Sigma J Eng & Nat Sci 36 (2), 2018, 433-440

Sigma Journal of Engineering and Natural Sciences
Sigma Mühendislik ve Fen Bilimleri Dergisi

Research Article

A NOTE FOR FINDING EXACT SOLUTIONS OF SOME NONLINEAR DIFFERENTIAL EQUATIONS

Zehra PINAR*[1], Turgut ÖZİŞ[2]

[1]*Namık Kemal University, Department of Mathematics, TEKIRDAĞ;* ORCID:0000-0002-9344-7308
[2]*Ege University, Department of Mathematics, Bornova-İZMIR;* ORCID:0000-0001-6718-1668

Received: 12.06.2017 Revised: 09.04.2018 Accepted: 23.04.2018

ABSTRACT

In this note, we show that a particular solution of Bernoulli equation is also the solutions of various second and high order nonlinear ordinary differential equations. The differential equations having solution as a particular solution of Bernoulli equation are listed accordingly. We have exemplified some of nonlinear equations having solution as a particular solution of Bernoulli equation.
Keywords: Nonlinear differential equations, Bernoulli equation, partial differential equations, exact solution.

1. INTRODUCTION

Recently, Kudryashov [1] uses logistic function (the sigmoid function) [1, 2] for finding exact solutions of nonlinear differential equations. Clearly seen that logistic function,

$$Q(\zeta)=\frac{1}{1+e^{-\zeta}}=\frac{1}{2}\tanh\left(\frac{\zeta}{2}\right)+\frac{1}{2} \tag{1}$$

where ζ is independent variable on the complex plane, is the solution of the first order differential equation so called Riccati equation [1,2]:

$$Q_\zeta-Q+Q^2=0\,. \tag{2}$$

where Q_ζ is the derivative of Q respect to ζ. In this study, motivated by Kudryashov [1], we extend his idea to that of a particular solution of Bernoulli equation

$$Q_\zeta=AQ+BQ^k,k>1 \tag{3}$$

One can easily see that the function

* Corresponding Author: e-mail: zpinar@nku.edu.tr, tel: (282) 250 27 06

$$Q(\zeta) = \frac{1}{\left(\frac{-B + c_1 A e^{-A(k-1)\zeta}}{A}\right)^{\left(\frac{1}{k-1}\right)}} \tag{4a}$$

is a particular solution of the Bernoulli equation and the equivalent of Eq.(4a) can also be presented in form of hyperbolic tangent functions in the following formula:

$$Q(\zeta) = \left(\frac{B - B\tanh\left(\frac{A(k-1)\zeta}{2}\right)^2 - c_1 A - c_1 A\tanh\left(\frac{A(k-1)\zeta}{2}\right)^2 + c_1 A\tanh\left(A(k-1)\zeta\right) + c_1 A\tanh\left(A(k-1)\zeta\right)\tanh\left(\frac{A(k-1)\zeta}{2}\right)^2}{A\left(-1 + \tanh\left(\frac{A(k-1)\zeta}{2}\right)^2\right)} \right) \tag{4b}$$

Furthermore, for $k = 2$, Eq. (4a) reduces to the logistic function with appropriate choices of coefficients and in the same manner Eq. (3) reduces to celebrity Riccati equation with, for example, for $A = 1, B = -1$.

Hence, the aim of this note, inspired by Kudryashov [1], is to find some nonlinear ordinary differential equations of the higher orders with exact solutions in the form of Eq. (4a) (or Eq. (4b)). Also, to show that there are nonlinear partial differential equations having solutions in the form of Eq. (4a) or equivalently as Eq. (4b). Recently, Bernoulli equation is considered as an auxiliary equation to obtain the exact solutions of the higher order nonlinear partial differential equations [21].

2. DETERMINATION OF THE NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS OF THE SECOND ORDER WITH IDENTICAL SOLUTION

Differentiating the Eq. (3) with respect to ζ , the following second order differential equation

$$Q_{\zeta\zeta} = AQ_{\zeta} + BkQ^{k-1}Q_{\zeta}, k > 1 \tag{5}$$

is obtained. It is noticeable that the Eq. (4a) (or Eq. (4b)) satisfies Eq. (5) for fitting choices of the coefficients.

In addition to Eq. (5), by using Eq. (3) one can achieve three additional ordinary differential equations

$$Q_{\zeta\zeta} = AQ + ABQ^k + BkQ^{k-1}Q_{\zeta}, k > 1 \tag{6}$$

$$Q_{\zeta\zeta} = AQ_{\zeta} + ABkQ^k + B^2kQ^{2k-1}, k > 1 \tag{7}$$

$$Q_{\zeta\zeta} = AQ + ABkQ^k + BkQ^{2k-1}, k > 1 \tag{8}$$

which have the solutions in the form of Eq. (4a) or equivalently as Eq. (4b) . Supplementary second order ordinary differential equations having solutions in form of Eq. (4a) (or Eq. (4b)) may be presented in the following manner.

$$Q_{\zeta\zeta} - AQ_{\zeta} - BkQ^{k-1}Q_{\zeta} + F(Q, Q_{\zeta}, \ldots)(Q_{\zeta} - AQ - BQ^k) = 0, k > 1 \tag{9}$$

$$Q_{\zeta\zeta} - AQ - ABQ^k - BkQ^{k-1}Q_{\zeta} + F(Q, Q_{\zeta}, \ldots)(Q_{\zeta} - AQ - BQ^k) = 0, k > 1 \tag{10}$$

$$Q_{\zeta\zeta} - AQ_{\zeta} - ABkQ^k - B^2kQ^{2k-1} + F(Q, Q_{\zeta}, \ldots)(Q_{\zeta} - AQ - BQ^k) = 0, k > 1 \tag{11}$$

$$Q_{\zeta\zeta} - AQ - ABkQ^{k} - BkQ^{2k-1} + F(Q,Q_{\zeta},...)(Q_{\zeta} - AQ - BQ^{k}) = 0, k > 1 \tag{12}$$

where $F(Q,Q_{\zeta},...)$ are some dependencies on Q,Q_{ζ} and so on. Now, let us confer the solution method for finding nonlinear ordinary differential equation in the form of Eq. (4).

3. THE METHODOLOGY

Now a day, there is collection of methods for finding exact solutions of nonlinear differential equations. For the sake of limited space, we will mention only a few of them: the tanh method [3-7], auxiliary equation method [8-13], simplest equation method [14-16], (G'/G)- expansion method [17-20] etc.

However, the aim of this note is to present exact solutions of nonlinear partial differential equations in terms of the list of the Eqs. (9)-(12) and give some examples of the application of the methodology suggested by Kudryashov [1].

Hence, consider a nonlinear partial differential equation

$$P(u,u_{x},u_{t},u_{xx},u_{tx},u_{tt},...) = 0 \tag{13}$$

Using travelling wave transformation $\zeta = \mu(x - ct)$, and assuming $u(x,t) = y(\zeta)$ we get the nonlinear ordinary differential equation:

$$R(y,y_{\zeta},y_{\zeta\zeta},y_{\zeta\zeta\zeta}...) = 0 \tag{14}$$

Assuming $y = a_0 Q(\zeta)$ in Eq. (14) and comparing the latest form of Eq. (14) with a suitable equation of the list (9)-(12) we can find the solution in form of equation Eq. (4a)(or Eq. (4b)) with free parameters.

Now let us exemplify the solutions of selected nonlinear partial differential equations in form of the Eq. (4a).

3.1. The Burgers equation

We now show that Burgers' equation has a solution in form of Eq. (4a). The Burgers' equation can be given as

$$u_t + 2uu_x - \upsilon u_{xx} = 0 \tag{15}$$

Using the transformation $\zeta = \mu(x - ct)$, where $c \neq 0$ and $\mu \neq 0$ and assuming $u(x,t) = y(\zeta)$ we get following ordinary differential equation

$$cy' + 2yy' - \upsilon\mu^2 y'' = 0 \tag{16}$$

Eq. (16) is in the similar form of Eq. (9) where $F(Q) = 0$,

$$Q_{\zeta\zeta} - AQ_{\zeta} - BkQ^{k-1}Q_{\zeta} = 0$$

As a result of the comparison we find the solution of the Burgers Eq. (15) in the form

$$Q(\zeta) = \frac{-a + \sqrt{2}\tanh\left(\frac{\sqrt{c_1 bk}\,(\zeta + c_2)\sqrt{2}}{2}\right)\sqrt{c_1 bk}}{bk}$$

where $k=2, A=\frac{c}{\upsilon\mu^2}, B=\left(\frac{1}{\upsilon\mu^2}\right)^2, a_0=\frac{1}{\upsilon\mu^2}$, so Eq.. (9) and Eq. (15) are the same in the case.

3.2. Burgers–Fisher equation

The Burger-Fisher equation can be given as

$$u_t + uu_x = u_{xx} - \beta u - \delta u^2 \tag{17}$$

Using the transformation $\zeta=\mu(x-ct)$, where $c\neq 0$ and $\mu\neq 0$ and assuming $u(x,t)=y(\zeta)$ we get following ordinary differential equation

$$cy' + yy' - y'' + \beta y + \delta y^2 = 0 \tag{18}$$

Eq. (18) is in the similar form of Eq. (10) where $F(Q)=-c$,

$$Q_{\zeta\zeta} - AQ - ABQ^k - BkQ^{k-1}Q_\zeta + F(Q,Q_\zeta,...)(Q_\zeta - AQ - BQ^k) = 0, k>1$$

As a result of the comparison we find the solution of the Burgers-Fisher equation (Eq. (17)) in the form

$$Q(\zeta)=\frac{1}{\left(\dfrac{-\dfrac{1}{2}+c_1\dfrac{\beta}{1-\left(\dfrac{1}{2}-\delta\pm\dfrac{\sqrt{1+4\delta+4\delta^2-4\beta}}{2}\right)}e^{-\frac{\beta}{1-\left(\frac{1}{2}-\delta\pm\frac{\sqrt{1+4\delta+4\delta^2-4\beta}}{2}\right)}\zeta}}{\dfrac{\beta}{1-\left(\dfrac{1}{2}-\delta\pm\dfrac{\sqrt{1+4\delta+4\delta^2-4\beta}}{2}\right)}}\right)}$$

where $k=2, A=\frac{\beta}{1-c}, B=\frac{1}{2}, c=\frac{1}{2}-\delta\pm\frac{\sqrt{1+4\delta+4\delta^2-4\beta}}{2}, a_0=1$, so Eq. (17) and Eq. (10) are the same in the case.

3.3. Hyperbolic equation with power-law nonlinearity

The hyperbolic equation with power-law nonlinearity can be given as

$$u_{tt} = u_{xx} + \alpha u + \beta u^n + \gamma u^{2n-1} \tag{19}$$

Using the transformation $\zeta = \mu(x - ct)$, where $c \neq 0$ and $\mu \neq 0$ and assuming $u(x,t) = y(\zeta)$ we get following ordinary differential equation

$$\left(c^2 - \mu^2\right) y'' - \alpha y - \beta y^n - \gamma y^{2n-1} = 0 \tag{20}$$

Eq. (20) is in the similar form of Eq. (8),

$$Q_{\zeta\zeta} - AQ - ABkQ^k - BkQ^{2k-1} = 0, k > 1$$

As a result of the comparison we find the solution of the hyperbolic equation with power-law nonlinearity, Eq. (19), in the form

$$u(x,t) = \frac{1}{\left(\dfrac{-\dfrac{\beta^2}{\gamma\alpha^2} + c_1 \alpha e^{-\alpha(n-1)\zeta}}{\alpha} \right)^{\left(\frac{1}{n-1}\right)}}$$

where $k = n, A = \alpha, B = \dfrac{\beta^2}{\gamma\alpha^2}, a_0 = \sqrt[n-1]{\dfrac{\beta}{\alpha\gamma}}, c^2 - \mu^2 = 1$, so Eq. (19) and Eq. (8) are the same in the case.

3.4. The KdVequation with power- law nonlinearity

The Korteweg–de Vries (KdV) equation with power-law nonlinearity can be given as

$$u_t + u_{xxx} + au^n u_x = 0 \tag{21}$$

Using the transformation $\zeta = \mu(x - ct)$, where $c \neq 0$ and $\mu \neq 0$ and assuming $u(x,t) = y(\zeta)$ we get following ordinary differential equation

$$cy' + \mu^3 y''' + a\mu y^n y' = 0 \tag{22}$$

Eq. (22) is in the similar form of the following equation

$$Q_{\zeta\zeta\zeta} = A^2 Q_\zeta + ABk(k-1)Q^{k-1}Q_\zeta + B^2 k(2k-1)Q^{2(k-1)}Q_\zeta \tag{23}$$

which is obtained directly by differentiating Eq. (8).

For example, choosing $k = 1/2$ in Eq. (23) one easily gets

$$Q_{\zeta\zeta\zeta} - A^2 Q_\zeta + \frac{AB}{4} Q^{1/2} Q_\zeta = 0 \tag{24}$$

Taking $n = 1/2$ in Eq. (22) and as a result of the comparison, the parameters defined $\mu^3 = \dfrac{1}{a_0}, n = k = 1/2, c = -A^2, a\mu = \dfrac{AB}{4}$, Eq. (24) and Eq. (21) are the same in the case.

3.5. The KdV type equation

The KdV type equation can be given as

$$u_t + u_{xxx} + au^n u_x - bu^{2n} u_x = 0 \tag{25}$$

Using the transformation $\zeta = \mu(x - ct)$, where $c \neq 0$ and $\mu \neq 0$ and assuming $u(x,t) = y(\zeta)$ we get following ordinary differential equation

$$cy' + \mu^3 y''' + a\mu y^n y' - b\mu y^{2n} y' = 0 \tag{26}$$

Eq. (26) is in the similar form of the following

$$Q_{\zeta\zeta\zeta} = A^2 Q_\zeta + ABk(k-1)Q^{k-1}Q_\zeta + B^2 k(2k-1)Q^{2(k-1)}Q_\zeta \tag{27}$$

which is obtained directly by differentiating Eq. (8). For $k - 1 = n$, the Eq. (27) is in the form

$$Q_{\zeta\zeta\zeta} = A^2 Q_\zeta + ABn(n+1)Q^n Q_\zeta + B^2(n+1)(2n+1)Q^{2n}Q_\zeta \tag{29}$$

As a result of the comparison, the parameters defined $\mu^3 = \dfrac{1}{a_0}, n = k-1, c = -A^2, a\mu = -ABk^2, bk(2k-1) = \mu$, Eq. (25) and Eq. (29) are the same in the case.

3.6. Kuramoto-Sivashinsky equation

The Kuramoto-Sivanhinsky equation can be given as

$$u_t - \alpha u^2 u_x - \gamma u_{xx} - \beta u^3 u_x + u_{xxxx} = 0 \tag{30}$$

Using the transformation $\zeta = \mu(x - ct)$, where $c \neq 0$ and $\mu \neq 0$ and assuming $u(x,t) = y(\zeta)$ we get following ordinary differential equation

$$cy' - \alpha\mu y^2 y' - \gamma\mu^2 y'' + \beta\mu y^3 y' + \mu^4 y^{(4)} = 0 \tag{31}$$

Eq. (31) is in the similar form of the following

$$Q_{\zeta\zeta\zeta\zeta} = AQ_{\zeta\zeta} + ABk(k-1)Q^{k-2}Q_\zeta + ABk(Ak - A + 1)Q^{k-1}Q_\zeta + AB^2 k(2k-1)(3k-2)Q^{2(k-1)}Q_\zeta + B^3 k(2k-1)(3k-2)Q^{3(k-1)}Q_\zeta \tag{32}$$

which is obtained directly differentiating Eq. (3).
For $k = 2$, the Eq. (32) reduces in to the form

$$Q_{\zeta\zeta\zeta\zeta} = AQ_{\zeta\zeta} + 2ABQ_\zeta + 2AB(A+1)QQ_\zeta + 24AB^2Q^2Q_\zeta + 24B^3Q^3Q_\zeta \tag{33}$$

As a result of the comparison, the parameters defined $\mu^4 = \dfrac{1}{a_0}, -\mu^2\gamma = 1, c = 2B, -\alpha\mu = 24B^2, \mu\beta = 24B^3, A = -1$, Eq. (30) and Eq. (33) are the same in the case.

4. CONCLUSION

In this note we have revealed that the solution of Bernoulli equation given in form of Eq. (4a) (or Eq. (4b)) is a solution of many nonlinear equations. This study, in one way, is the extended form of the logistic function solution of Riccati equation to Bernoulli equation which gives more responsive solutions compared to logistic function solutions. We have exemplified that the solutions of some of well-known nonlinear equations are expressed in form of Eq. (4a) (or Eq. (4b)). It is apparent that the solution of Bernoulli equation in form of Eq. (4a) (or Eq. (4b)) may be employed for producing exact solutions of various nonlinear differential equations.

REFERENCES

[1] Kudryashov, N. A. Logistic function as solution of many nonlinear differential equations, Applied Mathematical Modelling 39, 5733 – 5742,2015.
[2] Kudryashov, N.A. Polynomials in logistic function and solitary waves of nonlinear differential equations, Appl. Math. Comput. 219, 9245–9253,2013.
[3] Whitham, G. B. A general approach to linear and nonlinear waves using a Lagrangian, *J. Fluid Mech.*, 22, 273-283, 1965.
[4] Malfliet, W. Solitary wave solutions of nonlinear wave equations, Am. J. Phys. 60, 650–654, 1992.
[5] Malfliet, W. Hereman, W. The tanh method: I. Exact solutions of nonlinear evolution and wave equations, Phys. Scripta 54, 563–568, 1996.
[6] Parkes, E.J. Duffy, B.R. An automated tanh-function method for finding solitary wave solutions to nonlinear evolution equations, Comput. Phys. Commun. 98, 288–300, 1996.
[7] Malfliet, W. Hereman, W. The tanh method: II. Perturbation technique for conservative systems, Phys. Scripta 54, 569–575, 1996.
[8] Yan, Z. Y. An improved algebra method and its applications in nonlinear wave equations,Chaos Solitons & Fractals 21, 1013-1021, 2004.
[9] Hereman, W. and Nuseir, A. Symbolic methods to construct exact solutions of nonlinear partial differential equations, Mathematics and Computers in Simulation, 43, 13-27, 1997.
[10] Wazwaz, A.M. Integrable (2 + 1)-dimensional and (3 + 1)-dimensional breaking soliton equations, Phys. Scr., 81, 035005, 2010.
[11] Ma, S.-H. Peng, J. Zhang, C. New exact solutions of the (2+1)-dimensional breaking soliton system via an extended mapping method, Chaos, Solitons and Fractals, 46, 210-214, 2009.
[12] Pinar,Z. Ozis, T. An Observation on the Periodic Solutions to nonlinear Physical models by means of the auxiliary equation with a sixth-degree nonlinear term, Commun Nonlinear Sci Numer Simulat, 18, 2177-2187, 2013.
[13] Pinar,Z. Ozis, T. The Periodic Solutions to Kawahara Equation by Means of the Auxiliary Equation with a Sixth-Degree Nonlinear Term, Journal of Mathematics, 106349, 2013.
[14] Kudryashov, N.A. Simplest equation method to look for exact solutions of nonlinear differential equations, Chaos Soliton Frac. 24, 1217–1231, 2005.
[15] Kudryashov, N.A. Loguinova, N.B. Extended simplest equation method for nonlinear differential equations, Appl. Math. Comput. 205, 396–402, 2008.
[16] Kudryashov, N.A. Meromorphic solutions of nonlinear ordinary differential equations, Commun. Nonlinear Sci. Numer. Simul. 15, 2778–2790, 2010.
[17] Aslan, I. Özis, T. Analytic study on two nonlinear evolution equations by using the (G'/G)-expansion method, Applied Mathematics and Computation, 209, 425–429, 2009

[18] Aslan, I. Özis, T. On the validity and reliability of the (G'/G)-expansion method by using higher-order nonlinear equations, Applied Mathematics and Computation, 211, 531-536, 2009.

[19] Öziş, T. Aslan, İ. Symbolic computations and exact and explicit solutions of some nonlinear evolution equations in mathematical physics, *Commun. Theor. Phys*. 51, 577-580, 2009.

[20] Öziş, T. Aslan, İ. Symbolic computations and construction of new exact traveling wave solutions to Fitzhugh-Nagumo and Klein-Gordon equations, Z. Naturforsch 64, 1-6, 2009.

[21] Pinar, Z. Kocak, H. Exact solutions for the third-order dispersive-Fisher equations. Nonlinear Dyn, 91(1), 421-426, 2018.