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Research Article

SOME RESULTS ON DELTA–PRIMARY SUBMODULES OF MODULES

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ABSTRACT

In this paper we investigate δ-primary submodules which unify prime submodules and primary submodules. Our motivation is to extend the concept of δ-primary ideals into δ-primary submodules of modules over commutative rings. A number of main results about prime and primary submodules are extended into this general framework.
Keywords: Expansion of submodules δ-primary submodules, multiplication modules.

1. INTRODUCTION

Throughout this paper all rings will be commutative with non-zero identity and all modules will be unitary. In [3], δ-primary ideals have been investigated by Zhao Dongsheng. In this paper, Z. Dongsheng extented a number of main results about prime ideals and primary ideals. In this study, our motivation is to extend the concept of δ-primary ideals into δ-primary submodules of modules over commutative rings. Then various properties of δ-primary submodules are considered in our paper.

Now we define the concepts that we will use. If R is a ring and N is a submodule of an R-module M, the ideal $\{r \in R | rM \subseteq N\}$ will be denoted by $(N: M)$.

An expansion of ideals, or briefly an ideal expansion is a function δ_R which assigns to each ideal I of a ring R to another ideal $\delta_R(I)$ of the same rings such the following conditions are satisfied: (i): $I \subseteq \delta_R(I)$, (ii): $P \subseteq Q$ implies $\delta_R(P) \subseteq \delta_R(Q)$. [see, 3]

A submodule N of M is called prime if $N \neq M$ and whenever $r \in R$, $m \in M$, and $rm \in N$, then $m \in N$ or $r \in (N: M)$. A submodule $N$of M is called primary if $N \neq M$and whenever $r \in R$, $m \in M$, and $rm \in N$, then $m \in N$ or $r^n \in (N: M)$ for some positive integer n. In recent years, prime and primary submodules have attracted a good deal of attentions. [see, 2-5].

In this study, firstly we introduce a new concept "δ-primary submodule" which is defined as follow: Let R be a ring, M be an R-module and N be a submodule of M. A submodule $N(\neq M)$ of M is called δ-prim ary if $rm \in N, m \notin N \Longrightarrow r \in \delta_R((N: M))$. Then we have numerous results as

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following: If we get a collection of δ-primary submodules, the union of the collection is δ-primary submodule. Moreover, under multiplication module assumption, we obtain some results as followings: If N is δ-primary, then $(N:M)$ is δ_R-primary [see, Lemma 2.4]. Under some special conditions, we characterize δ-primary submodule, i.e., N is δ-primary submodule if and only if for any submodules N_1 and N_2 of M , if $N_1N_2 \subseteq N$ and $N_1 \nsubseteq N$, then $N_2 \subseteq \delta(N)$ [see, Theorem 2.2]. As [3, Theorem 2.5], we obtain that N is a δ-primary submodule of M if and only if every zero divisor of M/N is δ-nilpotent [see, Theorem 2.5]. Finally, under special conditions, we show that a module homomorphism can preserve the concept of δ-primary submodule, i.e., N is a δ-primary submodule of M if and only if the homomorphic image of N is δ-primary submodule [see Proposition 2.2].

2. EXPANSION OF SUBMODULES AND δ-PRIMARY SUB-MODULES

Definition 2 1 Given an expansion of δ_R of ideals, an ideal I of R is called δ_R-primary if for every $a, b \in R, ab \in I$ and $a \notin I \Longrightarrow b \in \delta_R(I)$ or if $ab \in I$ and $b \notin I \Longrightarrow a \in \delta_R(I)$.

Definition 2 2 Let N be a submodule of an R-module M such that $N \neq M$. N is called δ-primary if if $rm \in N, m \notin N \Longrightarrow r \in \delta_R((N:M))$ or if $rm \in N, r \notin \delta_R((N:M)) \Longrightarrow m \in N$for all $r \in R, m \in M$.

Example 2.3

1. Let $\delta_R(I) = I$ which is an expansion of ideals be a function of ideals of R.
A submodule N is δ-primary if and only if it is prime.

2. Let $\delta_R(I) = \sqrt{I}$ which is an expansion of ideals be a function of ideals of R. A submodule N is δ-primary if and only if it is primary.

Proposition 2.4 1. Let M be an R-module. If δ_R and γ_R are two ideal expansions and $\delta_R((N:M)) \subseteq \gamma_R((N:M))$ for each submodule N, then every δ_R-primary submodule is also γ_R-primary submodule.

2. Let M be an R-module and $\{N_i | i \in \lambda\}$ be a directed collection of δ-primary submodule of M, then $N = \bigcup_{i\in\lambda} N_i$ is δ -primary submodule.

Proof 1. Let N be a δ_R-primary submodule of M. Assume that $rm \in N, m \notin N$ where $r \in R$, $m \in M$. Then $r \in \delta_R((N:M)) \subseteq \gamma_R((N:M))$ since N is a δ_R-primary submodule. So N is a γ_R-primary.

2. It is clear that N is a submodule of M. We must indicate that it is δ-primary. Let $rm \in N, r \notin \delta_R((N:M))$. Then there is a submodule N_i such that $rm \in N_i, r \notin \delta_R((N_i:M))$ for some $i \in \lambda$. Then $m \in N_i$ and so $m \in N$. Thus N is δ-primary submodule.

Hence the set of all δ-primary submodules is a direct complete poset with respect to the inclusion order. Generally, the intersection of two δ-primary submodules is not a δ-primary since the intersection of two δ_R-primary ideals is not δ_R-primary.

Lemma 2.5 Let Nbe a submodule of an R-module M such that $N \neq M$. If N is a δ-primary, then $(N:M)$ is δ_R-primary.

Proof Suppose $ab \in (N:M)$ and $a \notin (N:M)$ where $a, b \in R$. Then $abM \subseteq N$ and $aM \nsubseteq N$. Thus there exists $m \in M$ such that $abm \in N$ and $am \notin N$. Since N is δ-primary, we have $b \in \delta_R\big((N:M)\big)$. Consequently, $(N:M)$ is a δ_R-primary ideal of R.

Lemma 2.6 (see [3, Lemma 1.8]) An ideal P is δ_R-primary if and only if for any two ideals I and J, if $IJ \subseteq P$ and $I \nsubseteq P,$ then $J \subseteq \delta_R(P)$.

Lemma 2.7 Let N be a submodule of M with $N \neq M$. Then N is δ-primary if and only if for any ideal I of R and for any submodule N' of Mif $IN' \subseteq N$ and $N' \nsubseteq N$, then $I \subseteq \delta_R((N:M))$.

Proof Let N be δ-primary. Suppose $IN' \subseteq N$ and $N' \nsubseteq N$. Let $a \in I$. There exists $n' \in N' \backslash N$such that $an' \in IN' \subseteq N$. Since N is δ-primary, then we have $a \in \delta_R\big((N:M)\big)$. Hence $I \subseteq \delta_R((N:M))$. Conversely, suppose that $rn' \in N, n' \notin N$. Therefore $(r)(n') \subseteq N$ and $(n') \nsubseteq N$. Hence $r \in (r) \subseteq \delta_R((N:M))$. Consequently, N is δ-primary.

Definition 2.8 Let R be a ring and M be an R-module. M is called multiplication module if for every submodule N of M there exists an ideal I of R such that $N = IM$.

Lemma 2.9 Let R be a ring, M be a multiplication R-module and N be a submodule of M such that $N \neq M$. N is δ-primary if and only if $(N:M)$ is δ_R-primary.

Proof Suppose that N is δ-primary. By Lemma 2.5, $(N:M)$ is δ_R-primary. Conversely, suppose that $(N:M)$ is δ_R-primary. Assume if $IN' \subseteq N$ and $N' \nsubseteq N$, for any submodule N' of M and for any ideal I of R. Since M is a multiplication R-module, then there exists an ideal J of R such that $N' = JM$. Thus $IJM \subseteq N$ implies $IJ \subseteq (N:M)$. Since $(N:M)$ is δ_R-primary and $J \subseteq (N:M)$, we have $I \subseteq \delta_R((N:M))$. Hence by Lemma 2.7, we conclude that N is δ-primary.

Theorem 2.9 Let R be a ring, M be an R-module and N be a submodule of Msuch that $N \neq M$.

1. If N is a δ-primary and I is an ideal with $I \nsubseteq \delta_R((N:M))$, then $(N:I) = N$ where $(N:I) = \{m \in M | mI \subseteq N\}$is an R-module.
2. For any δ-primary submodule N' and any subset X of M, $(N:X)$ is δ_R-primary where $(N':X) = \{r \in R | rX \subseteq N'\}$ is a δ-primary.

Proof

1. Clearly $N \subseteq (N:I)$. On the other hand, $(N:I)I \subseteq N$. Since N is δ-primary, by the hypothesis $I \nsubseteq \delta_R((N:M))$ we have $(N:I)I \subseteq N$. Hence $(N:I)I = N$.
2. Suppose $ab \in (N':X)$ for any two elements $a, b \in R$, and $a \notin (N':X)$. Thus there exists $n \in X$ such that $abn \in N'$ and $an \notin N'$. Since N is δ-primary, then $b \in \delta_R\big((N':M)\big)$. Furthermore $(N:M) \subseteq (N:X)$ implies $\delta_R((N:M)) \subseteq \delta_R(\,(N:X))$. This implies $b \in \delta_R\big((N':X)\big)$. Hence $(N':X)$ is δ_R-primary.

Definition 2.10 An ideal expansion δ_R is intersection preserving if it satisfies

$$\delta_R(I \cap J) = \delta_R(I) \cap \delta_R(J)$$

for any ideals I and J in R.

Lemma 2.11 Let δ_R be an intersection preserving ideal expansion. If $Q'_1, Q'_2, \ldots, Q'_n$ are δ-primary submodules of M and $\delta_R((Q'_i:M)) = P'$ for all i, then $Q' = \bigcap_{i=1}^{n} Q'_i$ is δ-primary.

Proof Suppose that $rm \in Q', m \notin Q'$. Then there exists k such that $rm \in Q'_k, m \notin Q'_k$. Since Q'_k is δ-primary, then $r \in \delta_R\big((Q'_k:M)\big) = P'$. Since δ_R is an intersection preserving ideal expansion and $(Q':M) = (\bigcap_{i=1}^{n} Q'_i:M) = \bigcap_{i=1}^{n}(Q'_i:M)$, then we have $\delta_R((Q':M)) = \delta_R((\bigcap_{i=1}^{n} Q'_i:M)) = \bigcap_{i=1}^{n} \delta_R((Q'_i:M)) = P'$. Thus $r \in \delta_R((Q':M))$. Hence Q' is δ-primary.

Definition 2.12 An expansion δ_R is said to be global if for any ring homomorphism $f: R \longrightarrow S$, $\delta_R(f^{-1}(I)) = f^{-1}(\delta_R(I))$ for all ideal I of S.

Definition 2.13 Let M be an R-module. An expansion δ is a function that assings to each submodule $N$of M to another submodule $\delta(N)$ of M.

Definition 2.14 Let R be a ring and M be a multiplication R-module. An expansion δ is multiplication preserving if it satisfies $\delta_R(I)M = \delta(IM)$ for any ideal I of R.

Definition 2.15 Let R be a ring and M be a multiplication R-module. An expansion δ is quotient preserving if it satisfies $\delta((N:M)) = \delta_R((N:M))$ for any submodule N of M such that $N \neq M$.

Definition 2.16 Let M be a multiplication R-module and let N and K be submodules of M such that $N = IM$ and $K = JM$ for some ideal I and J of R. The product of N and K is denoted by NK

and is defined by IJM. For $m, m' \in M$, by mm', we mean the product of Rm and Rm', which is equal to IJM for every presentation ideals I and J of m and m', respectively.

Theorem 2.17 Let R be a ring, M be a multiplication R-module and N be a submodule of M such that $N \neq M$. Let δ be a quotient and multiplication pre- serving expansion. Then N is a δ-primary if and only if for any two submodules N_1 and N_2, if $N_1N_2 \subseteq N$ and $N_1 \nsubseteq N$, then $N_2 \subseteq \delta(N)$.

Proof Suppose that N is a δ-primary submodule of M. Let $N_1N_2 \subseteq N$ and $N_1 \nsubseteq N$ for any submodules N_1 and N_2 of M. Since M is a multiplication R-module, there exist ideals J_1 and J_2 such that $N_1 = J_1M$ and $N_2 = J_2M$. As $N_1 \nsubseteq N$, then $J_1 \nsubseteq (N_1 : M)$. Since $(N: M)$ is R-primary, $N_1N_2 = J_1J_2M \subseteq N$ and $J_1J_2 \subseteq (N: M)$, it follows that $J_2 \subseteq \delta_R((N: M))$ Then $J_2M \subseteq \delta_R((N: M))M$. Since δ is multiplication preserving, then we have $N_2 = J_2M \subseteq \delta_R\big((N: M)\big)M = \delta(N)$.

Conversely, suppose that N' is a submodule of M and I is an ideal of R such that$IN' \subseteq N, N' \nsubseteq N$. Since M is a multiplication R-module, there exists an ideal J such that $N' = JM$. Then $IN' = IJM = (IM)(JM) \subseteq N$. Therefore $IM \subseteq \delta(N)$ by hypothesis. Thus $I \subseteq ((\delta(N): M))$. Hence, $I \subseteq \delta_R((N : M))$. Consequently, N is δ-primary.

Corollary 2.18 Let R be a ring, M be a multiplication R-module and N be a submodule of M such that $N \neq M$. Let δ be a quotient and multiplication preserving expansion. Then N is a δ-primary if and only if $mm' \subseteq N$ and $m \nsubseteq N$, then $m' \subseteq \delta(N)$ for any $m, m' \in M$.

Proof Let N be a δ-primary. The necessary part is clear from Theorem 2.17. For the sufficient part, suppose that $N_1N_2 \subseteq N$ and $N_1 \nsubseteq N$ for any submodules N_1 and N_2 of M. Let $m' \in N_2$. Then there exists $m \in N_1 \backslash N$ such that $mm' \subseteq N_1N_2 \subseteq N$. Therefore, by assumption $m' \in \delta(N)$. Consequently, $N_2 \subseteq \delta(N)$ and so N is δ-primary.

Definition 2.19 An element of a ring R is called δ_R-nilpotent if $a \in \delta_R(\{0_R\})$.

Theorem 2.20 (see, [3, Theorem 2.5]) Let δ_R be a global expansion. An ideal I of R is δ_R-primary if and only if every zero divisor of the quotient ring R/I is δ_R-nilpotent.

Theorem 2.21 Let δ_R be a global expansion and M be a multiplication R- module. Let N be a submodule of Msuch that $N \neq M$. A submodule N is δ_R-primary if and only if every zero divisor of R/J where $J = (N: M)$ is δ_R-nilpotent.

Proof N is a δ-primary submodule of M if and only if $(N: M)$ is a δ_R-primary by Lemma 2.9. Thus $(N: M)$ is δ_R-primary if and only if $R/(N: M)$ is δ_R-nilpotent by Theorem 2.20.

Definition 2.22 Let R be a ring and M be a multiplication R-module and N be a submodule of M. Then,

1. N is called nilpotent if $N^k = 0$ for some positive integer k, where N^k means the product of N, k times;
2. An element $m \in M$ is called nilpotent if $m^k = 0$ for some positive integer k.

Definition 2.23 An element m of a multiplication R-module M is called δ-nilpotent if $m \in \delta(\{0_M\})$.

Definition 2.24 Let M be a multiplication R-module. A zero divisor in M is an element $0_M \neq a \in M$ for which there exists $b \in M$ with $b \neq 0_M$ such that $ab = RaRb = 0_M$.

Definition 2.25 An expansion δ is said to be global-homomorphism if for any module homomorphism $f: M \longrightarrow M'$, $\delta(f^{-1}(N)) = f^{-1}(\delta(N))$ for all submodule N of M'.

Theorem 2.26 Let R be a ring, M be a multiplication R-module and N be a submodule of M such that $N \neq M$. Let δ be a global-homomorphism, quotient and multiplication preserving expansion. Then N is δ-primary if and only if every zero divisor of M/N is δ-nilpotent.

Proof Let N be a δ-primary submodule. If $\widetilde{m} = m + N$ is a zero divisor, then there is a $\tilde{s} = s + N \neq N$ with $\widetilde{m}\tilde{s} = ms + N = N$. This means that $ms \in N, s \notin N$. By the assumption, N is δ-

primary, so $m \in \delta(N)$, that is, $\widetilde{m} \in \delta(N)/N$. Let $q: M \longrightarrow M/N$ be natural quotient homomorphism. As δ is a global-homomorphism expansion, we have:

$$\delta(N) = \delta(q^{-1}(\{0_{M/N}\})) = q^{-1}(\delta(\{0_{M/N}\})).$$

As q is onto, so $\delta(N)/N = q(\delta(N)) = \delta(\{0_{M/N}\})$. Hence we get $\widetilde{m} \in \delta(\{0_{M/N}\}))$, i.e. $\widetilde{m}$ is δ-nilpotent.

Conversely, suppose every zero divisor of M/N is δ-nilpotent. Let $m, n \in M$ with $mn \in N$ and $m \notin N$. Then $\widetilde{m}\tilde{n} = 0_{M/N}$ and $\widetilde{m} \neq 0_{M/N}$. So $\tilde{n}$ is zero divisor element of M/N. By the assumption, $\tilde{n} \in \delta(\{0_{M/N}\}) = \delta(N)/N$. Then there is an $n' \in \delta(N)$ such that $n - n' \in N$. So $n - n'$ is in $\delta(N)$ also. It follows that $n = (n - n') + n' \in \delta(N)$. Hence N is δ-primary.

Lemma 2.27 Let M and M' be multiplication R-module and $f: M \longrightarrow M'$ be a surjective module homomorphism. Let δ be a global-homomorphism, quotient and multiplication preserving expansion. Then $f^{-1}(N)$ is δ-primary submodule of M for any δ-primary submodule N of M'.

Proof Assume that $N_1N_2 \subseteq f^{-1}(N)$ and $N_2 \nsubseteq f^{-1}(N)$ for any submodules N_1 and N_2 of M. Since M is a multiplication R-module, there exist ideals I and J such that $N_1 = IM$ and $N_2 = JM$. By hypothesis $(IM)(JM) = (IJ)M \subseteq f^{-1}(N)$ and $JM \nsubseteq f^{-1}(N)$, it follows that $f((IJ)M) \subseteq N$ and $f(JM) \nsubseteq N$, as f is surjective. Then $IJf(M) \subseteq N$ and $Jf(M) \nsubseteq N$, that is, $IJM' \subseteq N$ and $JM' \nsubseteq N$. Since N is δ-primary, then $IM' \subseteq \delta(N)$ and so $f(IM) \subseteq \delta(N)$. Thus $IM \subseteq f^{-1}\big(\delta(N)\big) = \delta(f^{-1}(N))$ since δ is a global-homomorphism. Consequently, $f^{-1}(N)$ is δ-primary submodule of M.

Proposition 2.28 Let M and M' be multiplication R-module, N be a submodule of M that contains $ker(f)$ and $f: M \longrightarrow M'$ be a surjective module homomorphism. Let δ be a global-homomorphism, quotient and multiplication preserving expansion. Then N is δ-primary if and only if $f(N)$ is δ-primary.

Proof ($\Longleftarrow$): Let $f(N)$ be a δ-primary submodule of M. Since N contains $ker(f), f^{-1}(f(N)) = N$ and N is δ-primary by Lemma 2.27.

($\Longrightarrow$): Let N be a δ-primary submodule of M. Suppose that $m_1m_2 \subseteq f(N)$ and $m_2 \notin f(N)$ for any $m_1, m_2 \in M'$. Consider presentation ideals I_1 and I_2 of m_1 and m_2, respectively. Then $m_1m_2 = (I_1I_2)M' \subseteq f(N)$, since f is surjective, $(I_1I_2)M = (I_1M)(I_2M) \subseteq N$ and $I_2M \nsubseteq N$. By hypothesis, $I_1M \subseteq \delta(N)$. Then it follows that $f(I_1M) = I_1f(M) = I_1M' \subseteq f\big(\delta(N)\big)$, that is, $m_1 \in f(\delta(N))$. Now, we must prove that $f\big(\delta(N)\big) = \delta(f(N))$. Since f is surjective, then $\delta(N) = \delta(f^{-1}\big(f(N)\big)) = f^{-1}(\delta(f(N)))$, so it is proved and $m_2 \in \delta(f(N))$.

Corollary 2.29 Let M be a multiplication R-modul, K and N be two submodules of M such that $N \subseteq K$ and δ be a global-homomorphism, quotient and multiplication preserving expansion. Then K/N is a δ-primary submodule of M/N iff K is a δ-primary submodule of M.

Proof It is obvious from Lemma 2.27 and Proposition 2.28.

As conclusion, under special conditions, (such as multiplication module, quotient-multiplication preserving expansion and global-homomorphism) we obtain some results as followings:

We characterize δ-primary submodule, i.e. N is δ-primary submodule if and only if for any two submodules N_1 and N_2, if $N_1N_2 \subseteq N$ and $N_1 \nsubseteq N$, then $N_2 \subseteq \delta(N)$ [See, Theorem 2.17]. Then, we get that N is δ-primary if and only if every zero divisor of M/N is δ-nilpotent [See, Theorem 2.26]. Finally, we obtain that a module homomorphism can preserve the concept δ-primary submodule, i.e. N is δ-primary if and only if the homomorphic image N is δ-primary [See, Proposition 2.28].

REFERENCES

[1] Ameri R., (2003) On the Prime Submodules of Multiplication Modules, Inter. J.of Math. and Math. Sci., Hindawi Publishing Corp., 27, 1715-1724.

[2] Barnard A., (1981) Multiplication Modules, J. Algebra 71(1), 174–178.

[3] Dongsheng Z., (2001) δ-primary Ideals of Commutative Rings, Kyungpook Math. J., 41, 17-22.

[4] Oral K. H., Tekir U. and Agargun A. G., (2011) On Graded Prime and Primary Submodules, Turk J. Math., 35, 159 – 167.

[5] Tekir U., (2006) A Note on Multiplication Modules, Inter. J. of Pure and Appl. Math., 27(1), 103-107.

[6] Tekir U., (2007) On Multiplication Modules, International Mathematical Forum, 29, 1415-1420.

[7] J. Nezhad R. and Naderi M. H., (2009) On Prime and Semiprime Submodules of Multiplication Modules, International Mathematical Forum, 4(26), 1257 - 1266.

[8] Nader M. H. and J. Nezhad R., (2009) Weak Primary Submodules of Multiplication Modules and Intersection Theorem, Int. J. Contemp. Math. Sciences, 4(33), 1645 - 1652.

[9] Khaksari A., (2011) Weakly Pure Submodules of Multiplication Modules, International Journal of Algebra, 5(5), 247 - 250.