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Research Article

DESIGNING A RESPONSE APPROACH IN CHAOTIC SYSTEMS

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ABSTRACT

In this paper, we propose a generalized method by designing new response systems for solving synchronization problems of coupled chaotic identical and non-identical dynamical systems. We extend our study by considering two new techniques for constructing a chaotic synchronization between two identical or non-identical dynamical systems. The first one is based on the classical Lyapunov stability theory. The proposed method is analyzed by means of equilibrium points, eigenvalue structures, and Lyapunov functions. The second one requires the nonlinear part of the response system to be smooth enough and uses the expansion of such a function. The designed controller functions enable the state variables of the drive system to globally synchronize with the state variables of the response system in both methods. The global convergence of the proposed methods have been discussed by giving two theoretical results. To show the effectiveness and feasibility of those approaches, various numerical simulations have been carried out.
Keywords: Dynamical systems, synchronization, chaotic system, stabilization, Lyapunov theory, numerical analysis.

1. INTRODUCTION

Chaotic synchronization is an interesting phenomenon characterizing many processes in natural systems. The original work on synchronization was introduced in coupled pendulum by Huygens [6] and has been extensively studied in the last two decades. Pecora and Carroll [12] proposed that synchronization can be observed even in chaotic systems and also stated that the behavior of synchronization is appearing when the distance between the corresponding states converges to zero as time tends to infinity. After this discovery, periodic synchronization has attracted attention of many researchers in biological science, chemical reaction, ecological systems, secure communication and so on. Several types of synchronization were discovered such as: Identical or complete synchronization appears as the coincidence of states of interacting systems, phase synchronization which means the phases of chaotic oscillators in a closely controlled phase relationship, lag synchronization appears as having a parameter mismatch in

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mutually coupled chaotic oscillator. This type of lag synchronizations has important technological implications in engineering systems.

The problem of chaotic synchronization is related to trajectories starting arbitrarily and close to each other as the time tends to infinity. Identical synchronization of two chaotic systems may occur when the systems are coupled or when one chaotic system drives another chaotic system [14,13]. However, many real systems are in general non-identical due to the parameters of two coupled systems do not match, or the coupled systems belong to different classes. So, the possibility of the transformation between drive and response dynamical variables these include the generalized synchronization can be very complicated. This issue may pose a trouble in practical application of synchronized chaos. In the literature, a large number of researchers have extensively concentrated on the identical synchronization, the generalized synchronization [15], active control methods such as adaptive control, feedback control, sliding mode control, adaptive lag synchronization for chaotic system [20], impulsive control and fuzzy control and so on [19, 11, 4].

In this paper, we propose to analyze a synchronization of a coupled chaotic identical and non-identical dynamical systems producing generalized synchronization in drive-response systems. Thus, we have investigated general methods to detect the existence of the transformation and study this kind of synchronous behavior. In the case of the drive-response methods, efforts to a systematic method that guides the development of solutions to synchronization problems, when trajectories of driving and response systems are strongly connected, then two close states in the state space of the response system correspond to two close states in the space of the driving system. Here, we consider two approaches for constructing chaotic unidirectional synchronization between the two systems. The systems are either both identical or both non-identical or each one different from the other. First, we apply the classical Lyapunov stability theory in synchronization of real systems. Secondly, we study a case when the nonlinear part of response system is required to be smooth enough. Then, we use the expansion of such a function to establish the global synchronization of the chaotic dynamical systems. We present that, these techniques can be implemented directly to any experiments and does not require mutual feedback.

This paper is organized as follows. In Section 2, we give a general approach for a synchronization problem. We describe a generalized synchronization for constructing response systems. In Section 3, we apply the proposed method to three standard chaotic systems. The first one concerns two identical Memristor systems. The second is related to two non-identical systems which are Lorenz and Rossler systems. The last one presents the behavior of synchronization in biological dynamic systems between two identical neural systems.

2. GENERALIZED SYNCHRONIZATION OF CHAOTIC DYNAMICAL SYSTEMS

Pecora and Carroll [12] introduced a method for studying a chaotic system of oscillators. They split the system into two subsystems. The first one is the driver system and the second one is the response system and may be given in the following form:

$$\begin{cases} \dot{x}(t) = F\big(x(t)\big) & \text{driver,} \\ \dot{y}(t) = G\big(y(t)\big) + u\big(x(t), y(t)\big) & \text{response,} \end{cases} \tag{1}$$

where the functions $F: \mathbb{R}^n \to \mathbb{R}^n$ and $G: \mathbb{R}^m \to \mathbb{R}^m$ are continuous vector valued functions. The vector $x(t) \in \mathbb{R}^n$ represents the driving signal and $y(t) \in \mathbb{R}^m$ represents the response signal. The functions F and G may be written as sum of linear and nonlinear parts as

$$F(x(t)) = A\,x(t) + f\big(x(t)\big) \text{ and } G\big(y(t)\big) = B\,y(t) + g\big(y(t)\big),$$

where the matrices A and B of size $n \times n$ and $m \times m$, respectively, are assumed to consist of constants and the functions $f: \mathbb{R}^n \to \mathbb{R}^n$ and $g: \mathbb{R}^m \to \mathbb{R}^m$ represent the nonlinear parts of F and G, respectively. The function $u: \mathbb{R}^k \to \mathbb{R}^m$ is a controller function, here $k = n + m$. Some of the outputs from the driver system are used to drive the response system. This means that, there exists a relation between the two coupled systems, which could be a smooth function $\Phi: \mathbb{R}^n \to \mathbb{R}^m$, action transforms the trajectories on the attractor of the first system into those on the attractor of the second system. We assume that the driver system in (1) is unstable at their equilibrium points. It is suitable to introduce the error system e given by

$$e(t) = y(t) - \Phi\big(x(t)\big).$$

Definition 2.1: System (1) is generalized synchronous with respect to vector function Φ, if the controller function u exists and satisfies the following property:

$$\lim_{t\to\infty}\|e(t)\| = \lim_{t\to\infty}\big\|y(t) - \Phi\big(x(t)\big)\big\| = 0,$$

for all initial conditions.

One can consider the Lyapunov function given by

$$V(t) = \frac{1}{2}\, e(t)^T P\, e(t). \tag{3}$$

The notation $(\ \ )^T$ stands for the transpose operator and P is a positive symmetric definite matrix and is independent of time. We consider the matrix P as identity matrix in all practical examples. We assume that the error system $e(t)$ is small enough and satisfies a differential equation of the form

$$\dot{e}(t) = -M(t)e(t), \tag{4}$$

where M is an appropriate matrix. We have

$$\dot{e}(t) = \dot{y}(t) - \mathcal{J}_\Phi\big(x(t)\big)\dot{x}(t) = By(t) + g\big(y(t)\big) = u\big(x(t), y(t)\big) - \mathcal{J}_\Phi\big(x(t)\big)F\big(x(t)\big), \tag{5}$$

where $\mathcal{J}_\Phi$ is the Jacobian matrix of the function Φ. According to condition (4) it follows that the corresponding controller function u exists and is given by

$$u(x(t), y(t)) = -M(t)\, e(t) + \mathcal{J}_\Phi(x(t))F(x(t)) - By(t) - g(y(t)). \tag{6}$$

Then, system (1) becomes

$$\begin{cases} \dot{x}(t) = F\big(x(t)\big) & \text{driver,} \\ \dot{y}(t) = -M(t)e(t) + \mathcal{J}_\Phi\big(x(t)\big)F\big(x(t)\big) & \text{response.} \end{cases} \tag{7}$$

Thus, we have the following results:

Theorem 2.1 Assume that

(i) Φ is a continuously differentiable function,

(ii) The matrix $M^T(t)\ P + P\ M(t)$ is a positive definite matrix.

Then, system (1) is globally generalized synchronous with respect to the vector function Φ.

Proof

The derivative of the Lyapunov function V is given by

$$\dot{V}(t) = \frac{1}{2}(\dot{e}(t)^T\ P\ e(t) + \ e(t)^T\ P\ e(t)) = \frac{-1}{2}((M(t)e(t))^T\ P + \ e^T\ P\ M(t)e(t))$$
$$= \frac{-1}{2}\ e^T(t)\ \ (M^T(t)\ \ P + \ P\ M(t))\ e(t)\ . \tag{8}$$

Since $M^T(t)\ P + P\ M(t)$ is positive definite matrix and from the Lyapunov stability theory, it follows that $\|e(t)\| \to 0$ as $t \to 0$ and system (1) is globally generalized synchronous with respect to the vector function Φ. ■

It is also possible to consider another hypothesis which guarantees the global synchronization of chaotic systems. We assume that function g in (1) is sufficiently smooth and $\|e(t)\|$ is small enough. So, we have the following expansion,

$$g(y(t)) = g\left(\Phi(x(t)) + e(t)\right) = g\left(\Phi(x(t))\right) + \mathcal{J}_g\left(\Phi(x(t))\right) e(t) + o(\|e(t)\|),$$

where $\mathcal{J}_g\left(\Phi(x(t))\right)$ is the Jacobian matrix of g at point $\Phi(x(t))$. It follows that $g(y(t))$ may be approximated by the sum $g\left(\Phi(x(t))\right) + \mathcal{J}_g\left(\Phi(x(t))\right) e(t)$. The response system in (1) may be approximated by

$$\dot{y}(t) = By(t) + g\left(\Phi(x(t))\right) + J_g\left(\Phi(x(t))\right) e(t) + u(x(t), y(t)).$$

Then,

$$\begin{aligned}\dot{e}(t) &= By(t) + g\left(\Phi(x(t))\right) + J_g\left(\Phi(x(t))\right) e(t) + u(x(t), y(t)) - \mathcal{J}_\Phi(x(t)) F(x(t)) \\ &= -M(t)e(t).\end{aligned}$$

One can thus obtain

$$u(x(t), y(t)) = -\mathcal{J}_g\left(\Phi(x(t)) + M(t)\right) e(t) + \mathcal{J}_\Phi(x(t)) F(x(t)) - By(t) - g\left(\Phi(x(t))\right). \quad (9)$$

Here, system (1) becomes

$$\begin{cases}\dot{x}(t) = F(x(t)) & \text{driver,} \\ \dot{y}(t) = g(y(t)) - g(\Phi(x(t)) - \mathcal{J}_g\left(\Phi(x(t)) + M(t)\right) e(t) + \mathcal{J}_\Phi(x(t)) F(x(t)) & \text{response.}\end{cases} \quad (10)$$

Now, if we se $E(t) = M(t) - B$ and if matrix $M(t)$ commutes with B then $E(t)$ commutes with B and the solution of the differential equation (4) is given by

$$e(t) = e^{-E(t)} w(t), \quad (11)$$

where $w(t)$ is the solution of the differential system

$$\dot{w}(t) = -Bw(t).$$

The solution $w(t)$ satisfies the condition

$$\left\|e^{-E(t)}\right\| \le C_1 e^{\lambda_L t}, \quad (12)$$

where λ_L denotes the maximum Lyapunov exponent of the response system in (1) and C_1 is a positive constant. Furthermore, we assume that matrix E satisfies the condition

$$\left\|e^{-E(t)}\right\| \le C_2 e^{\psi(t)}, \quad (13)$$

where C_2 is a positive constant and the function ψ is assumed to be a non-negative function satisfying the following property

$$\lim_{t\to+\infty} \frac{\psi(t)}{t} = \ell > \lambda_L. \quad (14)$$

Thus, we reach the following results:

Theorem 2.2: Assume that

(i) Φ and g are continuously differentiable functions,
(ii) Matrices B and $M(t)$ commute and matrix $E(t) = M(t) - B$ satisfies conditions (13)-(14).

Then, system (10) is globally generalized synchronous with respect to vector function Φ.

Proof

From (11), we have

$$\left\|e^{-E(t)}\right\| \le \left\|e^{-E(t)}\right\| \| w(t) \|.$$

According to (12) and (13) it follows that

$$\left\|e^{-E(t)}\right\| \le C\ e^{\lambda_L t - \psi(t)}, \quad (15)$$

where C is a positive constant. The property (14) gives that $\|e(t)\| \to 0$ as $t \to 0$, for any set of initial conditions. Hence we have completed the proof of system (10) that it is globally generalized synchronous with respect to the vector function Φ. ■

Remark. In a practical example we select matrix E to be independent of time in the form $E = k\, I_m$, where I_m is the identity matrix of size $m \times m$ and k is a coupling parameter of synchronization. So, matrix E commutes with any matrix and we have $M = B + kI_m$. It follows that

$$e^{-Et} = e^{-kt} \; I_m,$$

and

$$\|e^{-Et}\| = e^{-kt}.$$

In this case, we define the function $\psi(t) = kt$. After, condition (13) is satisfied. Then, we obtain the condition (15) as

$$\|e(t)\| \le C\, e^{(\lambda_L - k)t}.$$

Condition (14) is satisfied for

$$k > \lambda_L,$$

where the maximum Lyapunov exponent is approximately equal to the largest eigenvalue of matrix B. To ensure that $\|e(t)\|$ is small enough for all t, the value of the parameter k must be large enough.

3. NUMERICAL RESULTS

In this section, we consider three numerical examples from physical and biological problems. We demonstrate the effectiveness of the proposed control function. We have addressed the problem of synchronization of identical and nonidentical chaotic systems. All computations have been carried out using MATLAB 2015 on a workstation with 16 significant decimal digits. We solved the drive and response systems (7)-(9) by using the fourth order Runge-Kutta scheme with initial conditions $x(0)$ and $y(0)$. The interval time $[t_0, T]$ is partitioned into N subintervals $[t_n, t_{n+1}]$ with $t_n = t_0 + n\,\Delta t$ for $n = 0, \dots, N$, $\Delta t = \frac{T - t_0}{N}$. Let x_n and y_n denote the approximation of the vectors $x(t_n)$ and $y(t_n)$, respectively. We consider the relative error R_e as

$$t_n \text{ a } \; R_n(t_n) = \sqrt{\frac{\sum_{n=0}^{N} \|y_n - \Phi(x_n)\|^2}{\sum_{n=0}^{N} \|\Phi(x_n)\|^2}}, \qquad (16)$$

and the partial relative error function r_e defined by

$$t_n \text{ a } \; r_e(t_n) = \frac{\|y_n - \Phi(x_n)\|}{\|\Phi(x_n)\|}. \qquad (17)$$

The globally generalized synchronization with respect to function F is also confirmed by the simulation results.

3.1. Memristor system

Now, we provide the first experimental example to illustrate the result given in the previous section. We show the effectiveness of the proposed control function. We address the problem of synchronization of identical systems with $n = m = 4$ concerning the Memristor chaotic systems.

The Memristor was postulated as the fourth nonlinear circuit element by Chua [10]. This Memristor system may be described via the following nonlinear differential equations with respect to the fundamental basic circuit elements, resistance, capacitance, inductance and Memristor [2]. We then have

$$\begin{cases} C_1 \dfrac{dv_1}{dt} = i_3 - W(\varphi)v_1 \\ L \dfrac{di_3}{dt} = v_2 - v_1 \\ C_2 \dfrac{dv_2}{dt} = -i_3 + Gv_2 \\ \dfrac{d\varphi}{dt} = v_1, \end{cases} \tag{18}$$

where the parameters v_1, v_2, v_3, v_4 are the voltages and i_1, i_2, i_3, i_4 are the currents. Nonlinear function W is called the Memristance. We set $x_1 = v_1$, $x_2 = i_3$, $x_3 = v_2$, $x_4 = \varphi$, $\alpha = \frac{1}{C_1}$, $\beta = \frac{1}{C_2}$, $\gamma = \frac{G}{C_2}$ and $L = 1$. Then, system (18) can be transformed to a first order differential equation system as

$$\begin{cases} \dfrac{dx_1}{dt} = \alpha\big(x_2 - W(x_4)\big) \\ \dfrac{dx_2}{dt} = x_3 - x_1 \\ \dfrac{dx_3}{dt} = -\beta\, x_2 + \gamma x_2 \\ \dfrac{dx_4}{dt} = x_1, \end{cases} \tag{19}$$

where function $W(x_4)$ is defined as

$$W(x_4) = \begin{cases} a & \text{if } |x_4| < 1 \\ b & \text{if } |x_4| > 1. \end{cases} \tag{20}$$

From previous observations [10], the state orbits of the Memristor system (19) has a chaotic attractor portrayed for the parameter values fixed as $\alpha = 4$, $\beta = 1$, $\gamma = 0.65$, $a = 0.2$ and $b = 10$, as shown in Figure 1.

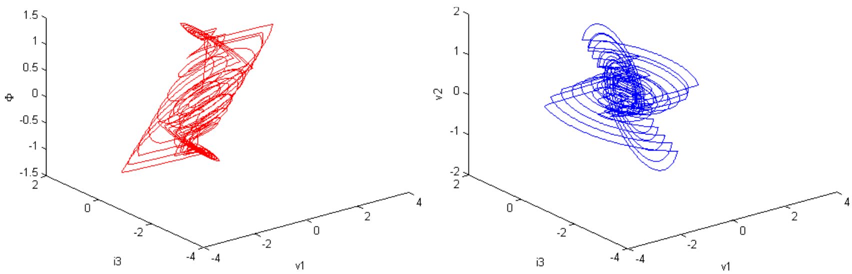

Figure 1. Chaotic attractors of system (19)

The same systems, the Memristor system (19), is taken for the response and given by

$$\begin{cases} \frac{dy_1}{dt} = \alpha\big(y_2 - W(y_4)\big) + u_1\big(x(t), y(t)\big) \\ \frac{dy_2}{dt} = y_3 - y_1 + u_2\big(x(t), y(t)\big) \\ \frac{dy_3}{dt} = -\beta\, x_2 + \gamma x_2 + u_3\big(x(t), y(t)\big) \\ \frac{dy_4}{dt} = y_1 + u_4\big(x(t), y(t)\big) \end{cases} \tag{21}$$

One can rewrite system (21) in the form $\dot{y}(t) = B\, y(t) + g\big(y(t)\big) + u\big(x(t), y(t)\big)$, where

$$B = \begin{bmatrix} 0 & \alpha & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -\beta & \gamma & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad g(y) = \begin{bmatrix} -\alpha W(y_4) y_1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

The control function $u(x, y)$ can be determined by (6), where $\Phi(x_1; x_2; x_3; x_4) = (y_1; y_2; y_3; y_4)$ is the identity function. We choose matrix $M = kI_4$. Thus by Theorem 1, systems (19)-(21) are globally generalized synchronous with respect to function Φ. We then solved the drive-response systems (19)-(21) by using the fourth order Runge-Kutta scheme with the initial conditions $x(0) = (1; 1; 1; 1)^T$ and $y(0) = (10; 20; 50; 20)^T$. The interval time is $[t_0, T] = [0, 20]$ with $N = 50000$.

Figure 2 shows the shape of relative error given by (16). The shape of the partial relative error r_e given by (17) is shown in Figure 3 and it illustrates the behavior of the error. Figure 4 shows the shape between different components of the systems for different values of the parameter k. The synchronization is observed when k the becomes larger and larger. We note that the error decreases as k increases for the synchronization motion between the systems (19) and (21).

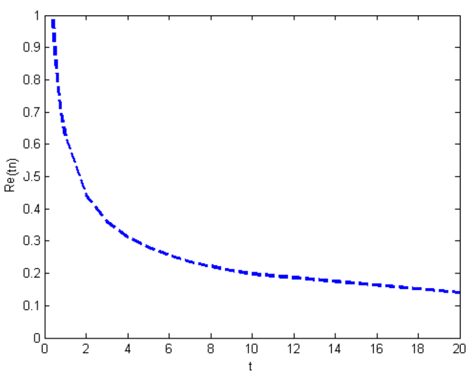

Figure 2. Behaviour of the relative error $R_e(t_n)$

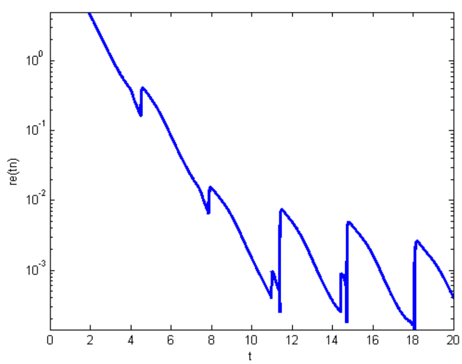

Figure 3. Behaviour of the partial relative error $r_e(t_n)$

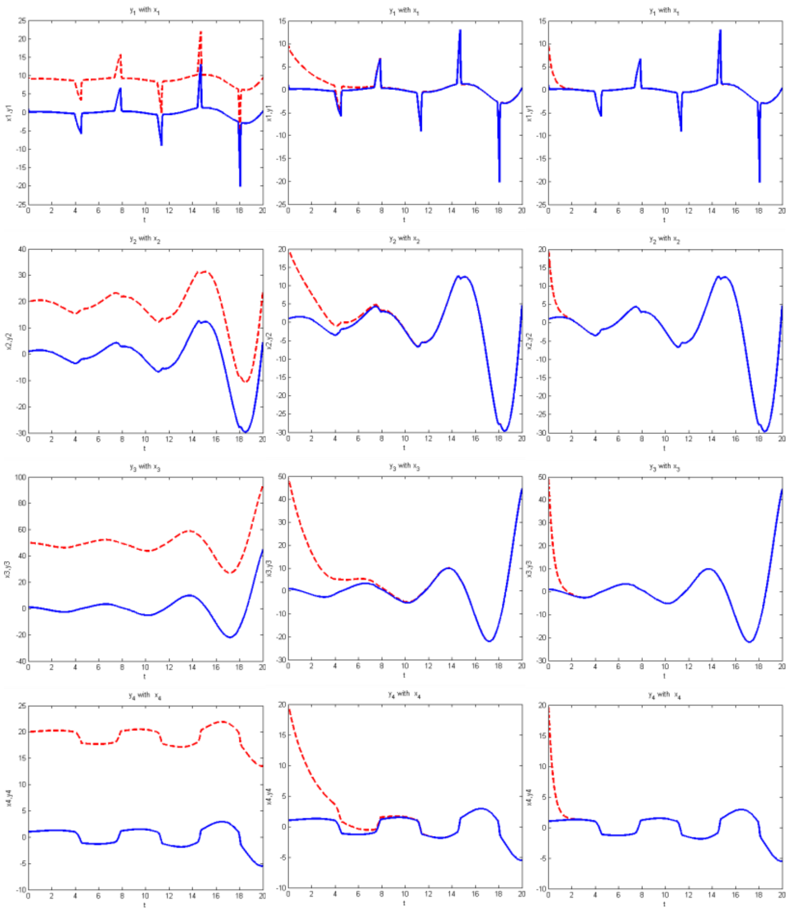

Figure 4. Time series for $x_i(t)$, $y_i(t)$ $(i = 1,2,3,4)$ at versus coupling constant k =0,0.5,1.5

3.2. Lorenz and Rössler systems

Here, we consider two nonidentical chaotic systems in both cases of $n < m$ and $n > m$. We take the well systems of Lorenz and Rössler [5, pp686-708]. The Lorenz system is defined by the three dimensional ordinary differential equations as follows

$$\begin{cases} \frac{dx_1}{dt} = \alpha(x_2 - x_1) + u_1\big(x(t), y(t)\big) \\ \frac{dy_2}{dt} = \gamma x_1 - x_2 + x_1 x_3 + u_2\big(x(t), y(t)\big) \\ \frac{dx_3}{dt} = -\beta\, x_3 + x_1 x_2 + u_3\big(x(t), y(t)\big) \end{cases} \tag{22}$$

The Rössler system is designed by four nonlinear ordinary differential equations.

One can see that the Lyapunov exponents are positive, and showing that the famous Lorenz and Rössler systems exhibit the chaotic attractor as seen in Figuer 5, for the parameter values $\alpha = 10$, $\beta = 8/3$, $\gamma = 28$, $a = 0.25$, $b = 3$, $c = 0.5$ and $d = 0.05$. The initial conditions are $x(0) = (10; 10; 10)^T$ and $y(0) = (1; 1; 1; 1)^T$, respectively; in the interval time $[t_0, T] = [0, 20]$.

$$\begin{cases} \frac{dy_1}{dt} = -y_2 - y_3 + u_1(x(t), y(t)) \\ \frac{dy_2}{dt} = y_1 + \alpha y_2 + y_4 + u_2(x(t), y(t)) \\ \frac{dy_3}{dt} = y_1 y_3 + b + u_3(x(t), y(t)) \\ \frac{dy_4}{dt} = -c y_3 + d y_4 + u_4(x(t), y(t)) \end{cases} \tag{23}$$

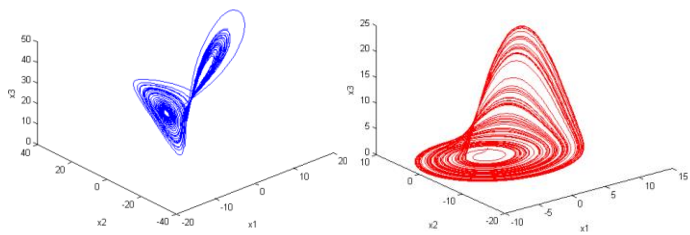

Figure 5. The chaotic attractor of the Lorenz and Rössler systems

To illustrate Theorem 2 for the first case $n < m$, the Lorenz system is chosen as the drive system and the Rössler as the response system which can be redefined as

$$\dot{y}(t) = g(y(t) - g(\Phi(x(t)) - \left(\mathcal{J}_g\left(\Phi(x(t))\right) + M(t)\right)e(t) + \mathcal{J}_\Phi(x(t))\, f(x(t)),$$

where

$$B = \begin{bmatrix} 0 & -1 & -1 & 0 \\ 1 & a & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -c & d \end{bmatrix} \quad \text{and} \quad g(y) = \begin{bmatrix} 0 \\ 0 \\ y_1 y_3 + b \\ 0 \end{bmatrix}.$$

The control function u(x,y) can be determined by (9). The vector function is given by $\Phi(x_1; x_2; x_3) = (y_1; y_2; y_3; y_1 + y_2 + y_3)^T$ which is the nonidentity function. We choose the matrix to be $M = B + E$, where $E = kI_4$. To produce simulation results, the fourth order Runge Kutta scheme is also used to solve the drive-response systems with the time step $\Delta t = \frac{T - t_0}{N}$ and $N = 50000$. Notice that the theoretical results of Theorem 2 is also confirmed by the numerical results, and we obtain the behaviour of synchronization by using the large value of coupling strength k. In Figure 6, we observe that the drive system is synchronized with the response system for the coupling $k \geq 7.6$. The graph of the relative error r_e given by (17) is illustrated in Figure 7 and it figures out the convergence of the results currently computed. The same systems were studied also in references [1,16,9,17]

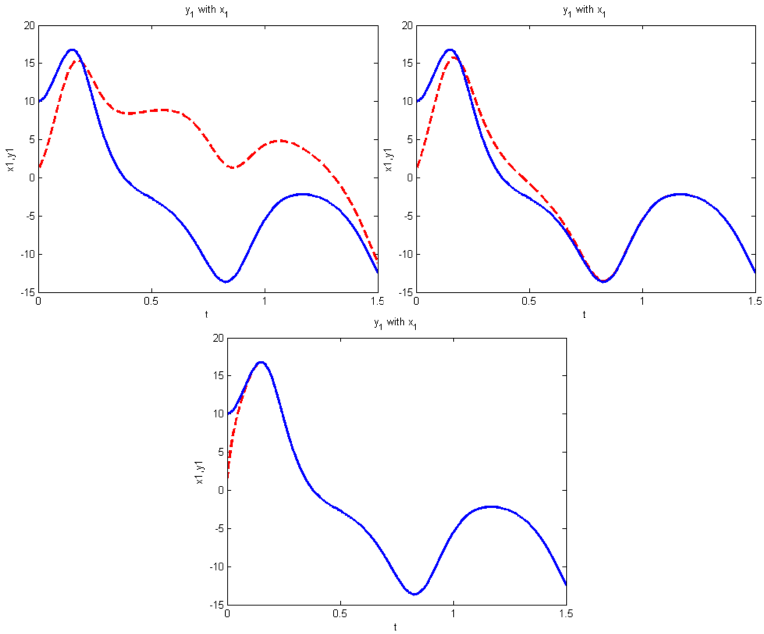

Figure 6. Time series for $x_i(t), y_i(t) (i = 1,2,3,4)$ at various coupling constant k =7.58, 10, 20

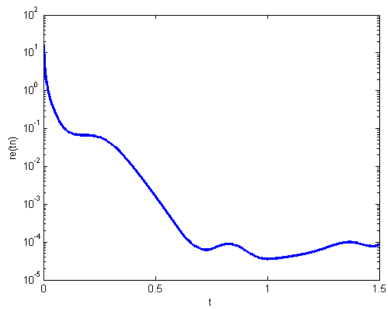

Figure 7. Behaviour of partial relative error $r_e(t_n)$

In the second case for $n > m$, we adopt the Rössler system as the drive, and making the Lorenz system as the response. Then, we rewrite the system in the form

$$\dot{y}(t) = -M(t)e(t) + \mathcal{J}_{\Phi}\big(x(t)\big)f\big(x(t)\big),$$

$$B = \begin{bmatrix} -\alpha & \alpha & 0 \\ \gamma & -1 & 0 \\ 0 & 0 & -\beta \end{bmatrix} \text{ and } g(y) = \begin{bmatrix} 0 \\ -y_1 y_3 \\ y_1 y_3 \end{bmatrix}.$$

The control function $u(x; y)$ can be calculated by (6), and $M = kI_3$. Here, we consider the nonidentity vector function given by $\Phi(x_1; x_2; x_3; x_4) = (x_1 + x_3; x_2 + x_3; x_1 + x_4)^T$. In our simulation, we set the coupling $k = 0.5$, while the initial conditions are chosen to be $x(0) =$

$(-5;-5;10;10)^{\mathrm{T}}$and $y(0)=(10;10;10)^{\mathrm{T}}$ and the time interval is taken to be $[t_0,T]=[0,2500]$. Figure 8 shows that the error state converges to zero, in this case; we extend the analysis to Theorem 1 to estimate the small coupling strength. The designed controller, the drive and response system are synchronized.

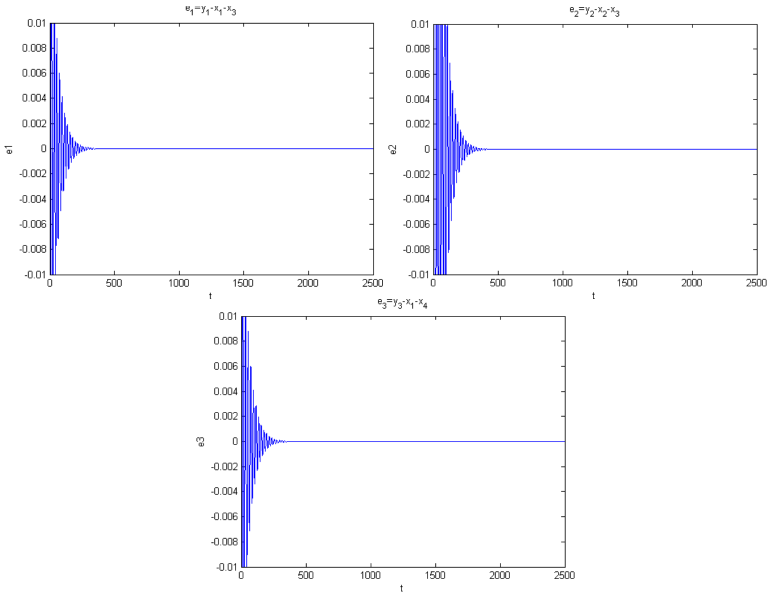

Figure 8. Error states (e_1,e_2,e_3)

3.3. Hindmarsh-Rose neuronal system

In this example, we mainly study the role of neural synchronization in physical diseases, and particularly in the case of heart attack, where the neural activity takes place in many parts of human body, such as the heart muscles. The problem of chaos in the heart muscles will decrease when neurons begin to convince to fire in synchronous with them. The dynamic variables during this process are the neurons membrane potential, which are changed and control a vast number of ionic channels. In general, it describes three different states of the membrane potential which can be Resting, Spiking and Bursting. Some papers investigated synchronization of two HR neurons [3, 18]. Hereafter, some information about neural activity and synchronization, we present the dynamics of the membrane potential in the axon of neuron with a three dimensional system which is known as the HR model

$$\begin{cases} \frac{dx_1}{dt} = x_2 + \alpha x_1{}^2 - x_1{}^3 - x_3 + I(t) \\ \frac{dx_2}{dt} = 1 - \beta x_1{}^2 - x_2 \\ \frac{dx_3}{dt} = \eta(\gamma(x_1 - C) - x_3), \end{cases} \tag{24}$$

where x_1, x_2 and x_3 represent the membrane potential, the recovery variable and the exchange of ions through slow ionic channels respectively. Here $I(t)$ is the externally applied current at time t, h is a recovery variable, which is very small. The parameter C is the x-coordinate of the left most equilibrium point of the model without adaptation. Parameters a,b, h and g are given in biological phenomena. The chaotic Bursting system (24) exhibits a chaotic attractor with parameter values as carried out in reference [8] for $\alpha = 3, \beta = 5, \gamma = 4$, C$= -8/5$, $I = 3.25$, and $\eta = 0.005$, and the initial conditions $x(0) = (-0.54, -1, 3)^T$ and $y(0) = (0.54, 1, -3)^T$ of the drive and response systems respectively as shown in Figure 9.

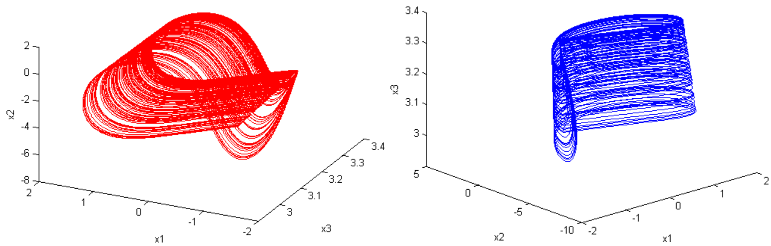

Figure 9. Chaotic bursting of neuronal system

To study synchronization motions of the two identity coupled HR neuronal systems, it is assumed that system (24) is considered to be the drive system, and the response system is given by

$$\begin{cases} \frac{dy_1}{dt} = y_2 + \alpha y_1^2 - y_1^3 - y_3 + I(t) + u_1(x(t), y(t)) \\ \frac{dy_2}{dt} = 1 - \beta y_1^2 - y_2 + +u_2(x(t), y(t)) \\ \frac{dy_3}{dt} = \eta(\gamma(y_1 - C) - y_3) + +u_3(x(t), y(t)). \end{cases} \tag{25}$$

System (25) can then be rewritten as

$$\dot{y}(t) = -M(t)e(t) + \mathcal{J}_\Phi(x(t))f(x(t)),$$

$$B = \begin{bmatrix} 0 & 1 & -1 \\ 0 & -1 & 0 \\ \eta\gamma & 0 & -\eta \end{bmatrix} \text{ and } g(y) = \begin{bmatrix} \alpha y_2^2 - y_2^3 + I \\ 1 - \beta y_2^2 \\ 0 \end{bmatrix}.$$

The control function $u(x, y)$ is given by (6), where $\Phi(x_1; x_2; x_3) = (y_1; y_2; y_3))^T$ is the identity function. We produce numerical results using the fourth order Runge-kutta method for the drive-response system, after $M = k\, I_3$ has been substituted. The time interval is taken to be $[t_0, T] = [0, 200]$. The hypothesis of Theorem 1 are satisfied and we have the synchronization analysis between the drive and response systems. The results are also confirmed by simulations for a coupling strength k which is small enough. Figure 10 shows the time series of x_1 component from drive system and y_1 component from the response system. The partial relative error $r_e$is presented in Figure 11. We observe that convergence of the relative error converge to zero.

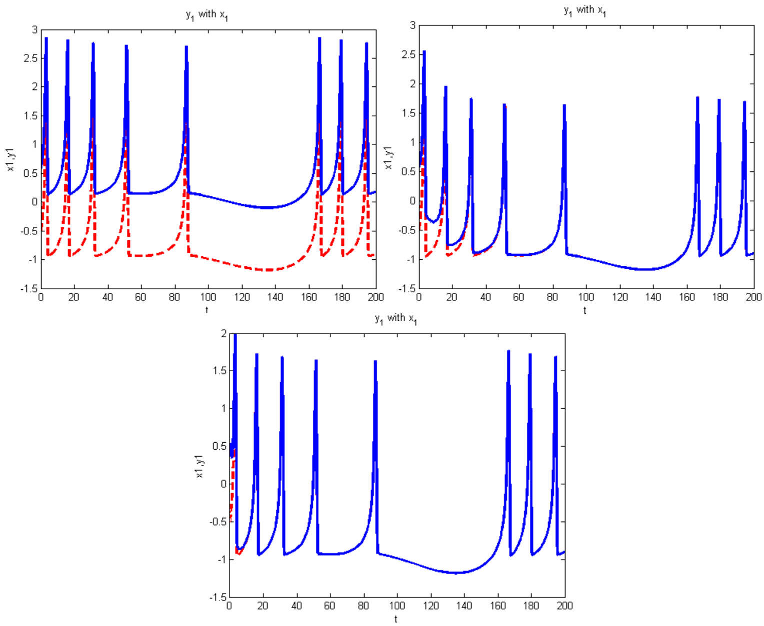

Figure 10. The synchronization between two identical HR neurons systems: plots of amplitudes y1 according x1 at various coupling strength k = 0, 0.1, 0.2

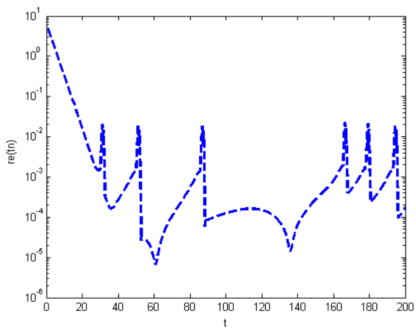

Figure 11. Behaviour of the partial relative error $r_e(t_n)$

3.4. Modeling Belousov-Zhabotinsky Reaction

In this example, we consider modeling of the Belousov-Zhabotinskii (BZ) reaction in chemistry. The reaction is mathematically important because it exhibits many characteristics of chaos. Considering the reaction rates and flow rate, the simple mathematical model consisting of two rate equations can be written as [7]

$$\begin{cases} \frac{dx_1}{dt} = (-x_1^3 - \mu x_1 + \gamma) - \lambda x_2 \\ \frac{dx_2}{dt} = \frac{(x_1 - x_2)}{\tau}, \end{cases} \tag{26}$$

where $x_1 = [HBrO2]$ and $x_2 = [Br^2]$. The characterization of chaos in the BZ reaction relied on the rate of parameters which are fed into the system. The chaotic attractor (26) exhibits with parameter values: $\mu = 0.000005$; $\gamma = 0.000009$; $\lambda = 10000$ and $\tau = 0.5$, and the initial condition $x(0) = (0; 0)$ as shown in Figure 12. In this figure, we obtain the chaotic attractors by choosing the sufficiently best parameter values in the model system (26).

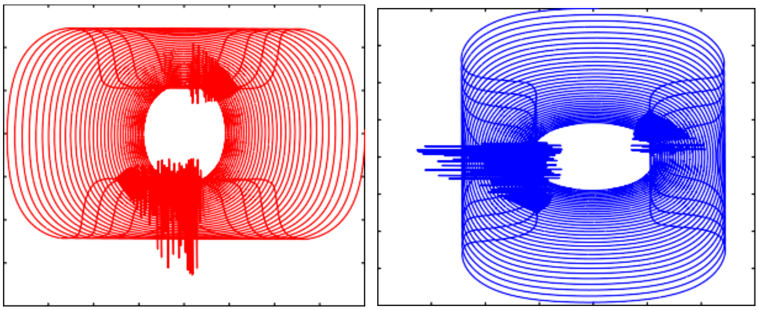

Figure 12. Chaotic modeling of the BZ reaction

Here, to study synchronization motions of the two identical systems of the BZ reaction system, it is assumed that system (26) is considered to be the drive system, and thus the response system is given by

$$\begin{cases} \frac{dy_1}{dt} = (-y_1^3 - \mu y_1 + \gamma) - \lambda y_2 + u_1(x(t), y(t)) \\ \frac{dy_2}{dt} = \frac{(y_1 - y_2)}{\tau} + u_2(x(t), y(t)). \end{cases} \quad (27)$$

Then, system (27) can then be rewritten as

$$\dot{y}(t) = -M(t)e(t) + \mathcal{J}_\Phi(x(t))f(x(t)),$$

where

$$B = \begin{bmatrix} -\mu & -\lambda \\ 1/\tau & -1/\tau \end{bmatrix} \text{ and } g(y) = \begin{bmatrix} -y_1^3 + \gamma \\ 0 \end{bmatrix}.$$

The control function $u(x, y)$ is given by (6), where $F(x_1, x_2) = (y_1; y_2)$ is the identity function. We have produced numerical results using the Runge-Kutta method for the drive-response system, after $M = KI_2$ has been substituted. The time interval is taken to be $[t_0, T] = [0, 1.5]$. Notice that, the advantages of the Theorem 1 are : the theoretical results are confirmed by the numerical results, the behaviour of synchronization are seen by using the small value of coupling strength k. And also, it is successful in designing of coupled functions which represent the fast synchronization in the mechanistic understanding of these complex reactions. In Figure 13, we observe that the drive system is synchronized with the response system for the coupling $k \geq 0.013$. The graph of the relative error $r_e(t_n)$ given by (17) is illustrated in Figure 14 and it figures out the convergence of the results have currently computed.

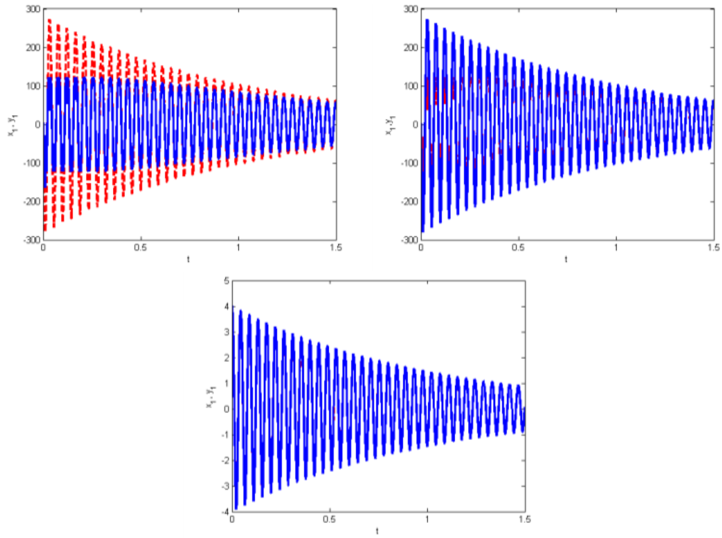

Figure 13. The synchronization between two identical systems BZ reaction systems: plots of amplitudes y_1 according to x_1 at various coupling strength k = 0.001,0.01,0.013

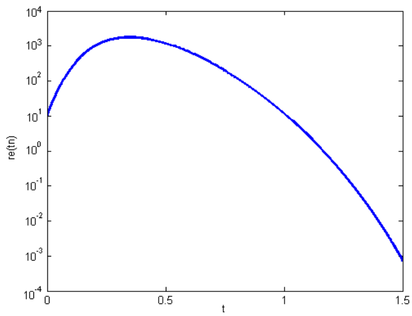

Figure 14. Behaviour of the partial relative error $r_e(t)$

4. CONCLUSION AND RECOMMANDATIONS

In this work, we have investigated the synchronization and stabilization of physical, biological and chemical problems which are well-intended chaotic systems. We have studied coupled systems that do not have mutual feedback but they were organized as a drive system and a response system through some communication channels between them. We have also discussed the asymptotic stability of the zero solutions of synchronization error system. A new response system proposed in each case of the presently generalized methods are given as: first, we have proposed that this phenomenon of chaotic synchronism may construct a response system via the Lyapunov stability theory to carry out the generalized synchronization with the derive system for a given smooth invertible function. We then considered a new hypothesis from the nonlinear part of the response system under some sufficient conditions which showed that the global generalized synchronization between chaotic systems. In this paper, we proved that analytic methods and theoretical results are effectively guaranteed for the stability of generalized chaotic synchronization. The methods may be implemented directly in any numerical simulations for synchronization of chaotic systems in various dimensions and have the fast synchronization speed. Numerical results also illustrated the effectiveness of the proposed approaches. For future research, while the theory of synchronization stays a challenging problem for networks of chaotic systems, and the phenomena of synchronization in coupled partial differential equations will be our focus.

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