Sigma J Eng & Nat Sci 9 (1), 2018, 23-29

Publications Prepared for the 13th Algebraic Hyperstructures and its Applications Conferences AHA 2017

Research Article

2-ABSORBING δ-PRIMARY FUZZY IDEALS OF COMMUTATIVE RINGS

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Received: 12.10.2017 Accepted: 29.10.2017

ABSTRACT

In this work, we define 2-absorbing δ-primary fuzzy ideals which is the generalization of 2-absorbing fuzzy ideal and 2-absorbing primary fuzzy ideals. Furthermore, we give some fundamental results concerning these notions.

Keywords: 2-absorbing fuzzy ideals, 2-absorbing δ-primary fuzzy ideals.

1. INTRODUCTION

The notion of fuzzy subset is defined by Zadeh in[9] and studied to apply fuzzy theory on algebraic structures. Then many researchers have studied on fuzzy ring theory. Liu [4] investigated the concept of fuzzy ideal of a ring. In fuzzy commutative algebra, prime ideals are the most significant structures. Mukherjee and Sen defined prime and primary fuzzy ideals in [5,6]. Also in [7] Sidky and Khatab studied on the notion of nil radical of fuzzy ideals.

The concept of 2-absorbing ideal, which is a generalization of prime ideal, was introduced by Badawi in [1]. At present, work on the 2-absorbing ideal theory is progresssing rapidly. It has studied extensively by many authors. Recently, Badawi et al. [2] introduced the concept of 2-absorbing primary ideals in commutative rings with $1 \neq 0$ and gave some characterizations related to it. Also in [8], Sönmez et al. introduced the notion of 2-absorbing primary fuzzy ideal, which is a generalization of prime and primary fuzzy ideals in commutative rings. Furthermore in [11], Zhao introduced the concept of expansion of ideals of R. In this paper, we introduce the notion of 2-absorbing δ-primary fuzzy ideal which is a generalization of 2-absorbing primary fuzzy ideal. Also recall that from [8], a nonconstant fuzzy ideal is said to be *2-absorbing primary* if if for any fuzzy points x_r, y_s, z_t of R, $x_r y_s z_t \in \mu$ implies that either $x_r y_s \in \mu$ or $x_r z_t \in \sqrt{\mu}$

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or $y_s z_t \in \sqrt{\mu}$. Based on this definition, a nonconstant fuzzy ideal μ of R is said to be a *2-absorbing* $\delta-$*primary fuzzy ideal* of R if for any fuzzy points x_r, y_s, z_t of R, $x_r y_s z_t \in \mu$ implies that either $x_r y_s \in \mu$ or $x_r z_t \in \delta(\mu)$ or $y_s z_t \in \delta(\mu)$.

2. PREMILINARIES

We assume throughout that all rings are commutative with $1 \neq 0$. Unless stated otherwise $L = [0,1]$ stands for a complete lattice . Let I be a proper ideal of R, the set

$\{I : I$ is an ideal of R $\}$ will be denoted by $I(R)$. Z denotes the ring of integers, $L(R)$ denotes the set of fuzzy sets of R and $LI(R)$ denotes the set of fuzzy ideals of R. For $\mu, \xi \in L(R)$, we say $\mu \subseteq \xi$ if and only if $\mu(x) \leq \xi(x)$ for all $x \in R$. When $r \in L$, $x, y \in R$ we define $x_r \in L(R)$ as follows :

$$x_r(y) = \begin{cases} r & x = y, \\ 0 & otherwise, \end{cases}$$

and x_r is referred to as fuzzy point of R.

Also, for $\mu \in L(R)$ and $t \in L$, define μ_t as follows :

$$\mu_t = \{x \in R : \mu(x) \geq t\}$$

Definition 2.1 [4] A fuzzy subset μ of a ring R is called a fuzzy ideal of R if $x, y \in R$ the following conditions are satisfied :

- $\mu(x-y) \geq \mu(x) \wedge \mu(y)$, $\forall x, y \in R$
- $\mu(xy) \geq \mu(x) \vee \mu(y)$, $\forall x, y \in R$

Let μ be any fuzzy ideal of R; $x, y \in R$, and 0 be the additive identity of R. Then it is easy to verify the following:

(i) $\mu(0) \geq \mu(x)$, $\mu(x) = \mu(-x)$ and $\mu_t \subset \mu_s$ where $s, t \in Im(\mu)$ and $t > s$.

(ii) If $\mu(0) = \mu(x-y)$, then $\mu(x) = \mu(y)$, $\mu(x) = s$ iff $x \in \mu_s$ and $x \notin \mu_t, \forall t > s$.

Definition 2.2 [3] Let μ be any fuzzy ideal of R. The ideal μ_t, ($\mu(0) \geq t$) is called level ideal of μ.

Definition 2.3 [6] A fuzzy ideal μ of R is called a prime fuzzy ideal if for any two fuzzy points x_r, y_s of R , $x_r y_s \in \mu$ implies either $x_r \in \mu$ or $y_s \in \mu$.

Definition 2.4 [5] Let μ be a fuzzy ideal of R. Then $\sqrt{\mu}$, called the radical of μ, is defined by $\sqrt{\mu}(x) = \bigvee_{n \geq 1} \mu(x^n)$.

Definition 2.5 [5] A fuzz y ideal μ of R is called a primary fuzzy ideal if for $x, y \in R$, $\mu(xy) > \mu(x)$ implies $\mu(xy) \leq \mu(y^n)$ for some positive integer n.

Theorem 2.6 [5] Let μ be a fuzzy ideal of a ring R. Then $\sqrt{\mu}$ is a fuzzy ideal of R.

Definition 2.7 [6] Let R be a ring. Then a nonconstant fuzzy ideal μ is said to be a weakly completely prime fuzzy ideal iff for $x, y \in R$, $\mu(xy) = max\{\mu(x), \mu(y)\}$.

Theorem 2.8 [7] If μ and ξ are two fuzzy ideals of R, then $\sqrt{\mu \cap \xi} = \sqrt{\mu} \cap \sqrt{\xi}$.

Theorem 2.9 [7] Let $f : R \to S$ be a ring homomorphism and let μ be a fuzzy ideal of R such that μ is constant on $Kerf$ and ξ be a fuzzy ideal of S. Then ,

- $\sqrt{f(\mu)} = f(\sqrt{\mu})$,
- $\sqrt{f^{-1}(\xi)} = f^{-1}(\sqrt{\xi})$.

Definition 2.10 [1] A nonzero proper ideal I of a commutative ring R with $1 \neq 0$ is called a 2-absorbing ideal if whenever $a, b, c \in R$ with $abc \in I$, then either $ab \in I$ or $ac \in I$ or $bc \in I$.

Definition 2.11 [2] A proper ideal I of R is called a 2-absorbing primary ideal of R if whenever $a, b, c \in R$ with $abc \in I$, then either $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$.

Theorem 2.12 [2] If I is a 2-absorbing primary ideal of R, then $\sqrt{I}$ is a 2-absorbing ideal of R.

Definition 2.13 [8] A nonconstant fuzzy ideal is called a 2-absorbing primary fuzzy ideal if if for any fuzzy points x_r, y_s, z_t of R, $x_r y_s z_t \in \mu$ implies that either $x_r y_s \in \mu$ or $x_r z_t \in \sqrt{\mu}$ or $y_s z_t \in \sqrt{\mu}$

Definition 2.14 [11] A function $\delta : I(R) \to I(R)$ is called an expansion function of ideals of R if whenever P, Q are ideals of R with $P \subseteq Q$, then $P \subseteq \delta(P)$ and $\delta(P) \subseteq \delta(Q)$

Definition 2.15 [11] A proper ideal I of R is said to be a δ -primary ideal of R if $xy \in I$ implies that $x \in I$ or $y \in \delta(I)$ for any $x, y \in R.$

Theorem 2.16 [11] Let δ_0 be an identity function, where $\delta(I) = I$ for every $I \in Id(R).$ Then an ideal I is δ_0 -primary ideal if and only if I is a prime ideal.

Theorem 2.17 [11] Let δ_1 be a function, which is defined $\delta_1(I) = \sqrt{I}$ for every $I \in Id(R)$. Then an ideal I is a δ_1 -primary if and only if it is primary ideal.

3. 2-ABSORBING $\delta-$PRIMARY FUZZY IDEALS

Definition 3.1 Let μ be a nonconstant fuzzy ideal of R. Then μ is called a 2-absorbing $\delta-$primary fuzzy ideal of R if for any fuzzy points x_r, y_s, z_t of R, $x_r y_s z_t \in \mu$ implies that either $x_r y_s \in \mu$ or $x_r z_t \in \delta(\mu)$ or $y_s z_t \in \delta(\mu)$.

Example 3.2 Let δ_0 be identity function where $\delta_0(\mu) = \mu$ for every $\mu \in LI(R)$. Then μ is a 2-absorbing δ_0-primary fuzzy ideal if and only if it is a 2-absorbing fuzzy ideal.

Example 3.3 Let δ_1 be a function such that δ_1 is defined by $\delta_1(\mu) = \sqrt{\mu}$, the fuzzy radical of fuzzy ideal μ. Then μ is a 2-absorbing δ_1-primary fuzzy ideal if and only if it is a 2-absorbing primary fuzzy ideal.

Theorem 3.4 Let μ be a fuzzy ideal of R. Assume that δ is a fuzzy ideal expansion and ideal expansion of R such that $\delta(\mu_t) = \delta(\mu)_t$ for $t \in [0, \mu(0)]$. If μ is a 2-absorbing δ-primary fuzzy ideal, then the level ideal μ_t, $t \in [0, \mu(0)]$, is a 2-absorbing δ-primary ideal of R.

Proof Assume that μ is a 2-absorbing δ-primary fuzzy ideal of R such that $\delta(\mu_t) = \delta(\mu)_t$. Let $xyz \in \mu_t$ where $t \in [0, \mu(0)]$. Then $x_t y_t z_t \in \mu$. Since μ is a 2-absorbing δ-primary fuzzy ideal of R, then we have $x_t y_t \in \mu$ or $x_t z_t \in \delta(\mu)$ or $y_t z_t \in \delta(\mu)$. Thus we conclude that $xy \in \mu_t$ or $xz \in \delta(\mu)_t = \delta(\mu_t)$ or $yz \in \delta(\mu)_t = \delta(\mu_t)$. Hence, μ_t is a 2-absorbing δ-primary ideal of R.

Theorem 3.5 Let γ and δ be two fuzzy ideal expansions of R. If $\delta(\mu) \subseteq \gamma(\mu)$ for each fuzzy ideal μ, then every 2-absorbing δ-primary fuzzy ideal of R is also 2-absorbing γ-primary fuzzy ideal of R. Thus, in particular, a 2-absorbing fuzzy ideal of R is a 2-absorbing δ-primary fuzzy ideal for every fuzzy ideal expansion δ.

Proof Assume that μ is a 2-absorbing δ-primary fuzzy ideal of R. We show that μ is a 2-absorbing γ-primary fuzzy ideal of R. Let $x_r y_s z_t \in \mu$. Since μ is a 2-absorbing δ-primary fuzzy ideal and $\delta(\mu) \subseteq \gamma(\mu)$, then we conclude that $x_r y_s \in \mu$ or $x_r z_t \in \delta(\mu) \subseteq \gamma(\mu)$ or $y_s z_t \in \delta(\mu) \subseteq \gamma(\mu)$. Hence μ is a 2-absorbing γ-primary fuzzy ideal.

For each δ fuzzy ideal expansion, if μ is a 2-absorbing fuzzy ideal, then we have μ is a 2-absorbing δ-primary fuzzy ideal by the definition of 2-absorbing fuzzy ideals.

Definition 3.6 A fuzzy ideal expansion δ is called intersection preserving if it is satisfies $\delta(\mu \cap \xi) = \delta(\mu) \cap \delta(\xi)$ for any $\mu, \xi \in \mathrm{LI(R)}$.

Definition 3.7 An expansion is said to be global if for any ring homomorphism $f : R \to S$, $\delta(f^{-1}(\mu)) = f^{-1}(\delta(\mu))$ for all $\mu \in \mathrm{LI(S)}$.

Note that the expansions δ_0 and δ_1 are both intersection preserving and global.

Theorem 3.8 Let δ be an intersection preserving fuzzy ideal expansion of R. If $\mu_1, \dots \mu_n$ are 2-absorbing δ -primary fuzzy ideals of R and $\xi = \delta(\mu_i)$ for all i, then

$$\mu = \bigcap_{i=1}^{n} \mu_i$$

is a δ -primary fuzzy ideal.

Proof Suppose that $x_r y_s z_t \in \mu$ and $x_r y_s \notin \mu$. Then $x_r y_s \notin \mu_j$ for some $n \geq j \geq 1$ and $x_r y_s z_t \in \mu_j$ for all $n \geq j \geq 1$. Since μ_j is a 2-absorbing δ -primary fuzzy ideal, then we have

$$y_s z_t \in \delta(\mu_j) = \bigcap_{i=1}^{n} \delta(\mu_i) = \delta(\bigcap_{i=1}^{n} \mu_i) = \delta(\mu)$$

or

$$x_r z_t \in \delta(\mu_j) = \bigcap_{i=1}^{n} \delta(\mu_i) = \delta(\bigcap_{i=1}^{n} \mu_i) = \delta(\mu).$$

Thus μ is a 2-absorbing δ -primary ideal of R.

Theorem 3.9 If δ is a global and $f : R \to S$ is a ring homomorphism, then for any 2-absorbing δ -primary fuzzy ideal μ of S, $f^{-1}(\mu)$ is a 2-absorbing δ -primary fuzzy ideal of R.

Proof Assume that $x_r y_s z_t \in f^{-1}(\mu)$ where x_r, y_s, z_t are any fuzzy points of R. Then $r \wedge s \wedge t \leq f^{-1}(\mu)(xyz) = \mu(f(xyz)) = \mu(f(x) f(y) f(z))$.

Let $f(x) = a, \ f(y) = b, f(z) = c \in S$. Thus we get that $r \wedge s \wedge t \leq \mu(abc)$ and $a_r b_s c_t \in \mu$. Since μ is a 2-absorbing primary fuzzy ideal, then we get

$$a_r b_s \in \mu \text{ or } a_r c_t \in \delta(\mu) \text{ or } b_s c_t \in \delta(\mu).$$

If $a_r b_s \in \mu$, then we have

$$r \wedge s \leq \mu(ab) = \mu(f(x) f(y)) = \mu(f(xy)) = f^{-1}(\mu)(xy).$$

Hence we conclude that $x_r y_s \in f^{-1}(\mu)$.

If $a_r c_t \in \delta(\mu)$, then we get

$$r \wedge t \leq \delta(\mu)(ac) = \delta(\mu)(f(x) f(z)) = \delta(\mu) f(xz) = f^{-1}(\delta(\mu))(xz).$$

Since δ is a global, then we have

$$r \wedge t \leq \delta(f^{-1})(xz) \text{ so } x_r z_t \in \delta(f^{-1}(\mu)).$$

By the similar way, it can be see that $y_s z_t \in \delta(f^{-1}(\mu))$.

Theorem 3.10 Let $f : R \to S$ be a surjective ring homomorphism and let δ be a global fuzzy ideal expansion. Then a fuzzy ideal μ of R which is constant on $Kerf$ is a 2-absorbing δ-primary if and only if $f(\mu)$ is a 2-absorbing δ-primary fuzzy ideal of S.

Proof Suppose that $a_r b_s c_t \in f(\mu)$ where a_r, b_s, c_t are any fuzzy points of S. Since f is a surjective ring homomorphism, we have there exist $x, y, z \in R$ such that $f(x) = a, f(y) = b, f(z) = c$. Thus, $a_r b_s c_t(abc) = r \wedge s \wedge t \leq f(\mu)(abc) = f(\mu)(f(x)f(y)f(z)) = f(\mu)(f(xyz)) = \mu(xyz)$ because μ is constant on $Kerf$. Then we get $x_r y_s z_t \in \mu$. Since μ is a 2-absorbing δ-primary fuzzy ideal, we conclude $x_r y_s \in \mu$ or $x_r z_t \in \delta(\mu)$ or $y_s z_t \in \delta(\mu)$. Thus,

$$r \wedge s \leq \mu(xy) = f(\mu)(f(xy)) = f(\mu)(f(x)f(y)) = f(\mu)(ab) \text{ so}$$
$$a_r b_s \in f(\mu) \text{ or}$$
$$r \wedge t \leq \delta(\mu)(xz) = \delta(f(\mu))(f(xz)) = \delta(f(\mu)f(x)f(z)) = \delta(f(\mu))(ac) \text{ so}$$
$$a_r c_t \in \delta(f(\mu))$$

By the similar way it is easy to see that $b_s c_t \in \delta(f(\mu))$ if $y_s z_t \in \delta(\mu)$.

Theorem 3.11 Let μ be a fuzzy ideal of R. If $\delta(\mu)$ is a prime fuzzy ideal of R, then μ is a 2-absorbing primary fuzzy ideal of R.

Proof Assume that $x_r y_s z_t \in \mu$ and $x_r y_s \notin \mu$ for any $x, y, z \in R$ and $r, s, t \in [0,1]$. Since $x_r y_s z_t \in \mu$ and R is a commutative ring then

$$x_r y_s z_t z_t = (x_r z_t)(y_s z_t) \in \mu \subseteq \delta(\mu).$$

Thus $x_r z_t \in \delta(\mu)$ or $y_s z_t \in \delta(\mu)$ since $\delta(\mu)$ is a prime fuzzy ideal of R. Hence, we conclude that μ is a 2-absorbing primary fuzzy ideal of R.

4. CONCLUSION

In this work the theoretical point of view of 2-absorbing primary fuzzy ideals is discussed. We introduce the definition of 2-absorbing δ-primary fuzzy ideal which is a generalization of 2-absorbing primary fuzzy ideal. Furthermore, under a ring homomorphism, these ideals are investigated. In order to extend this study, one could study other algebraic structures and do some further study on properties them.

Acknowledgement

This work was supported by Research Fund of the Yildiz Technical University. Project Number: 868.

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