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Research Article

THE EMBEDDING OF AN ORDERED SEMIHYPERGROUP IN TERMS OF FUZZY SETS

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ABSTRACT

In this paper we have investigated an embedding theorem of ordered semihypergroups in terms of fuzzy sets. We prove that an ordered semihypergroup R is embedded in the set $F(R)$ of all fuzzy subsets of R, which is an *poe*-semigroup with the ordered relation and the multiplication and addition defined in this paper.
Keywords: Semihypergroup, ordered semihypergroup, fuzzy sets.

1. INTRODUCTION AND PRELIMINARIES

Algebraic hyperstructures are a suitable generalization of classical algebraic structures. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. The concept of hyperstructure was first introduced by Marty [18] at the eighth Congress of Scandinavian Mathematicians in 1934. A comprehensive review of the theory of hyperstructures can be found in [3, 4, 12, 7, 23]. Let S be a non-empty set and $P^*(S)$ be the family of all non-empty subsets of S. A mapping $\circ: S \times S \to P^*(S)$ is called a *hyperoperation* on S. A *hypergroupoid* is a set S together with a (binary) hyperoperation. In the above definition, if A and B are two non-empty subsets of S and $x \in S$, then we denote

$$A \circ B = \bigcup_{\substack{a\in A \\ b\in B}} a \circ b,\ x \circ A = \{x\} \circ A \text{ and } B \circ x = B \circ \{x\}.$$

A hypergroupoid $(S,\circ)$ is called a *semihypergroup* if for every x, y, z in S, $x \circ (y \circ z) = (x \circ y) \circ z$. That is,

$$\bigcup_{u\in y\circ z} x \circ u = \bigcup_{v\in x\circ y} v \circ z.$$

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A non-empty subset K of a semihypergroup S is called a *subsemihypergroup* of S if $K \circ K \subseteq K$. A hypergroupoid $(S,\circ)$ is called a *quasihypergroup* if for every $x \in S$, $x \circ S = S = S \circ x$. This condition is called the reproduction axiom. The couple $(S,\circ)$ is called a *hypergroup* if it is a semihypergroup and a quasihypergroup. A non-empty subset K of S is a *subhypergroup* of S if $K \circ a = a \circ K = K$, for every $a \in K$. A hypergroup $(S,\circ)$ is called *commutative* if $x \circ y = y \circ x$, for every $x, y \in S$.

Bakhshi and Borzooei [1] introduced the notion of ordered polygroups. Heidari and Davvaz introduced and studied ordered semihypergroup in [13]. Chvalina [5] and Hort [15] used ordered structures for the construction of hypergroups. Chvalina [5] started the concept of ordered semihypergroups as a special class of hypergroups in 1994. The concept of ordered semihypergroups is a generalization of the concept of ordered semigroups. Several authors have recently studied different aspects of ordered semihypergroups, for instance, Changphas and Davvaz [2], Davvaz et al. [6], Gu and Tang [14], Tang et al. [21], and many others. By an ordered semihypergroup, we mean an algebraic hyperstructure $(S,\circ,\leq)$, which satisfies the following conditions: (1) $(S,\circ)$ is a semihypergroup together with a partial order $\leq$; (2) If x, y and z are elements of S such that $x \leq y$, then $z \circ x \leq z \circ y$ and $x \circ z \leq y \circ z$. Here, $z \circ x \leq z \circ y$ means for any $a \in z \circ x$ there exists $b \in z \circ y$ such that $a \leq b$. The case $x \circ z \leq y \circ z$ is defined similarly.

Let $(R,\circ,\leq_R)$ and $(T,\circ',\leq_T)$ be two ordered semihypergroups. The map $\varphi: R \to T$ is called a homomorphism if for all $x, y \in R$, the following conditions hold:

1. $\varphi(x \circ y) \subseteq \varphi(x) \circ' \varphi(y)$,
2. φ is isotone, that is, $x \leq_R y$ implies $\varphi(x) \leq_T \varphi(y)$.

Also, φ is called a good homomorphism if in the previous conditions (1) and (2), the equality is valid. Note that if $(R,\circ,\leq_R)$ and $(T,\circ',\leq_T)$ are two ordered semigroups, then the notions of homomorphism and good homomorphisms coincide.

The map φ is called reverse isotone if $x, y \in R, f(x) \leq_T f(y)$ implies $x \leq_R y$. It is clear that each reverse isotone mapping is one-to-one. In fact: let $x, y \in R, f(x) = f(y)$. Since $f(x) \leq_T f(y)$, we have $x \leq_R y$. Since $f(y) \leq_T f(x)$, we have $y \leq_R x$. The map φ is called an isomorphism if it is onto, good homomorphism and reverse isotone.

The concept of a fuzzy set, introduced by Zadeh in his classic paper [25], provides a natural framework for generalizing some of the notions of classical algebraic structures and of abstract set theory. Fuzzy semigroups have been first considered by Kuroki [17]. After the introduction of the concept of fuzzy sets by Zadeh, several researches conducted the researches on the generalizations of the notions of fuzzy sets with huge applications in computer, logics and many branches of pure and applied mathematics. Fuzzy set theory has been shown to be a useful tool to describe situations in which the data are imprecise or vague. Fuzzy sets handle such situations by attributing a degree to which a certain object belongs to a set. In 1971, Rosenfeld [22] defined the concept of fuzzy group. Since then many papers have been published in the field of fuzzy algebra. Recently fuzzy set theory has been well developed in the context of algebraic hyperstructure theory. In [8], Davvaz introduced the concept of fuzzy hyperideals in a semihypergroup. In 2009, Davvaz [9] gave the concept of fuzzy hyperideals in ternary semihyperrings. The relationships between the fuzzy sets and algebraic hyperstructures have been considered by Corsini, Davvaz, Leoreanu, Zhan, Zahedi, Ameri, Cristea and many other researchers. More on fuzzy hyperstructures one can find in [10] and [11].

Let S be a non-empty set. A mapping $f: S \to [0,1]$ is called a fuzzy subset of the universe S. If f is not a constant function, then f is called a non-empty fuzzy subset. Let $F(S)$ denote the set of all fuzzy sets in S.

There are various kinds of embedding theorem of different algebraic structures in others. In [16] and [24], some embedding theorems of ordered semigroups and semihypergroups respectively, in terms of fuzzy sets are considered. In this paper we have investigated an embedding theorem of ordered semihypergroups in terms of fuzzy sets. We prove that an ordered semihypergroup R is embedded in the set $F(R)$ of all fuzzy subsets of R, which is *poe*-semigroup with the ordered relation and the multiplication defined in this paper.

2. MAIN RESULT

Definition 2.1 *Let $(R,\circ,\leq_R)$ and $(T,\circ',\leq_T)$ be two ordered semihypergroups. $(R,\circ,\leq_R)$ is called embedded in $(T,\circ',\leq_T)$ if there exists a mapping $\phi: R\to T$ which is a good homomorphism and reverse isotone.*

Let we prove the following theorem.

Theorem 2.2 *Let $(R,\circ,\leq_R)$ be an ordered semihypergroup. Then the set $F(R)$ of all fuzzy subsets of R with the multiplication $*$ and the order relation $\subseteq$ on $F(R)$ defined as follows:*

$$*: F\times F\to F$$
$$(f,g)\mapsto f*g$$

where $f*g$ is the mapping of R into $[0,1]$ defined by

$$(f*g)(z)=\begin{cases}\sup\limits_{z\leq x\circ y}\{\min\{f(x),g(y)\}\}\\ 0 \quad \text{otherwise}\end{cases}$$
$$f\preccurlyeq g \Leftrightarrow f(x)\leq g(x), \forall x\in R,$$

is an poe-semigroup and R is embedded in $F(R)$.

Proof. We remark first the set $F(R)$ is non-empty set. One can easily prove that the operation $*$ on $F(R)$ is well-defined. We consider the ordered relation $\preccurlyeq$ on $F(R)$ defined by $f\preccurlyeq g\Leftrightarrow f(x)\leq g(x),\forall x\in R$, then $(F(R),*,\preccurlyeq)$ is an *poe*-semigroup. Indeed:

1. We show that $(F(R),*)$ is a semigroup, that is, $*$ is associative on $F(R)$. For any $z\in R$, $f,g,h\in F(R)$, we have to show that

$$((f*g)*h)(z)=(f*(g*h))(z).$$

We have the following cases:

A. If there exist $x,y\in R$ such that $z\leq x\circ y$, then

$$((f*g)*h)(z)=\sup_{z\leq x\circ y}\{\min\{(f*g)(x),h(y)\}\},$$
$$(f*(g*h))(z)=\sup_{z\leq x\circ y}\{\min\{f(x),(g*h)(y)\}\}.$$

We take

$$t=\sup_{z\leq x\circ y}\{\min\{(f*g)(x),h(y)\}\}$$
$$s=\sup_{z\leq x\circ y}\{\min\{f(x),(g*h)(y)\}\}.$$

Let we prove that $t\geq \min\{f(x),(g*h)(y)\}$ for every $x,y\in R$ such that $z\leq x\circ y$. Then we have

$$t\geq \sup_{z\leq x\circ y}\{\min\{f(x),(g*h)(y)\}\}=s$$

Similarly, we prove that $t\leq s$ and thus $t=s$.

Let $c,d\in R, z\leq c\circ d$. Then $t\geq \min\{f(c),(g*h)(d)\}$. Indeed:

(a). Let $t\geq f(c)$. Then, since $f(c)\geq \min\{f(c),(g*h)(d)\}$, we have $t\geq \min\{f(c),(g*h)(d)\}$.

(b). Let $t<f(c)$.

(b1) If there do not exist $d_1,d_2\in R$ such that $d\leq d_1\circ d_2$, then $g*h)(d)=0$. Since f is a fuzzy set in R, we have $f(c)\geq 0$. Then $\min\{f(c),(g*h)(d)\}=0$. Since $t\geq 0$, we have $t\geq \min\{f(c),(g*h)(d)\}$.

(b2) If there exist $d_1, d_2 \in R$ such that $d \leq d_\circ d_2$, we prove that $t \geq \min\{g(d_1), h(d_2)\}$ for every $d_1, d_2 \in R$ such that $d \leq d_1 \circ d_2$. Then we have

$$t \geq \sup_{d \leq d_1 \circ d_2} \{\min\{g(d_1), h(d_2)\}\} = (g * h)(d).$$

Then, since $(g * h)(d) \geq \min\{f(c), (g * h)(d)\}$, we have $t \geq \min\{f(c), (g * h)(d)\}$. Let $d_1, d_2, \in R, d \leq d_1 \circ d_2$. Then $t \geq \min\{g(d_1), h(d_2)\}$. Indeed:

(i) If $t \geq g(d_1)$, then since $g(d_1) \geq \min\{g(d_1), h(d_2)\}$, we have $t \geq \min\{g(d_1), h(d_2)\}$.

(ii) If $t < g(d_1)$, since $z \leq c \circ d, d \leq d_1 \circ d_2$, then $z \leq c \circ (d_1 \circ d_2) = (c \circ d_1) \circ d_2$. Thus $t \geq \sup_{k \leq c \circ d_1} \{\min\{(f * g)(k), h(d_2)\}\}$. On the other hand $(f * g)(k) \geq \min\{f(c), g(d_1)\}$. Since $t < f(c)$ and $t < g(d_1)$, we have $t < min\{f(c), g(d_1)\}$. Then $(f * g)(k) > t$. Thus $h(d_2) \leq t$. Otherwise, if $t < h(d_2)$, then $\min\{(f * g)(k), h(d2)\} > t$. Thus $\sup_{k \leq c \circ d_1} \{\min\{(f * g)(k), h(d_2)\}\} > t$. It is impossible.

B. If there do not exist $x, y \in R$ such that $z \leq x \circ y$, then $((f * g) * h)(z) = f * (g * h)(z) = 0$.

2. Let we show now that the ordered relation ° on $F(R)$ is compatible with the operation $*$. In other words, if $f, g, h \in F(R), f \preccurlyeq g$, then $f * h \preccurlyeq g * h$, $h * f \preccurlyeq h * g$. To show $f * h \preccurlyeq g * h$, we only show that $(f * h)(z) \leq (g * h)(z), \forall z \in R$. For this, we have the following cases:

A. If there are not $x, y \in R$ such that $z \leq x \circ y$, then $(f * h)(z) = 0, (g * h)(z) = 0$, and $(f * h)(z) \leq (g * h)(z), \forall z \in R$.

B. If there are $x, y \in R$ such that $z \leq x \circ y$, then

$$(f * h)(z) = \sup_{z \leq x \circ y} \{\min\{f(x), h(y)\}\},$$
$$(g * h)(z) = \sup_{z \leq x \circ y} \{\min\{g(x), h(y)\}\}.$$

We take

$$t = \sup_{z \leq x \circ y} \{\min\{f(x), h(y)\}\}$$
$$s = \sup_{z \leq x \circ y} \{\min\{g(x), h(y)\}\}.$$

We will prove that $\min\{f(x), h(y)\} \leq t, \forall x, y \in R, z \leq x \circ y$. Then we have $\sup_{z \leq x \circ y} \{\min\{f(x), h(y)\}\} \leq t$ and thus $s \leq t$.

Let $x, y \in R, z \leq x \circ y$. Then $\min\{f(x), h(y)\} \leq t$. Indeed: By hypothesis, $t \geq \min\{g(x), h(y)\}$, then $t \geq g(x)$ or $t \geq h(y)$.

If $t \geq g(x)$, then since $f \preccurlyeq g$, we get $f(x) \leq g(x)$. Since $f(x) \geq \min\{f(x), h(y)\}$, we get $t \geq \min\{f(x), h(y)\}$.

If $t \geq h(y)$, then since $h(y) \geq \min\{f(x), h(y)\}$, we get $t \geq \min\{f(x), h(y)\}$.

In similar way, we prove that $(h * f)(x) \leq (h * g)(x), \forall x \in R$, that is $h * f \preccurlyeq h * g$.

3. Clearly the mapping $1_F: R \to [0,1] | x \mapsto 1_F(x) = 1$ is the greatest element of $F(R)$.

Until now, we have proved that $(F(R), *, \preccurlyeq)$ is an ordered semigroup.

4. We will prove now that $(R, \circ, \leq)$ is embedded in $(F(R), *, \preccurlyeq)$. Indeed:

Let A be a nonempty subset of R and f_A is defined by

$$f_A(x) = \begin{cases} 1, & \text{if } x \leq A \\ 0, & \text{if } x \nleq A \end{cases}$$

which is called the characteristic function of A.

Let we consider the mapping $\phi: R \to F(R) | x \mapsto f_{\{x\}}$.

(1) It is clear that ϕ is well-defined and ϕ is one-to-one.

(2) We will prove that ϕ is a good homomorphism.

Let $a, b \in R, x \in R$. Then $f_{a \circ b}(x) = (f_a * f_b)(x)$. Indeed:

For this, we have the following cases:

A. If there are not $c, d \in R$ such that $x \leq c \circ d$, then $(f_a * f_b)(x) = 0$. Since $y \nleq c \circ d$, we have $f_{a \circ b}(x) = 0$.

B. If there are $c, d \in R$ such that $x \leq c \circ d$, then $(f_a * f_b)(x) = \sup_{x \leq c \circ d} \{\min\{f_a(c), f_b(d)\}\}$. Let $t = \sup_{x \leq c \circ d} \{\min\{f_a(c), f_b(d)\}\}$.

b1. Let $x \leq a \circ b$. Then $f_{a \circ b}(x) = 1$. On the other hand, $s \leq \min\{f_a(a), f_b(b)\} = 1$. Since $s \in [0,1], s \leq 1$, we have $s = 1$, and $f_{a \circ b}(x) = (f_a * f_b)(x) = 1$.

b2. Let $x \nleq a \circ b$. Then $f_{a \circ b}(x) = 0$. We will prove that

$$\min\{f_a(c), f_b(d)\} = 0, \forall c, d \in R, x \leq c \circ d.$$

Then $\sup_{x \leq c \circ d} \{\min\{f_a(c), f_b(d)\}\} = 0$ and $f_{a \circ b}(x) = (f_a * f_b)(x) = 0$.

For this, let $c, d \in R, x \leq c \circ d$. If $f_a(c) \neq 0$ and $f_b(d) \neq 0$, then $f_a(c) = f_b(d) = 1, c = a, b = d$ and $x \leq a \circ b$, which is impossible.

Thus we have $f_a(c) = 0$ or $f_b(d) = 0$. If $f_a(c) = 0$, then, since $f_b(d) \geq 0$, we have $\min\{f_a(c), f_b(d)\} = 0$. If $f_b(d) = 0$, then, since $f_a(c) \geq 0$, we have $\min\{f_a(c), f_b(d)\} = 0$.

Now let we prove that ϕ is reverse isotone. That is, if $a, b \in R, f_a \preccurlyeq f_b$, then $a \leq b$. Indeed: We have that $1 = f_a(a) \leq f_b(a)$. Since $f_b \in F$, then $f_b(a) \leq 1$. Therefore we have $f_b(a) = 1$ and $a \leq b$.

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