_1},\ldots,g_n^{\beta_n}\right) \in U_g$ and $\mathbf{K} \in D_n(R)$ we define thehomomorphism

$$\Pi_{gf} : M_f \to M_g \quad by \quad \Pi_{gf}\left(\frac{m}{f_1^{\alpha_1},...,f_n^{\alpha_n}}\right) = \frac{|\mathbf{K}|m}{\left(g_1^{\beta_1},...,g_n^{\beta_n}\right)}$$

such that

$$\left[g_1^{\beta_1}\ldots g_n^{\beta_n}\right]^T \in \mathbf{K}\left[f_1^{\alpha_1}\ldots f_n^{\alpha_n}\right]^T. \tag{1}$$

Also set $\left\{M_f, \Pi_{fg} \middle| f \leq g \quad and \quad f, g \in U\right\}$ is a direct limit on direct set U .

Proof. According to the above comments, there exist $\mathbf{K} \in D_n(R)$ and $\beta_1,\ldots,\beta_n \in \square$ such that the relation (1) holds. So, let there exist $\mathbf{K},\mathbf{K}' \in D_n(R)$, $\beta_1,\ldots,\beta_n \in \square$ and $\beta_1',\ldots,\beta_n' \in \square$ such that

$$\left[g_1^{\beta_1}\ldots g_n^{\beta_n}\right]^T \in \mathbf{K}\left[f_1^{\alpha_1}\ldots f_n^{\alpha_n}\right]^T \text{ and } \left[g_1^{\beta_1'}\ldots g_n^{\beta_n'}\right]^T \in \mathbf{K}'\left[f_1^{\alpha_1}\ldots f_n^{\alpha_n}\right]^T. \tag{2}$$

For $1 \leq i \leq n$ consider $\gamma_i = \max\{\beta_i, \beta'_i\}$, $\mathbf{D} = \operatorname{diag}(\gamma_1 - \beta_1, \ldots, \gamma_n - \beta_n)$ and $\mathbf{D}' = \operatorname{diag}(\gamma_1 - \beta'_1, \ldots, \gamma_n - \beta'_n)$. Thus

$$\left[g_1^{\gamma_1} \ldots g_n^{\gamma_n}\right]^T \in \mathbf{D}\left[g_1^{\beta_1} \ldots g_n^{\beta_n}\right]^T \subseteq \mathbf{DK}\left[f_1^{\alpha_1} \ldots f_n^{\alpha_n}\right]^T; \quad (3)$$

and

$$\left[g_1^{\gamma_1} \ldots g_n^{\gamma_n}\right]^T \in \mathbf{D}'\left[g_1^{\beta'_1} \ldots g_n^{\beta'_n}\right]^T \subseteq \mathbf{D}'\mathbf{K}'\left[f_1^{\alpha_1} \ldots f_n^{\alpha_n}\right]^T. \quad (4)$$

Then, we obtain

$$\frac{|\mathbf{D}|m}{\left(g_1^{\gamma_1}, \ldots, g_n^{\gamma_n}\right)} = \frac{m}{\left(g_1^{\beta_1}, \ldots, g_n^{\beta_n}\right)} \quad \text{and} \quad \frac{|\mathbf{D}'|m}{\left(g_1^{\gamma_1}, \ldots, g_n^{\gamma_n}\right)} = \frac{m}{\left(g_1^{\beta'_1}, \ldots, g_n^{\beta'_n}\right)}.$$

So

$$\frac{|\mathbf{KD}|m}{\left(g_1^{\gamma_1}, \ldots, g_n^{\gamma_n}\right)} = \frac{|\mathbf{K}|m}{\left(g_1^{\beta_1}, \ldots, g_n^{\beta_n}\right)} \quad \text{and} \quad \frac{|\mathbf{K}'\mathbf{D}'|m}{\left(g_1^{\gamma_1}, \ldots, g_n^{\gamma_n}\right)} = \frac{|\mathbf{K}'|m}{\left(g_1^{\beta'_1}, \ldots, g_n^{\beta'_n}\right)}.$$

Also, according to the relations (4) and (1) we obtain

$$\frac{|\mathbf{KD}|m}{\left(g_1^{\gamma_1}, \ldots, g_n^{\gamma_n}\right)} = \frac{m}{\left(f_1^{\alpha_1}, \ldots, f_n^{\alpha_n}\right)} = \frac{|\mathbf{K}'\mathbf{D}'|m}{\left(g_1^{\gamma_1}, \ldots, g_n^{\gamma_n}\right)}.$$

Hence

$$\Pi_{gf}\left(\frac{m}{\left(f_1^{\alpha_1}, \ldots, f_n^{\alpha_n}\right)}\right) = \frac{|\mathbf{K}|m}{\left(g_1^{\beta_1}, \ldots, g_n^{\beta_n}\right)} = \frac{|\mathbf{K}'|m}{\left(g_1^{\beta'_1}, \ldots, g_n^{\beta'_n}\right)} = \Pi_{fg}\left(\frac{m}{\left(f_1^{\alpha_1}, \ldots, f_n^{\alpha_n}\right)}\right).$$

Therefore, the above function is independent of the choice $\mathbf{K}$ and $\beta_1, \ldots, \beta_n$. Now, let in M_f, $\frac{m}{\left(f_1^{\alpha_1}, \ldots, f_n^{\alpha_n}\right)} = \frac{m'}{\left(f_1^{\alpha'_1}, \ldots, f_n^{\alpha'_n}\right)}$. Then, there exist $\mathbf{H}, \mathbf{H}' \in D_n(R)$ and $\gamma_1, \ldots, \gamma_n \in$ □ such that

$$\varnothing \neq |\mathbf{H}|m - |\mathbf{H}'|m' \cap \left(\sum_{i=1}^{n-1} R f_i^{\gamma_i}\right)M,$$

and

$$\left[f_1^{\gamma_1} \ldots f_n^{\gamma_n}\right]^T \in \mathbf{H}\left[f_1^{\alpha_1} \ldots f_n^{\alpha_n}\right]^T \cap \mathbf{H}'\left[f_1^{\alpha'_1} \ldots f_n^{\alpha'_n}\right]^T. \quad (5)$$

Then there exist $\mathbf{K} \in D_n(R)$, $\beta_1, \ldots, \beta_n \in$ □ such that

$$\left[g_1^{\beta_1} \ldots g_n^{\beta_n}\right]^T \in \mathbf{KH}\left[f_1^{\alpha_1} \ldots f_n^{\alpha_n}\right]^T \cap \mathbf{KH}'\left[f_1^{\alpha'_1} \ldots f_n^{\alpha'_n}\right]^T.$$

Furthermore, according to the relations (1) and (5), we obtain

$$\varnothing \neq |\mathbf{KH}|m - |\mathbf{KH}'|m' \cap \left(\sum_{i=1}^{n-1} R\,|\mathbf{K}|K f_{i}^{\alpha_i} \right) M \ .$$

By Lemma 2.2, we have

$$\varnothing \neq |\mathbf{KH}|m - |\mathbf{KH}'|m' \cap \left(\sum_{i=1}^{n-1} R\, g_{i}^{\beta_i} \right) M \ .$$

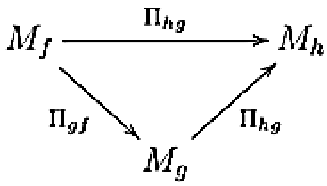
Therefore,

$$\Pi_{gf}\left(\frac{m}{\left(f_1^{\alpha_1},\dots,f_n^{\alpha_n}\right)} \right) = \frac{|\mathbf{KH}|m}{\left(g_1^{\beta_1},\dots,g_n^{\beta_n}\right)} \quad \text{and} \quad \Pi_{gf}\left(\frac{m'}{\left(f_1^{\alpha_1},\dots,f_n^{\alpha_n}\right)} \right) = \frac{|\mathbf{KH}'|m'}{\left(g_1^{\beta_1},\dots,g_n^{\beta_n}\right)} \ .$$

Now, we have

$$\Pi_{gf}\left(\frac{m'}{\left(f_1^{\alpha_1},\dots,f_n^{\alpha_n}\right)} \right) = \frac{|\mathbf{KH}|m}{\left(g_1^{\beta_1},\dots,g_n^{\beta_n}\right)} = \frac{|\mathbf{KH}'|m}{\left(g_1^{\beta_1},\dots,g_n^{\beta_n}\right)} = \Pi_{fg}\left(\frac{m'}{\left(f_1^{\alpha'_1},\dots,f_n^{\alpha'_n}\right)} \right) .$$

Then Π_{gf} is well-defined. Similarly, we can prove that Π_{gf} is a homomorphism. Now, we show that $\left\{ M_f, \Pi_{fg} \mid f \le g \text{ and } f, g \in U \right\}$ is a direct limit on direct set U. If $f \in U$, It is clear that $\Pi_{ff} : M_f \to M_f$ is identity mapping. Let $f, g, h \in U$ such that $f \le g \le h$, we show that diagram



is commutative. Let $\dfrac{m}{\left(f_1^{\alpha_1},\dots,f_n^{\alpha_n}\right)} \in M_f$, $\mathbf{H},\mathbf{K},\mathbf{L} \in D_n(R)$, $\beta_1,\dots,\beta_n \in \square$, $\gamma_1,\dots,\gamma_n \in \square$ and $\mathrm{r}_1,\dots,\mathrm{r}_n \in \square$ such that

$$\left[g_1^{\beta_1}\dots g_n^{\beta_n}\right]^T \in \mathbf{K}\left[f_1^{\alpha_1}\dots f_n^{\alpha_n}\right]^T \quad \text{and} \quad \Pi_{gf}\left(\frac{m}{\left(f_1^{\alpha_1},\dots,f_n^{\alpha_n}\right)} \right) = \frac{|\mathbf{K}|m}{\left(g_1^{\beta_1},\dots,g_n^{\beta_n}\right)} \ ,$$

$$\left[h_1^{\gamma_1}\dots h_n^{\gamma_n}\right]^T \in \mathbf{L}\left[g_1^{\beta_1}\dots g_n^{\beta_n}\right]^T \quad \text{and} \quad \Pi_{hg}\left(\frac{|\mathbf{K}|m}{\left(g_1^{\beta_1},\dots,g_n^{\beta_n}\right)} \right) = \frac{|\mathbf{LK}|m}{\left(h_1^{\gamma_1},\dots,h_n^{\gamma_n}\right)} \ ,$$

$$\left[h_1^{\gamma_1}\dots, h_n^{\gamma_n}\right]^T \in \mathbf{H}\left[f_1^{\alpha_1}\dots f_n^{\alpha_n}\right]^T \quad \text{and} \quad \Pi_{hf}\left(\frac{m}{\left(f_1^{\alpha_1},\dots,f_n^{\alpha_n}\right)} \right) = \frac{|\mathbf{H}|m}{\left(h_1^{\gamma_1},\dots,h_n^{\gamma_n}\right)} \ ,$$

i.e., we have

$$\left[h_1^{\gamma_1}\ldots h_n^{\gamma_n}\right]^T \in \mathbf{LK}\left[f_1^{\alpha_1}\ldots f_n^{\alpha_n}\right]^T \cap \mathbf{H}\left[f_1^{\alpha_1}\ldots f_n^{\alpha_n}\right]^T,$$

$$\Pi_{hf}\left(\frac{m}{\left(f_1^{\alpha_1},\ldots,f_n^{\alpha_n}\right)}\right)=\frac{|\mathbf{LK}|m}{\left(h_1^{\gamma_1},\ldots,h_n^{\gamma_n}\right)}=\frac{|\mathbf{H}|m}{\left(h_1^{\gamma_1},\ldots,h_n^{\gamma_n}\right)},$$

$$\Pi_{hg}\circ\Pi_{gf}\left(\frac{m}{\left(f_1^{\alpha_1},\ldots,f_n^{\alpha_n}\right)}\right)=\Pi_{hf}\left(\frac{m}{\left(f_1^{\alpha_1},\ldots,f_n^{\alpha_n}\right)}\right).$$

Then, $\left\{M_f, \Pi_{fg} \mid f \le g \quad and \quad f, g \in U\right\}$?is a direct limit on direct set U. Therefore, there exists the direct limit M_f, i.e., $\varinjlim M_f$. □

Let X be the disjoint union $\cup M_f$, define an equivalence relation on X by $a_f \sim a_g$, $a_f \in M_f$, $a_g \in M_g$ if there exists an index $h \ge f, g$ with $\pi_{fh} a_f = \phi_{gh} a_g$. The equivalence class of a_f is denoted by $[a_f]$.Suppose that $X/\sim$ is the set of all equivalence classes. It isclear that $a_f \sim \pi_{fg} a_g$ for $g \ge f$ in $X/\sim$. Now, for $r \in A$ and $[a_f],[a_g] \in X/\sim$ we define

$$[a_f]\oplus[a_g]=\left\{[x] \,\middle|\, x \in a_h + a'_h \text{ where } a_h=\phi_h^f a_f,\ a'_h=\phi_h^g a_g \text{ for some } h \ge f, g\right\},$$

$$r\circ[a_f]=\left\{[x] \,\middle|\, x \in r\,a_f\right\}.$$

Lemma 3.2. [12] The hyperoperations $\oplus$ and $\circ$ are well-defined.

Lemma 3.3. $(X/\sim, \oplus)$ is a canonical hypergroup.

Proof. Suppose that $[a_f],[a'_g],[a''_h] \in X/\sim$ and $k \ge f, g, h$. Since M_k is associative, it follows that

$$\left(\pi_{fk}a_f+\pi_{gk}a'_g\right)+\pi_{hk}a''_h=\phi_{fk}a_f+\left(\pi_{gk}a'_g+\pi_{hk}a''_h\right).$$

So, we have

$$\left([a_f]+[a'_g]\right)+[a''_h]=[a_f]+\left([a'_g]+[a''_h]\right).$$

Therefore, $(X/\sim, \oplus)$ is associative. other conditions of canonical hypergroup can obtain easily. □

Proposition 3.4. $(X/\sim, \oplus, \circ)$ is hypermodule over R.

Proof. Suppose that $r_1, r_2 \in R$ and $[a_f], [a'_g] \in X/\sim$ and $k \geq f, g, h$. In the hypermodule M_k, we have

$$r_1\left(\left(\pi_{fk}a_f + \pi_{gk}a'_g\right)\right) = \left(r_1\pi_{fk}a_f + r_1\pi_{gk}a'_g\right),$$

$$\left(r_1 + r_2\right)\pi_{fk}a_f = r_1\pi_{fk}a_f + r_2\pi_{fk}a_f,$$

$$\left(r_1r_2\right)\pi_{fk}a_f = r_1\left(r_2\pi_{fk}a_f\right).$$

Therefore, we have

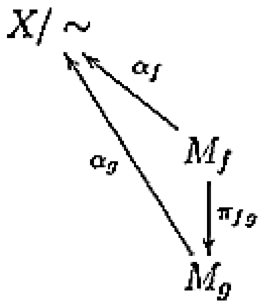
$$r_1 \circ \left(\left([a_f] \oplus [a'_g]\right)\right) = \left(r_1 \circ [a_f] + r_1 \circ [a'_g]\right),$$

$$\left(r_1 + r_2\right)[a_f] = r_1 \circ [a_f] + r_2 \circ [a_f],$$

$$\left(r_1r_2\right) \circ [a_f] = r_1 \circ \left(r_2 \circ [a_f]\right). \square$$

Proposition 3.5. Let $\left(M_f, \pi_{fg}\right)$ be a direct system of hypermodules indexed by I. Then, the hypermodule $U^{-n}M$ is $\varinjlim M_f$.

Proof. First, we show that hypermodule $X/\sim$ is $\varinjlim M_f$. We define $\alpha_f : M_i \to X/\sim$ given by $a_f \mapsto [a_f]$, and consider the following diagram



So, $\alpha_g\left(\pi_{fg}a_f\right) = \left[\pi_{fg}a_f\right] = [a_f] = \alpha_f\left(a_f\right)$. Therefore, $\alpha_g\pi_{fg} = \alpha_f$. Hence,the diagram is commutative. Now, let M be a hypermodule and $\left\{\varphi_f \mid \varphi_f : M_f \to M_f\right\}$ be a family of homomorphisms with $\varphi_f = \varphi_g\pi_{fg}$. Now, we define $\beta : \frac{X}{\sim} \to M$ by $[a_f] \mapsto \varphi_f a_f$. We show that β is a homomorphism and so the universal mapping property holds. First, we show that β is well-defined. Suppose that $[a_f] = [b_g]$. Then, there exists $h \geq f, g$ such that $\pi_{fg}a_f = \pi_{gh}b_g$. Hence,

$\varphi_h \pi_{f\,h} a_f = \varphi_h \pi_{g\,h} a_g$ and so $\varphi_f a_f = \varphi_g b_g$. Therefore, β is well-defined. Now, let $[a_f],[b_g] \in X/\,\square$ and $r \in R$, then

$$\beta([a_f] \oplus [b_g]) = \{\beta(x) \mid x \in a_h + a'_h \text{ where } a_h = \pi_{f\,h} a_f, a'_h = \pi_{g\,h} b_g \text{ for some } h \geq f, g\}$$

$$= \{\varphi_h x \mid x \in a_h + a'_h \text{ where } a_h = \pi_{f\,h} a_f, a'_h = \pi_{g\,h} b_g \text{ for some } h \geq f, g\}$$

$$= \varphi_h (a_h + a'_h) \text{ where } a_h = \pi_{f\,h} a_f, a'_h = \pi_{g\,h} b_g = \varphi_h a_h + \varphi_h a'_h$$

$$= \varphi_h \pi_{f\,h} a_f + \varphi_h \pi_{g\,h} b_g = \varphi_f a_f + \varphi_g b_g = \beta([a_f]) + \beta([b_g]),$$

and

$$\beta(r \circ [a_f]) = \beta([ra_f]) = \varphi_f (ra_f) = r\varphi_f (a_f) = r\beta([a_f]).$$

Therefore, β is a homomorphism and $\beta\alpha_f = \varphi_f$. Now, let L is the set of all equivalence classes. We show that $L \,\square\, U^{-n}M$. First, we prove that for every

$f \in U$ and $X_f \in M_f$, $[X_f] = \{X_f\}$ .Let $X_f = \frac{m}{(f_1^{\alpha_1},...,f_n^{\alpha_n})} \in M_f$,

$Y_g = \frac{m}{(g_1^{\beta_1},...,g_n^{\beta_n})} \in M_g$ and $X_f \,\square\, Y_g$. Then there exists $g, f \leq h \in U$ such that $\Pi_{hf}(X_f) = \Pi_{hg}(X_g)$, i.e., there is $\mathbf{H},\mathbf{K},\mathbf{H}',\mathbf{K}' \in D_n(R)$, $\gamma_1,\ldots,\gamma_n \in \square$ and $\gamma'_1,\ldots,\gamma'_n \in \square$ such that

$$[h_1 \ldots h_n]^T \in \mathbf{H}[f_1 \ldots f_n]^T \cap \mathbf{K}[g_1 \ldots g_n]^T,$$

$$[h_1^{\gamma_1} \ldots h_n^{\gamma_1}]^T \in \mathbf{H}'[f_1^{\alpha_1} \ldots f_n^{\alpha_n}]^T \text{ and } \Pi_{hf}\left(\frac{m}{(f_1^{\alpha_1},...,f_n^{\alpha_n})}\right) = \frac{|\mathbf{H}'|m}{(h_1^{\gamma_1},...,h_n^{\gamma_n})}, \tag{7}$$

$$[h_1^{\gamma'_1} \mathrm{K}\, h_n^{\gamma'_1}]^T \in \mathbf{K}'[g_1^{\beta_1} \mathrm{K}\, g_n^{\beta_n}]^T \text{ and } \Pi_{hg}\left(\frac{n}{(g_1^{\beta_1},\mathrm{K}, g_n^{\beta_n})}\right) = \frac{|\mathbf{K}'|m}{(h_1^{\gamma'_1},...,h_n^{\gamma'_n})}, \tag{8}$$

and

$$\frac{|\mathbf{H}'|m}{(h_1^{\alpha_1},...,h_n^{\alpha_1})} = \frac{|\mathbf{K}'|n}{(h_1^{\gamma'_1},...,h_n^{\gamma'_1})}.$$

Without loss of generality it can be assumed for $1 \leq \mathrm{i} < \mathrm{n}$, $\gamma_i = \gamma'_i$. Because of relations (7) and (9) we have

$$\emptyset \neq |\mathbf{H}'|m - |\mathbf{K}'|n \cap \left(\sum_{j=1}^{n-1} R\, h_j^{\gamma_j}\right)M \quad ,$$

and because of (7) and (8) we obtain

$$\left[h_1^{\gamma_1} \ldots h_n^{\gamma_n}\right]^T \in \mathbf{H}'\left[f_1^{\alpha_1} \ldots f_n^{\alpha_n}\right]^T \cap \mathbf{K}'\left[g_1^{\beta_1} \ldots g_n^{\beta_n}\right]^T .$$

Therefore,

$$X_f = \frac{m}{\left(f_1^{\alpha_1},\ldots,f_n^{\alpha_n}\right)} = \frac{n}{\left(g_1^{\beta_1},\ldots,g_n^{\beta_n}\right)} = Y_g \,,$$

so $\left[X_f\right] = \left\{X_f\right\}$. Let $\varphi : U^{-n}M \to L$ with $\dfrac{m}{u} \mapsto \left[\dfrac{m}{u}\right]$. It is clear that φ is one to one. If $r \in A$ and $\dfrac{m}{u}, \dfrac{m}{v} \in U^{-n}M$, then for $w \in U$ and $\mathbf{H},\mathbf{K} \in D_n(R)$ we have:

$$w^T \in \mathbf{H}u^T \cap \mathbf{K}v^T \quad \text{and} \quad \frac{m}{u} + \frac{n}{v} = \frac{|\mathbf{H}|m + |\mathbf{K}|n}{w}.$$

Hence

$$\varphi\left(\frac{m}{u} + \frac{n}{v}\right) = \varphi\left(\frac{|\mathbf{H}|m + |\mathbf{K}|n}{w}\right) = \left[\frac{|\mathbf{H}|m + |\mathbf{K}|n}{w}\right] = \left[\frac{|\mathbf{H}|m}{w} + \frac{|\mathbf{K}|n}{w}\right] =$$

$$\left[\Pi_{wu}\left(\frac{m}{u}\right) + \Pi_{wu}\left(\frac{n}{v}\right)\right] = \left[\frac{m}{u}\right] + \left[\frac{n}{v}\right] = \varphi\left(\frac{m}{u}\right) + \varphi\left(\frac{n}{v}\right).$$

and

$$\varphi\left(r\frac{m}{u}\right) = \left[\frac{rm}{u}\right] + r\left[\frac{m}{u}\right] = r\varphi\left(\frac{m}{u}\right).$$

Therefore, φ is a homomorphism. Clearly, φ is onto. □

4. THE FUNDAMENTAL RELATIONS γ^*, ε^* AND DIRECT SYSTEMS

Let M be a hypermodule over a hyperring R. The relation γ^* is the smallest equivalence relation on R such that the quotient $\frac{R}{\gamma^*}$ is a ring. The relation γ^* is called the fundamental equivalence relation on R and $\frac{R}{\gamma^*}$ is called the fundamental ring,see [22]. The fundamental relation ε^* on M over R is the smallest equivalence relation such that M/ε^* is a module over the ring $\frac{R}{\gamma^*}$ [23]. Denote ν the set of all expressions consisting of finite hyperoperations of either on R and M or the external hyperoperation applied on finite set of R and M. The relation ε can be defined on M whose transitive closure is the fundamental relation

ε^*. The relation ε is as follow:

$$x\varepsilon y \Leftrightarrow \{x, y\} \subseteq u \quad \text{for some } u \in \nu .$$

Suppose that $\gamma^*(r)$ is the equivalence class containing $r \in R$ and $\varepsilon^*(x)$ is the equivalence class containing $x \in M$. On M/ε^* the sum $\oplus$ and the product $\odot$ using the γ^* classes in R, are defined as follow:

$$\varepsilon^*(x) \oplus \varepsilon^*(y) = \varepsilon^*(c), \text{ for all } c \in \varepsilon^*(x) + \varepsilon^*(y),$$
$$\gamma^*(r) \odot \varepsilon^*(x) = \varepsilon^*(d), \text{ for all } d \in \gamma^*(r).\varepsilon^*(y).$$

Now, we will prove one proposition concerning the fundamental relations γ^* and ε^*. Let ε_u^* be the fundamental equivalence relation on $U^{-n}M$ and ν_u is the set of all expression consisting of finite hyperoperations of either on R and $U^{-n}M$ or of external hyperoperation. In this case $U^{-n}M/\varepsilon_u^*$ is ab R/γ^*-module.

Proposition 4.1. There is an R/γ^*-homomorphism $f : M/\varepsilon^* \to U^{-n}M/\varepsilon_u^*$.

Proof. We define $f\left(\varepsilon^*(m)\right) = \varepsilon_u^*\left(\frac{m}{1}\right)$. First we prove that f is well-defined. Suppose that $\left(\varepsilon^*(m_1)\right) = \left(\varepsilon^*(m_2)\right)$. So, $m_1\varepsilon^* m_2 . m_1 \varepsilon^* m_2$ if and only if there exist $x_1,\dots,x_{m+1}; u_1,\dots,u_m \in \nu$ with $x_1 = m_1$ and $x_{m+1} = m_2$ such that $\{x_i, x_{i+1}\} \subseteq u$, $i = 1,\dots,m$ which implies that $\left\{\frac{x_i}{1}, \frac{x_{i+1}}{1}\right\} \subseteq \nu_u$. Therefore, we have $\frac{m_1}{1}\varepsilon_u^*\frac{m_2}{1}$ and so $\varepsilon_u^*\left(\frac{m_1}{1}\right) = \varepsilon_u^*\left(\frac{m_2}{1}\right)$. Thus, f is well-defined. Moreover, f is a homomorphism because

$$f\left(\varepsilon^*(a) \oplus \varepsilon^*(b)\right) = f\left(\varepsilon^*(c)\right) = \varepsilon_u^*\left(\frac{c}{1}\right), \text{ for all } c \in \varepsilon^*(a) + \varepsilon^*(b),$$

$$f\left(\varepsilon^*(a) \otimes \varepsilon^*(b)\right) = \varepsilon_u^*\left(\frac{a}{1}\right) \otimes \varepsilon_u^*\left(\frac{b}{1}\right) = \varepsilon_u^*\left(\frac{d}{s}\right), \text{ for all } \frac{d}{s} \in \varepsilon_u^*\left(\frac{a}{1}\right) \times \varepsilon_u^*\left(\frac{b}{1}\right),$$

By setting $d = c \in a + b$ and $u = 1$, we obtain $f\left(\varepsilon^*(a) \oplus \varepsilon^*(b)\right) = f\left(\varepsilon^*(a) \otimes \varepsilon^*(b)\right)$. In addition, we have

$$f\left(\gamma^*(r)\square\ \varepsilon^*(m)\right)=f\left(\varepsilon^*(rm)\right)=\varepsilon_u^*\left(\frac{a}{1}\right),\ \text{for all}\quad a\in rm\,,$$
$$\gamma^*(r)\lozenge f\left(\varepsilon^*(m)\right)=\gamma^*(r)\lozenge\varepsilon_u^*\left(\frac{m}{1}\right)=\varepsilon_u^*\left(\frac{b}{1}\right),\ \text{for all}\quad b\in rm\,.$$

Therefore, we obtain $f\left(\gamma^*(r)\square\ \varepsilon^*(m)\right)=\gamma^*(r)\lozenge f\left(\varepsilon^*(m)\right)$. Therefore, f is a homomorphism of modules. □

Proposition 4.2. Let $\left(M_f\ ,\Pi_{fg}\right)$ be a direct system of hypermodules over a hyperring R indexed by a direct set I .Then $\left(M_i\ /\ \varepsilon_{f\ M_f}\ ,\Pi_{fg}\right)$ is a direct system of modules over the ring $R\ /\ \gamma^*$, where for $m\in M$,$\left(f_1^{\alpha_1},\ldots,f_n^{\alpha_n}\right)\in U_f$,

$$\Pi_{f\ g}^*:M_f\ /\ \varepsilon_{f\ M_f}^*\ \rightarrow M_g\ /\ \varepsilon_{g\ M_g}^*$$
$$\varepsilon_{f\ M_f}^*\left(\frac{m}{\left(f_1^{\alpha_1},\ldots,f_n^{\alpha_n}\right)}\right)\mapsto\varepsilon_{g\ M_g}^*\left(\Pi_{f\ g}\left(\frac{m}{\left(f_1^{\alpha_1},\ldots,f_n^{\alpha_n}\right)}\right)\right).$$

Proof. $\left(M_i\ /\ \varepsilon_{f\ M_f}^*\ ,\Pi_{fg}\right)$ is a family of $R\ /\ \gamma^*$ -modules and $R\ /\ \gamma^*$ -homomorphisms. It is clear that $\Pi_{f\ f}^*$ is the identity for all $f\in I$. Now, for $f\le g\le h$, we have

$$\left(\pi_{gh}\pi_{fg}\right)^*\left(\varepsilon^*_{fM_f}\left(\frac{m}{\left(f_1^{\alpha_1},\ldots,f_n^{\alpha_n}\right)}\right)\right)=\pi^*_{fh}\left(\varepsilon^*_{fM_f}\left(\frac{m}{\left(f_1^{\alpha_1},\ldots,f_n^{\alpha_n}\right)}\right)\right)$$
$$=\varepsilon^*_{kM_k}\left(\pi_{fh}\left(\frac{m}{\left(f_1^{\alpha_1},\ldots,f_n^{\alpha_n}\right)}\right)\right)$$
$$=\varepsilon^*_{kM_k}\left(\left(\pi_{gh}\pi_{fg}\right)\left(\frac{m}{\left(f_1^{\alpha_1},\ldots,f_n^{\alpha_n}\right)}\right)\right)$$
$$=\varepsilon^*_{kM_k}\left(\pi_{gh}\left(\pi_{fg}\frac{m}{\left(f_1^{\alpha_1},\ldots,f_n^{\alpha_n}\right)}\right)\right)$$
$$=\pi^*_{gh}\left(\varepsilon^*_{gM_g}\left(\pi_{fg}\frac{m}{\left(f_1^{\alpha_1},\ldots,f_n^{\alpha_n}\right)}\right)\right)$$
$$=\pi^*_{gh}\pi^*_{fg}\left(\varepsilon^*_{fM_f}\left(\frac{m}{\left(f_1^{\alpha_1},\ldots,f_n^{\alpha_n}\right)}\right)\right).$$

Therefore $\left(\pi_{gh}\pi_{fg}\right)^*=\pi^*_{fh}=\pi^*_{gh}\pi^*_{fg}$. □

Proposition 4.3. Let $\left[m,\left(f_1^{\alpha_1},\ldots,f_n^{\alpha_n}\right)\right],\left[n,\left(g_1^{\beta_1},\ldots,g_n^{\beta_n}\right)\right]\in X/\sim$, for $m,n\in M$, $\left(f_1^{\alpha_1},\ldots,f_n^{\alpha_n}\right)\in U_f$, $\left(g_1^{\beta_1},\ldots,g_n^{\beta_n}\right)\in U_g$, $\alpha_1,\ldots,\alpha_n\in\mathbb{N}$ and $\beta_1,\ldots,\beta_n\in\mathbb{N}$ as described in Proposition 3.4, we define

$$\left[m,\left(f_1^{\alpha_1},\ldots,f_n^{\alpha_n}\right)\right]\theta\left[n,\left(g_1^{\beta_1},\ldots,g_n^{\beta_n}\right)\right],$$

if there exists $h\geq f,g$ such that $\pi_{fh}\left(\left(\frac{m}{\left(f_1^{\alpha_1},\ldots,f_n^{\alpha_n}\right)}\right)\right)\varepsilon_{kM_k}\,\pi_{gh}\left(\left(\frac{n}{\left(g_1^{\beta_1},\ldots,g_n^{\beta_n}\right)}\right)\right).$

Then $\theta=\varepsilon_{X/\sim}$

Proof. The proof is similar to the proof of Proposition 3.2 in [12]. □

Proposition 4.4. Let $\left(M_f,\Pi_{fg}\right)$ be a direct system of hypermodules over a Krasner hypermodule R indexed by a direct set I , and let ε^* be the fundamental relation of $\lim_{\rightarrow}M_f$. Then

$$\varinjlim \left(M_f / \varepsilon^*_{f_{M_f}} \right) \cong_{R/\gamma^*} \left(\varinjlim M_f \right) / \varepsilon^* .$$

Proof. The proof is similar to the proof of Theorem 3.3 in [12]. □

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