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Research Article

COMMON COUPLED FIXED POINT THEOREMS FOR GENERALIZED NONLINEAR CONTRACTIONS ON METRIC SPACES INVOLVING A GRAPH

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ABSTRACT

New generalized forms of Banach and Kannan type contractions have been described and some coupled coincidence and coupled common fixed point results have been established in the setting of metric space involving a graph. Our theorems extend and generalize many existing results in the literature. Furthermore, an application to integral equations is presented to affirm the importance.
Keywords: Coupled coincidence point, coupled fixed point, metric space, edge preserving map, connected grap.
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1. INTRODUCTION

In [1], authors introduced the concept of coupled coincidence point and established some coupled coincidence point results for a pair of mappings with the mixed g −monotone property in the setting of partially ordered metric spaces.

Definition 1.1. [1] An element $(a,b) \in C \times C$ is said to be a coupled coincidence point of the mappings $F: C \times C \to C$ and $g: C \to C$ if $F(a,b) = ga$ and $F(b,a) = gb$.

Later, Choudhury and Kundu [2] presented the concept of compatible mappings in the context of coupled coincidence point problems and used the notion to flourish the results indicated in [1].

Definition 1.2. [2] The mappings $F: C \times C \to C$ and $g: C \to C$ are said to be compatible if

$$d\big(gF(a_n,b_n), F(ga_n, gb_n)\big) \to 0 \text{ as } n \to \infty$$

and

$$d\big(gF(b_n,a_n), F(gb_n, ga_n)\big) \to 0 \text{ as } n \to \infty$$

whenever $\{a_n\}$ and $\{b_n\}$ are sequences in C such that $\lim_{n\to\infty} F(a_n,b_n) = \lim_{n\to\infty} ga_n = a$ and $\lim_{n\to\infty} F(b_n,a_n) = \lim_{n\to\infty} gb_n = b$ and $a, b \in C$.

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Let $\mathcal{L}$ be the class of functions $\beta: [0,\infty) \to [0,1)$ with $\beta(f_n) \to 1$ implies $f_n \to 0$. By using the function $\beta \in \mathcal{L}$, Geraghty [3] established the following significant theorem.

Theorem 1.1. [3] Let (C,d) be a complete metric space and $T: C \to C$ be an operator. If T satisfies the following inequality

$$d(Ta,Tb) \leq \beta\big(d(a,b)\big)d(a,b), \text{ for any } a,b \in C,$$

where $\beta \in \mathcal{L}$, then T has a unique fixed point.

In [4], authors presented coupled coincidence theorems to generalized nonlinear contraction mappings with the mixed monotone property in partially ordered metric spaces. Afterwards, Kim and Chandok [5] utilized mixed g −monotone property to obtain coupled coincidence point theorems for Banach and Kannan type contractions with commute of F and g.

Jachymski [6] introduced the fixed point theorem using the context of metric spaces endowed with a graph. Other conclusions for single valued and multivalued operators in such metric spaces were studied by [7-19].

Let (C,d) be a metric space, Δ be a diagonal of $C \times C$, and G be a directed graph with no parallel edges such that the set $V(G)$ of its vertices coincides with the points of C and $\Delta \subseteq E(G)$, where $E(G)$ is the set of the edges of the graph. That is, G is denoted by $\big(V(G), E(G)\big)$. We will employ this concept of G throughout this paper.

In [11], authors established the concept of G −continuity for a mapping $F: C \times C \to C$ and the property A as follows.

Definition 1.3. [11] Let (C,d) be a complete metric space, G be a directed graph, and $F: C \times C \to C$ be a mapping. Then

(i) F is called G −continuous if for all $(a_*, b_*) \in C \times C$ and for any sequence $(n_i)_i \in \mathbb{N}$ of positive integers such that $F\big(a_{n_i}, b_{n_i}\big) \to a_*$, $F\big(b_{n_i}, a_{n_i}\big) \to b_*$ as $i \to \infty$ and $\Big(F\big(a_{n_i}, b_{n_i}\big), F\big(a_{n_i+1}, b_{n_i+1}\big)\Big), \Big(F\big(b_{n_i}, a_{n_i}\big), F\big(b_{n_i+1}, a_{n_i+1}\big)\Big) \in E(G)$, we get that

$$F\Big(F\big(a_{n_i}, b_{n_i}\big), F\big(b_{n_i}, a_{n_i}\big)\Big) \to F(a_*, b_*) \text{ as } i \to \infty$$

and

$$F\Big(F\big(b_{n_i}, a_{n_i}\big), F\big(a_{n_i}, b_{n_i}\big)\Big) \to F(b_*, a_*) \text{ as } i \to \infty;$$

(ii) (C,d,G) has the property A if for any sequence $(a_n)_{n\in\mathbb{N}} \subseteq C$ with $a_n \to a$ as $n \to \infty$ and $(a_n, a_{n+1}) \in E(G)$ for $n \in \mathbb{N}$, then $(a_n, a) \in E(G)$.

We define the set $CoinFix\,(F)$ of all coupled coincidence points of mappings $F: C \times C \to C$ and $g: C \to C$ and the set $(C \times C)_{Fg}$ as follows:

$$CoinFix(F) = \{(a,b) \in C \times C : ga = F(a,b) \text{ } and \text{ } gb = F(b,a)\}$$

and

$$(C \times C)_{Fg} = \big\{(a,b) \in C \times C : \big(ga, F(a,b)\big), \big(gb, F(b,a)\big) \in E(G)\big\}.$$

We also denote the set $ComFix(F)$ of all common fixed points of mappings $F: C \times C \to C$ and $g: C \to C$, that is,

$$ComFix(F) = \{(a,b) \in C \times C : a = ga = F(a,b) \text{ } and \text{ } b = gb = F(b,a)\}.$$

Recently, Suantai et al. [12] presented the notion of G −edge preserving map and the transitivity property.

Definition 1.4. [12] $F: C \times C \to C$ and $g: C \to C$ are G −edge preserving if

$$\begin{aligned}&[(ga, gc), (gb, gd) \in E(G)]\\ &\Rightarrow \big[\big(F(a,b), F(c,d)\big), \big(F(b,a), F(d,c)\big) \in E(G)\big].\end{aligned}$$

Definition 1.5. [12] Let (C, d) be a complete metric space, and $E(G)$ be the set of the edges of the graph. $E(G)$ satisfies the transitivity property iff for all $a, b, r \in C$,

$$(a, r), (r, y) \in E(G) \rightarrow (a, b) \in E(G).$$

Using the concepts listed above, in the present paper, some new coupled coincidence point results for generalized Banach and Kannan type contraction mappings on a metric space endowed with a directed graph are discussed.

2. MAIN RESULTS

Now we are ready to discuss the main results. First we study fixed point results for generalized Banach type contractions.

Definition 2.1. Let (C, d) be a complete metric space endowed with a directed graph G. The pair of mappings $F: C \times C \rightarrow C$ and $g: C \rightarrow C$ is called a generalized Banach type contraction if:

(1) F and g are $G-$edge preserving;
(2) There exists $\beta \in \mathcal{L}$ such that for all $a, b, c, d \in C$ satisfying $(ga, gc), (gb, gd) \in E(G)$,

$$d\big(F(a,b), F(c,d)\big) \leq \beta\left(2^{-1} \times \big(d(ga, gc) + d(gb, gd)\big)\right) \times \left(2^{-1} \times \big(d(ga, gc) + d(gb, gd)\big)\right). \quad (1)$$

Theorem 2.1. Let (C, d) be a complete metric space endowed with a directed graph G, and let $F: C \times C \rightarrow C$ and $g: C \rightarrow C$ be a generalized Banach type contraction. Suppose that:

(i) $g(C)$ is closed and g is continuous;
(ii) compatiblity of the pair (F, g) and $F(C \times C) \subseteq g(C)$;

(1) F is $G-$continuous, or
(2) the tripled (C, d, G) has property A;

(iii) $E(G)$ satisfies the transitivity property.

Under these conditions, $CoinFix(F) \neq \emptyset$ iff $(C \times C)_{Fg} \neq \emptyset$.

Proof. Let $a_0, b_0, t_0, s_0 \in C$. Since $F(C \times C) \subseteq g(C)$, by the assumption, we can choose $a_1, b_1, t_1, s_1 \in C$ such that $F(a_0, b_0) = ga_1$ and $F(b_0, a_0) = gb_1$, $F(t_0, s_0) = gt_1$ and $F(s_0, t_0) = gs_1$. Again, we can choose $a_2, b_2, t_2, s_2 \in C$ such that $F(a_1, b_1) = ga_2$ and $F(b_1, a_1) = gb_2$, $F(t_1, s_1) = gt_2$ and $F(s_1, t_1) = gs_2$.

Continuing this process, we can construct sequences $\{a_n\}$, $\{b_n\}$, $\{t_n\}$, $\{s_n\}$ in C for which

$$ga_n = F(a_{n-1}, b_{n-1}) \text{ and } gb_n = F(b_{n-1}, a_{n-1}) \quad (2)$$
$$gt_n = F(t_{n-1}, s_{n-1}) \text{ and } gs_n = F(s_{n-1}, t_{n-1}) \text{ for } n = 1,2, \cdots.$$

Assume that $CoinFix(F) \neq \emptyset$. Let $(c, d) \in CoinFix(F)$ be such that $\big(gc, F(c,d)\big) = (gc, gc)$, $\big(gd, F(d,c)\big) = (gd, gd) \in \Delta \subset E(G)$. Thus, $\big(gc, F(c,d)\big), \big(gd, F(d,c)\big) \in E(G)$. Then we obtain $(c, d) \in (C \times C)_{Fg}$, so $(C \times C)_{Fg} \neq \emptyset$.

Let $(C \times C)_{Fg} \neq \emptyset$. Consider $a_0, b_0 \in C$ such that $(a_0, b_0) \in (C \times C)_{Fg}$. Then $\big(ga_0, F(a_0, b_0)\big)$, $\big(gb_0, F(b_0, a_0)\big) \in E(G)$. From (2), we have $\big(ga_0, F(a_0, b_0)\big) = (ga_0, ga_1)$ and $\big(gb_0, F(b_0, a_0)\big) = (gb_0, gb_1) \in E(G)$. Moreover, since F and g are $G-$edge preserving, we get $\big(F(a_0, b_0), F(a_1, b_1)\big) = (ga_1, ga_2)$ and $\big(F(b_0, a_0), F(b_1, a_1)\big) = (gb_1, gb_2) \in E(G)$. Then, by mathematical induction, we shall get (ga_{n-1}, ga_n) and $(gb_{n-1}, gb_n) \in E(G)$ for each $n \in \mathbb{N}$.

Let (ga_n, gt_n) and $(gb_n, gs_n) \in E(G)$ for all $n \in \mathbb{N}$. From (1) and (2), we get that

$$d(ga_{n+1},gt_{n+1}) = d\big(F(a_n,b_n),F(t_n,s_n)\big) \tag{3}$$
$$\leq \beta\left(2^{-1}\times\big(d(ga_n,gt_n)+d(gb_n,gs_n)\big)\right)$$
$$\times\left(2^{-1}\times\big(d(ga_n,gt_n)+d(gb_n,gs_n)\big)\right)$$

and

$$d(gb_{n+1},gs_{n+1}) = d\big(F(b_n,a_n),F(s_n,t_n)\big) \tag{4}$$
$$\leq \beta\left(2^{-1}\times\big(d(gb_n,gs_n)+d(ga_n,gt_n)\big)\right)\left(\frac{d(gb_n,gs_n)+d(ga_n,gt_n)}{2}\right)$$

for all $n\in\mathbb{N}$. By (3) and (4), we obtain that

$$\begin{aligned} d(ga_{n+1},gt_{n+1}) + d(gb_{n+1},gs_{n+1}) & \\ \leq 2\times\beta\left(2^{-1}\times\big(d(ga_n,gt_n)+d(gb_n,gs_n)\big)\right) & \\ \times\left(2^{-1}\times\big(d(ga_n,gt_n)+d(gb_n,gs_n)\big)\right) & \\ \leq \beta\left(2^{-1}\times\big(d(ga_n,gt_n)+d(gb_n,gs_n)\big)\right) & \\ \times\big(d(ga_n,gt_n)+d(gb_n,gs_n)\big) & \\ \leq d(ga_n,gt_n)+d(gb_n,gs_n) & \end{aligned} \tag{5}$$

for all $n\in\mathbb{N}$, i.e.,

$$d(ga_{n+1},gt_{n+1}) + d(gb_{n+1},gs_{n+1}) \leq d(ga_n,gt_n)+d(gb_n,gs_n).$$

It follows that $\sigma_n \coloneqq d(ga_n,gt_n)+d(gb_n,gs_n)$ is a monotone decreasing sequence of non-negative real numbers. Hence, there is some $\sigma\geq 0$ such that $\sigma_n\to\sigma$ as $n\to\infty$. Assume $\sigma>0$. By (5), we have

$$\frac{\sigma_{n+1}}{\sigma_n} = \frac{d(ga_{n+1},gt_{n+1}) + d(gb_{n+1},gs_{n+1})}{d(ga_n,gt_n)+d(gb_n,gs_n)} \leq \beta\left(2^{-1}\times\big(d(ga_n,gt_n)+d(gb_n,gs_n)\big)\right) < 1,$$

which yields that $\beta\left(2^{-1}\times\big(d(ga_n,gt_n)+d(gb_n,gs_n)\big)\right)\to 1$ as $n\to\infty$. Since $\beta\in\mathcal{L}$, we have $d(ga_n,gt_n)\to 0$ and $d(gb_n,gs_n)\to 0$ as $n\to\infty$. Thus,

$$\lim_{n\to\infty}\sigma_n = \lim_{n\to\infty}\{d(ga_n,gt_n)+d(gb_n,gs_n)\} = 0 \tag{6}$$

which is a contradiction. Therefore

$$\lim_{n\to\infty} d(ga_n,gt_n) = 0 \text{ and } \lim_{n\to\infty} d(gb_n,gs_n) = 0. \tag{7}$$

Next, we shall prove that $\{ga_n\}$ and $\{gb_n\}$ are Cauchy sequences. Using a similar assertion as in the proof of Theorem 6 in [5] and by (7), the assumption *(iv)*, we conclude that $\{ga_n\}$ and $\{gb_n\}$ are Cauchy sequences. Since g is continuous and $g(C)$ is closed, there exists $c,d\in g(C)$ such that

$$\lim_{n\to\infty} F(a_n,b_n) = \lim_{n\to\infty} ga_n = c,$$
$$\lim_{n\to\infty} F(b_n,a_n) = \lim_{n\to\infty} gb_n = d. \tag{8}$$

Owing to compatiblity of the pair (F,g), by (8), we have

$$d\big(gF(a_n,b_n),F(ga_n,gb_n)\big)\to 0 \text{ as } n\to\infty$$
$$d\big(gF(b_n,a_n),F(gb_n,ga_n)\big)\to 0 \text{ as } n\to\infty \tag{9}$$

Let the assumption $(iii)_1$ holds. For all $n\geq 0$, we get

$$d\big(gc,F(ga_n,gb_n)\big) \leq d\big(gc,gF(a_n,b_n)\big) + d\big(gF(a_n,b_n),F(ga_n,gb_n)\big). \tag{10}$$

Letting $n\to\infty$ in (10), using the inequalities (8) and (9) and from assumption *(i)* and *(iii)*, we obtain that $d\big(gc,F(c,d)\big)=0$, i.e., $gc=F(c,d)$. Again, for all $n\geq 0$,

$$d\big(gd, F(gb_n, ga_n)\big) \le d\big(gd, gF(b_n, a_n)\big) + d\big(gF(b_n, a_n), F(gb_n, ga_n)\big). \tag{11}$$

Letting $n \to \infty$ in (11), using the inequalities (8) and (9) and from assumption *(i)* and *(iii)*, we obtain that $d\big(gd, F(d,c)\big) = 0$, i.e., $gd = F(d,c)$. Therefore, the element $(a,b) \in C \times C$ is a coupled coincidence point of the mappings $F: C \times C \to C$ and $g: C \to C$. Thus, $CoinFix(F) \neq \emptyset$.

Let the assumption $(iii)_2$ holds.

Consider $ga = c$ and $gb = d$ for some $a, b \in C$. In that case, we get (ga_n, ga) and $(gb_n, gb) \in E(G)$ for each $n \in \mathbb{N}$. Using the generalize Banach type contraction (1) we have

$$\begin{aligned} &d(F(a,b), ga_{n+1}) = d\big(F(a,b), F(a_n, b_n)\big) \\ &\le \beta\left(2^{-1} \times \big(d(ga, ga_n) + d(gb, gb_n)\big)\right) \\ &\times \left(2^{-1} \times \big(d(ga, ga_n) + d(gb, gb_n)\big)\right) \\ &\le \left(2^{-1} \times \big(d(ga, ga_n) + d(gb, gb_n)\big)\right) \ (as\ \beta \in \mathcal{L}). \end{aligned} \tag{12}$$

Letting $n \to \infty$ in (12), we have $d(F(a,b), ga) = 0$, i.e., $F(a,b) = ga$. In a similar way, $F(b,a) = gb$. Therefore, the element $(a,b) \in C \times C$ is a coupled coincidence point of the mappings $F: C \times C \to C$ and $g: C \to C$.

Corollary 2.1. Let (C,d) be a complete metric space endowed with a directed graph G, and let $F: C \times C \to C$ and $g: C \to C$ be two mappings. Suppose that F and g are $G-$edge preserving and there exists a $m \in [0,1)$ such that for all $a, b, c, d \in C$ satisfying $(ga, gc), (gb, gd) \in E(G)$,

$$d\big(F(a,b), F(c,d)\big) \le m\left(2^{-1} \times \big(d(ga, gc) + d(gb, gd)\big)\right). \tag{13}$$

Also suppose that the following conditions hold:

(i) $g(C)$ is closed and g is continuous;
(ii) compatiblity of the pair (F, g) and $F(C \times C) \subseteq g(C)$;

(1) F is $G-$continuous, or
(2) the tripled (C, d, G) has property A;

(iv) $E(G)$ satisfies the transitivity property.

Under these conditions, $CoinFix(F) \neq \emptyset$ iff $(C \times C)_{Fg} \neq \emptyset$.

Proof. To prove the above corollary it suffices to take $\beta(t) = m$, where $m \in [0,1)$ in Theorem 2.1.

Theorem 2.2. In addition to Theorem 2.1, suppose that

(iv) for any two elements $(a,b), (c,d) \in C \times C$, there exists $(t,s) \in C \times C$ such that $(ga, gt), (gc, gt), (gb, gs), (gd, gs) \in E(G)$.

Then, $ComFix(F) \neq \emptyset$ iff $(C \times C)_{Fg} \neq \emptyset$.

Proof. By Theorem 2.1, we have $(a,b) \in C \times C$ such that $ga = F(a,b)$ and $gb = F(b,a)$. Suppose that there exists another $(c,d) \in C \times C$ such that $gc = F(c,d)$ and $gd = F(d,c)$. Next, we shall show that $gc = ga$ and $gd = gb$. By *(v)*, there exists $(t,s) \in C \times C$ such that $(ga, gt), (gc, gt), (gb, gs), (gd, gs) \in E(G)$. Set $t = t_0$, $s = s_0$, $a = a_0$, $b = b_0$, $c = c_0$, $d = d_0$, Using (2), we have the sequences $\{t_n\}, \{s_n\}, \{a_n\}, \{b_n\}$, $\{c_n\}$ and $\{d_n\}$ in C for which

$$F(t_{n-1}, s_{n-1}) = gt_n, F(s_{n-1}, t_{n-1}) = gs_n, F(a_{n-1}, b_{n-1}) = ga_n,$$
$$F(b_{n-1}, a_{n-1}) = gb_n, F(c_{n-1}, d_{n-1}) = gc_n, F(d_{n-1}, c_{n-1}) = gd_n,$$

for all $n \in \mathbb{N}$. From the features of coincidence points,

$$\begin{gathered} a = a_n, b = b_n\ and\ c = c_n, d = d_n, \\ i.e., \\ ga_n = F(a,b), gb_n = F(b,a)\ and\ gc_n = F(c,d), gd_n = F(d,c) \end{gathered} \tag{14}$$

for all $n \in \mathbb{N}$. By virtue of $(ga, gt), (gb, gs) \in E(G)$, we obtain that $(ga, gt_0), (gb, gs_0) \in E(G)$. Owing to $G-$edge preserving property of F and g, we have $\big(F(a,b), F(t_0, s_0)\big) = (ga, gt_1)$ and $\big(F(b,a), F(s_0, t_0)\big) = (gb, gs_1) \in E(G)$. In the same way, $(ga, gt_n), (gb, gs_n) \in E(G)$. By (7), we know that $d(ga_n, gt_n) \to 0$ and $d(gb_n, gs_n) \to 0$ as $n \to \infty$. In the same way, by $d(gc, gt_n), d(gd, gs_n) \in E(G)$, we get that $d(gc, gt_n) \to 0$ and $d(gd, gs_n) \to 0$ as $n \to \infty$. By the triangular inequality we have $d(ga, gc) \le d(ga, gt_n) + d(gt_n, gc)$ and $d(gb, gd) \le d(gb, gs_n) + d(gs_n, gd)$ for all $n \in \mathbb{N}$. Taking $n \to \infty$ in these inequalities, we have $d(ga, gc) = 0$ and $d(gb, gd) = 0$. Thus, $gc = ga$ and $gd = gb$.

Corollary 2.2. In addition to Corollary 2.1, suppose that

(v) for any two elements $(a,b), (c,d) \in C \times C$, there exists $(t,s) \in C \times C$ such that $(ga, gt), (gc, gt), (gb, gs), (gd, gs) \in E(G)$.

Then, $ComFix(F) \neq \emptyset$ iff $(C \times C)_{Fg} \neq \emptyset$.

Proof. To prove the above corollary it suffices to take $\beta(t) = m$, where $m \in [0,1)$ in Theorem 2.2.

Next we discuss some fixed point results for generalized Kannan type contractions.

Definition 2.2. Let (C, d) be a complete metric space endowed with a directed graph G. The pair of mappings $F: C \times C \to C$ and $g: C \to C$ is called a generalized Kannan type contraction if:

(1) F and g are $G-$edge preserving;
(2) There exists $\beta \in \mathcal{L}$ such that for all $a, b, c, d \in C$ satisfying $(ga, gc), (gb, gd) \in E(G)$,

$$d\big(F(a,b), F(c,d)\big) \le \beta\big(K(a,b,c,d)\big)(K(a,b,c,d)) \tag{15}$$

where

$$K(a,b,c,d) = \left(4^{-1} \times \begin{pmatrix} d\big(ga, F(a,b)\big) + d\big(gb, F(b,a)\big) \\ + d\big(gc, F(c,d)\big) + d\big(gd, F(d,c)\big) \end{pmatrix} \right). \tag{16}$$

Theorem 2.3. Let (C, d) be a complete metric space endowed with a directed graph G, and let $F: C \times C \to C$ and $g: C \to C$ be a generalized Kannan type contraction. Suppose that:

(i) $g(C)$ is closed and g is continuous;
(ii) The pair (F, g) is compatible and $F(C \times C) \subseteq g(C)$;

(1) F is $G-$continuous, or
(2) the tripled (C, d, G) has property A;

(iii) $E(G)$ satisfies the transitivity property.

Under these conditions, $CoinFix(F) \neq \emptyset$ iff $(C \times C)_{Fg} \neq \emptyset$.

Proof. Following the proof of Theorem 2.1, we have sequences $\{a_n\}, \{b_n\}, \{t_n\}, \{s_n\}$ endowed by (2). Suppose that $CoinFix(F) \neq \emptyset$. Let $(c,d) \in CoinFix(F)$ be such that $\big(gc, F(c,d)\big) = (gc, gc)$ and $\big(gd, F(d,c)\big) = (gd, gd) \in \Delta \subseteq E(G)$. Therefore, $\big(gc, F(c,d)\big), \big(gd, F(d,c)\big) \in E(G)$. Then we get $(c,d) \in (C \times C)_{Fg}$, thus $(C \times C)_{Fg} \neq \emptyset$. Assume that $(C \times C)_{Fg} \neq \emptyset$. Let a_0, $b_0 \in C$ be such that $(a_0, b_0) \in (C \times C)_{Fg}$. Then $\big(ga_0, F(a_0, b_0)\big), \big(gb_0, F(b_0, a_0)\big) \in E(G)$. Also, we know that $\big(ga_0, F(a_0, b_0)\big) = (ga_0, ga_1)$ and $\big(gb_0, F(b_0, a_0)\big) = (gb_0, gb_1) \in E(G)$. Furthermore, since F and g are $G-$edge preserving, we have

$$\big(F(a_0, b_0), F(a_1, b_1)\big) = (ga_1, ga_2) \in E(G),$$
$$\big(F(b_0, a_0), F(b_1, a_1)\big) = (gb_1, gb_2) \in E(G),$$

and so, by induction, we shall obtain (ga_{n-1}, ga_n) and $(gb_{n-1}, gb_n) \in E(G)$ for each $n \in \mathbb{N}$.

Let (ga_n, gt_n) and $(gb_n, gs_n) \in E(G)$ for all $n \in \mathbb{N}$. From (15) and (16), we get that

$$\begin{aligned}
&(ga_{n+1}, gt_n) + (gb_{n+1}, gs_n) \\
&= d\big(F(a_n, b_n), F(t_{n-1}, s_{n-1})\big) + d\big(F(b_n, a_n), F(s_{n-1}, t_{n-1})\big) \\
&\leq 2\beta\left(\frac{1}{4}\begin{pmatrix} d\big(ga_n, F(a_n, b_n)\big) + d\big(gb_n, F(b_n, a_n)\big) \\ +d\big(gt_{n-1}, F(t_{n-1}, s_{n-1})\big) + d\big(gs_{n-1}, F(s_{n-1}, t_{n-1})\big)\end{pmatrix}\right) \\
&\times \frac{1}{4}\begin{pmatrix} d\big(ga_n, F(a_n, b_n)\big) + d\big(gb_n, F(b_n, a_n)\big) \\ +d\big(gt_{n-1}, F(t_{n-1}, s_{n-1})\big) + d\big(gs_{n-1}, F(s_{n-1}, t_{n-1})\big)\end{pmatrix} \\
&\leq \beta\left(\frac{1}{4}\begin{pmatrix} d\big(ga_n, F(a_n, b_n)\big) + d\big(gb_n, F(b_n, a_n)\big) \\ +d\big(gt_{n-1}, F(t_{n-1}, s_{n-1})\big) + d\big(gs_{n-1}, F(s_{n-1}, t_{n-1})\big)\end{pmatrix}\right) \\
&\times \frac{1}{2}\begin{pmatrix} d\big(ga_n, F(a_n, b_n)\big) + d\big(gb_n, F(b_n, a_n)\big) \\ +d\big(gt_{n-1}, F(t_{n-1}, s_{n-1})\big) + d\big(gs_{n-1}, F(s_{n-1}, t_{n-1})\big)\end{pmatrix} \\
&\leq \frac{1}{2}\begin{pmatrix} d\big(ga_n, F(a_n, b_n)\big) + d\big(gb_n, F(b_n, a_n)\big) \\ +d\big(gt_{n-1}, F(t_{n-1}, s_{n-1})\big) + d\big(gs_{n-1}, F(s_{n-1}, t_{n-1})\big)\end{pmatrix}
\end{aligned} \tag{17}$$

that is,

$$(ga_{n+1}, gt_n) + (gb_{n+1}, gs_n) \leq (ga_n, gt_{n-1}) + (gb_n, gs_{n-1}) \tag{18}$$

Let $G_n = (ga_{n+1}, gt_n) + (gb_{n+1}, gs_n)$, then $\{G_n\}$ is a monotone decreasing sequence of non-negative real numbers. Thus, there exists some $G \geq 0$ such that

$$\lim_{n\to\infty} G_n = \lim_{n\to\infty}\{(ga_{n+1}, gt_n) + (gb_{n+1}, gs_n)\} = G. \tag{19}$$

We claim that $G = 0$. Suppose to the contrary that $G > 0$.

By (17), we get that

$$\frac{G_{n+1}}{\frac{G_{n+1}+G_n}{2}} \leq \beta\left(\frac{1}{4}\begin{pmatrix} d\big(ga_n, F(a_n, b_n)\big) \\ +d\big(gb_n, F(b_n, a_n)\big) \\ +d\big(gt_{n-1}, F(t_{n-1}, s_{n-1})\big) \\ +d\big(gs_{n-1}, F(s_{n-1}, t_{n-1})\big)\end{pmatrix}\right) < 1. \tag{20}$$

Taking $n \to \infty$ in (20), we obtain that

$$\lim_{n\to\infty} \beta\left(\frac{1}{4}\begin{pmatrix} d\big(ga_n, F(a_n, b_n)\big) + d\big(gb_n, F(b_n, a_n)\big) \\ +d\big(gt_{n-1}, F(t_{n-1}, s_{n-1})\big) + d\big(gs_{n-1}, F(s_{n-1}, t_{n-1})\big)\end{pmatrix}\right) = 1.$$

Therefore, we have

$$\lim_{n\to\infty}\begin{pmatrix} d\big(ga_n, F(a_n, b_n)\big) + d\big(gb_n, F(b_n, a_n)\big) \\ +d\big(gt_{n-1}, F(t_{n-1}, s_{n-1})\big) + d\big(gs_{n-1}, F(s_{n-1}, t_{n-1})\big)\end{pmatrix} = 0$$

a contradiction. Hence, $G = 0$, that is,

$$\lim_{n\to\infty} G_n = \lim_{n\to\infty}\{(ga_{n+1}, gt_n) + (gb_{n+1}, gs_n)\} = 0. \tag{21}$$

Now, we shall show that $\{ga_n\}$ and $\{gb_n\}$ are Cauchy sequences. Using a similar assertion as in the proof of Theorem 10 in [5] and by (21), the assumption *(iv)*, we conclude that $\{ga_n\}$ and $\{gb_n\}$ are Cauchy sequences. By assumption *(i)*, there exists $c, d \in g(C)$ such that

$$\begin{aligned}
&\lim_{n\to\infty} ga_n = \lim_{n\to\infty} F(a_n, b_n) = c, \\
&\lim_{n\to\infty} gb_n = \lim_{n\to\infty} F(b_n, a_n) = d.
\end{aligned} \tag{22}$$

Since F and g are compatible mappings, by (22), we obtain

$$d\big(gF(a_n,b_n),F(ga_n,gb_n)\big)\to 0 \text{ as } n\to\infty,\qquad (23)$$
$$d\big(gF(b_n,a_n),F(gb_n,ga_n)\big)\to 0 \text{ as } n\to\infty$$

Now, assume that $(iii)_1$ holds. For all $n\geq 0$, we get

$$d\big(gc,F(ga_n,gb_n)\big)\leq d\big(gc,gF(a_n,b_n)\big)+d\big(gF(a_n,b_n),F(ga_n,gb_n)\big).\qquad (24)$$

Letting $n\to\infty$ in (24), using the inequalities (22) and (23) and from assumption *(i)* and *(iii)*, we obtain that $d\big(gc,F(c,d)\big)=0$, i.e., $gc=F(c,d)$. In the similar way, we also obtain that $gd=F(d,c)$. Hence, $CoinFix(F)\neq\emptyset$.

Assume that $(iii)_2$ holds. Let $ga=c$ and $gb=d$ for some $a,b\in C$. Then we get (ga_n,ga) and $(gb_n,gb)\in E(G)$ for each $n\in\mathbb{N}$. Using the generalized Kannan type contraction (15) and (16) we have

$$\begin{aligned}
&d(F(a,b),ga_{n+1})+d(F(b,a),gb_{n+1}) \qquad (25)\\
&=d\big(F(a,b),F(a_n,b_n)\big)+\big(F(b,a),F(b_n,a_n)\big)\\
&\leq 2\beta\left(\frac{1}{4}\begin{pmatrix} d\big(ga,F(a,b)\big)+d\big(gb,F(b,a)\big)\\ +d\big(ga_n,F(ga_n,gb_n)\big)+d\big(gb_n,F(gb_n,ga_n)\big)\end{pmatrix}\right)\\
&\times\frac{1}{4}\begin{pmatrix} d\big(ga,F(a,b)\big)+d\big(gb,F(b,a)\big)\\ +d\big(ga_n,F(ga_n,gb_n)\big)+d\big(gb_n,F(gb_n,ga_n)\big)\end{pmatrix}\\
&=\beta\left(\frac{1}{4}\begin{pmatrix} d\big(ga,F(a,b)\big)+d\big(gb,F(b,a)\big)\\ +d\big(ga_n,F(ga_n,gb_n)\big)+d\big(gb_n,F(gb_n,ga_n)\big)\end{pmatrix}\right)\\
&\times\frac{1}{2}\begin{pmatrix} d\big(ga,F(a,b)\big)+d\big(gb,F(b,a)\big)\\ +d\big(ga_n,F(ga_n,gb_n)\big)+d\big(gb_n,F(gb_n,ga_n)\big)\end{pmatrix}\\
&<\frac{1}{2}\begin{pmatrix} d\big(ga,F(a,b)\big)+d\big(gb,F(b,a)\big)\\ +d(ga_n,ga_{n+1})+d(gb_n,gb_{n+1})\end{pmatrix}.
\end{aligned}$$

Letting $n\to\infty$ in (25), we have $d(F(a,b),ga)+d\big(gb,F(b,a)\big)=0$, i.e., $F(a,b)=ga$. In a similar way, $F(b,a)=gb$. Therefore, the element $(a,b)\in C\times C$ is a coupled coincidence point of the mappings $F\colon C\times C\to C$ and $g\colon C\to C$.

Corollary 2.3. Let (C,d) be a complete metric space endowed with a directed graph G, and let $F\colon C\times C\to C$ and $g\colon C\to C$ be two mappings. Suppose that F and g are $G-$edge preserving and there exists a $m\in[0,1)$ such that for all $a,b,c,d\in C$ satisfying $(ga,gc),(gb,gd)\in E(G)$,

$$d\big(F(a,b),F(c,d)\big)\leq m\big(K(a,b,c,d)\big)\qquad (26)$$

where

$$K(a,b,c,d)=\left(4^{-1}\times\begin{pmatrix} d\big(ga,F(a,b)\big)+d\big(gb,F(b,a)\big)\\ +d\big(gc,F(c,d)\big)+d\big(gd,F(d,c)\big)\end{pmatrix}\right).\qquad (27)$$

Also suppose that the following conditions hold:

(i) $g(C)$ is closed and g is continuous;
(ii) compatiblity of the pair (F,g) and $F(C\times C)\subseteq g(C)$;

(1) F is $G-$continuous, or
(2) the tripled (C,d,G) has property A;

(iv) $E(G)$ satisfies the transitivity property.

Under these conditions, $CoinFix(F)\neq\emptyset$ iff $(C\times C)_{Fg}\neq\emptyset$.

Proof. To prove the above corollary it suffices to take $\beta(t) = m$, where $m \in [0,1)$ in Theorem 2.3.

Theorem 2.4. In addition to Theorem 2.3, suppose that

(v) for any two elements $(a,b),(c,d) \in C \times C$, there exists $(t,s) \in C \times C$ such that $(ga,gt),(gc,gt),(gb,gs),(gd,gs) \in E(G)$.

Then, $ComFix(F) \neq \emptyset$ iff $(C \times C)_{Fg} \neq \emptyset$.

Proof. By Theorem 2.3, we have $(a,b) \in C \times C$ such that $ga = F(a,b)$ and $gb = F(b,a)$. Suppose that there exists a $(c,d) \in C \times C$ such that $gc = F(c,d)$ and $gd = F(d,c)$. Now, we shall show that $gc = ga$ and $gd = gb$. By *(v)*, there exists $(t,s) \in C \times C$ such that $(ga,gt),(gc,gt),(gb,gs),(gd,gs) \in E(G)$. Set $t = t_0$, $s = s_0$, $a = a_0$, $b = b_0$, $c = c_0$, $d = d_0$, Using (2), we have the sequences $\{t_n\}$, $\{s_n\}$, $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ and $\{d_n\}$ in C for which $F(t_{n-1},s_{n-1}) = gt_n$, $F(s_{n-1},t_{n-1}) = gs_n$, $F(a_{n-1},b_{n-1}) = ga_n$, $F(b_{n-1},a_{n-1}) = gb_n$, $F(c_{n-1},d_{n-1}) = gc_n$, $F(d_{n-1},c_{n-1}) = gd_n$ for all $n \in \mathbb{N}$. By the properties of coincidence points, $a = a_n, b = b_n \; and \; c = c_n, d = d_n$, that is, $ga_n = F(a,b), gb_n = F(b,a) \; and \; gc_n = F(c,d), gd_n = F(d,c)$ for all $n \in \mathbb{N}$. Since $(ga,gt),(gb,gs) \in E(G)$, we have $(ga,gt_0),(gb,gs_0) \in E(G)$. Based on $G-$edge preserving of F and g, we have $\big(F(a,b),F(t_0,s_0)\big) = (ga,gt_1)$ and $\big(F(b,a),F(s_0,t_0)\big) = (gb,gs_1) \in E(G)$. Similarly, $(ga,gt_n),(gb,gs_n) \in E(G)$. By (21), we have $\lim_{n\to\infty} d(ga_n,gt_n) = 0$ and $\lim_{n\to\infty} d(gb_n,gs_n) = 0$. Similarly, from $(gc,gt_n),(gd,gs_n) \in E(G)$, we get that $\lim_{n\to\infty} d(gc \;\;, gt_n) = 0$ and $\lim_{n\to\infty} d(gd \;\;, gs_n) = 0$. We also $d(ga,gc) \leq d(ga,gt_n) + d(gt_n,gc)$ and $d(gb,gd) \leq d(gb,gs_n) + d(gs_n,gd)$ for all $n \in \mathbb{N}$. Letting $n \to \infty$ in these inequalities, we have $d(ga,gc) = 0$ and $d(gb,gd) = 0$. Hence, $gc = ga$ and $gd = gb$.

Corollary 2.4. In addition to Corollary 2.3, suppose that

(v) for any two elements $(a,b),(c,d) \in C \times C$, there exists $(t,s) \in C \times C$ such that $(ga,gt),(gc,gt),(gb,gs),(gd,gs) \in E(G)$.

Then, $ComFix(F) \neq \emptyset$ iff $(C \times C)_{Fg} \neq \emptyset$.

Proof. To prove the above corollary it suffices to take $\beta(t) = m$, where $m \in [0,1)$ in Theorem 2.4.

The following corollary is a conclusion of Theorem 2.1 and Theorem 2.2.

Corollary 2.5. Let $(C, \preccurlyeq)$ be a partially ordered set and suppose there exists a metric d on C such that (C,d) is a complete metric space. Let $F: C \times C \to C$ and $g: C \to C$ be two mappings having the mixed $g-$monotone property. Assume that there exists $\beta \in \mathcal{L}$ such that

$$d\big(F(a,b),F(c,d)\big) \leq \beta\left(2^{-1}\big(d(ga,gc) + d(gb,gd)\big)\right)\left(2^{-1}\big(d(ga,gc) + d(gb,gd)\big)\right)$$

for all $a,b,c,d \in C$ with $a \preccurlyeq c$ and $b \succcurlyeq d$. Assume that C has the following property:

(1) if a non-decreasing sequence $\{a_n\} \to a$, then $a_n \preccurlyeq a$ for all $n \in \mathbb{N}$;
(2) if a non-increasing sequence $\{b_n\} \to b$, then $b_n \succcurlyeq b$ for all $n \in \mathbb{N}$.

Then there exists $a, b \in C$ such that $ga = F(a,b)$ and $gb = F(b,a)$, that is, the pair (F,g) has a coupled coincidence point in C.

Proof. Let $G = \big(V(G),E(G)\big)$, where $V(G) = C$ and $E(G) = \{(a,b) \in C \times C: a \preccurlyeq b\}$. It is a straight forward conclusion that all conditions of Theorem 2.1 and Theorem 2.2. Thus, the pair (F,g) has a coupled coincidence point in C.

Corollary 2.6 is a consequence of Theorem 2.3 and Theorem 2.4.

Corollary 2.6. Let $(C,\preccurlyeq)$ be a partially ordered set and suppose there exists a metric d on C such that (C,d) is a complete metric space. Let $F: C\times C\to C$ and $g: C\to C$ be two mappings having the mixed g −monotone property. Assume that there exists $\beta\in\mathcal{L}$ such that

$$d\big(F(a,b),F(c,d)\big)\leq\beta\big(K(a,b,c,d)\big)\big(K(a,b,c,d)\big)$$

Where

$$K(a,b,c,d)=\Big(4^{-1}\times\Big(d\big(ga,F(a,b)\big)+d\big(gb,F(b,a)\big)+d\big(gc,F(c,d)\big)+d\big(gd,F(d,c)\big)\Big)\Big)$$

for all $a,b,c,d\in C$ with $a\preccurlyeq c$ and $b\succcurlyeq d$. Assume that C has the following property:

(1) if a non-decreasing sequence $\{a_n\}\to a$, then $a_n\preccurlyeq a$ for all $n\in\mathbb{N}$;
(2) if a non-increasing sequence $\{b_n\}\to b$, then $b_n\succcurlyeq b$ for all $n\in\mathbb{N}$.

Then there exists $a,b\in C$ such that $ga=F(a,b)$ and $gb=F(b,a)$, that is, the pair (F,g) has a coupled coincidence point in C.

Proof. Let $G=\big(V(G),E(G)\big)$, where $V(G)=C$ and $E(G)=\{(a,b)\in C\times C: a\preccurlyeq b\}$. We can easily conclude that all conditions of Theorem 2.3 and Theorem 2.4. Thus, the pair (F,g) has a coupled coincidence point in C.

Remark 2.1. *(i)* Corollary 2.5 and Corollary 2.6 are an improvement and extension of coupled fixed point results of Kim and Chandok (Theorem 6 and Theorem 10 in [5]) to a coupled coincidence point theorems for the pair (F,g) of compatible having the mixed g −monotone property.

(ii) If g is an identity mapping, then Corollary 2.5 and Corollary 2.6 generalize results of Choudhury and Kundu (Theorem 2.1 and Theorem 2.2 in [4]).

(iii) If $\beta(t)=m$, where $m\in[0,1)$ and $g=I$, the consequence of Bhaskar and Lakshmikantham [20] is a special case of Corollary 2.5 and Corollary 2.6.

(iv) If we take $\beta(t)=m$, where $m\in[0,1)$ and $g=I$ in Theorem 2.1, we have results of Chifu and Petrusel (Theorem 2.1 in [11]).

3. APPLICATION

Finally we discuss the application of our main results by establishing an existence and uniqueness theorem for the solution of a nolinear integral equation.

Consider the following nonlinear integral equation:

$$a(t)=v(t)+\int_0^T M(t,s)f\big(s,a(s),b(s)\big)ds, \tag{28}$$
$$b(t)=v(t)+\int_0^T M(t,s)f\big(s,b(s),a(s)\big)ds,$$

where $t\in[0,T]$ with $T>0$.

Let $C\coloneqq C([0,T],\mathbb{R}^n)$. Let $d(a,b)=\|a-b\|_\infty=sup_{t\in[0,T]}|a(t)-b(t)|$, for $a,b\in C$.

Consider the graph G with partial order relation by $a,b\in C$, $a\leq b\Leftrightarrow a(t)\leq b(t)$ for any $t\in[0,T]$. Then $(C,\|.\|)$ is a complete metric space endowed with a directed graph G.

Let $E(G)=\{(a,b)\in C\times C: a\leq b\}$. Thus $E(G)$ satisfies the transitivity property, and $(C,\|.\|,G)$ has property A.

We consider the following conditions:

(1) $f:[0,T]\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ and $v:[0,T]\to\mathbb{R}^n$ are continuous;
(2) for all $t,s\in[0,T]$, $a,b,c,d\in\mathbb{R}^n$, $a\leq c$, $d\leq b$,

$$|f(t,s,a,b)-f(t,s,c,d)|\leq\frac{1}{T}\ln(1+k_1|a-c|+k_2|b-d|) \tag{29}$$

where $k_1,k_2>0$;

(3) for all $t, s \in [0, T]$, there exists a continuous $M: [0, T] \times \mathbb{R} \to \mathbb{R}$ such that

$$sup_{t\in[0,T]} \int_0^T M(t,s)ds < 1; \tag{30}$$

(4) there exists $(a_0, b_0) \in C \times C$ such that

$$a_0(t) \le v(t) + \int_0^T M(t,s) f\big(s, a_0(s), b_0(s)\big)ds, \tag{31}$$
$$b_0(t) \ge v(t) + \int_0^T M(t,s) f\big(s, b_0(s), a_0(s)\big)ds,$$

where $t \in [0, T]$.

Theorem 3.1. Suppose that assumptions (1)-(4) are satisfied. Then (28) has a unique solution in $C([0,T], \mathbb{R}^n)$.

Proof. Let $F: C \times C \to C$, $(a, b) \to F(a, b)$, where

$$F(a,b)(t) = v(t) + \int_0^T M(t,s) f\big(s, a(s), b(s)\big)ds,\ t \in [0,T], \tag{32}$$

and define $g: C \to C$ by $ga(t) = a(t)$. Thereby, (28) can be expressed as

$$ga = F(a,b) \text{ and } gb = F(b,a). \tag{33}$$

Let $a, b, c, d \in C$ be such that $ga \le gc$ and $gd \le gb$. Then, for each $t \in [0, T]$,

$$\begin{aligned}
F(a,b)(t) &= v(t) + \int_0^T M(t,s) f\big(s, a(s), b(s)\big)ds \\
&= v(t) + \int_0^T M(t,s) f\big(s, g(a)(s), g(b)(s)\big)ds \\
&\le v(t) + \int_0^T M(t,s) f\big(s, g(c)(s), g(d)(s)\big)ds \\
&= v(t) + \int_0^T M(t,s) f\big(s, c(s), d(s)\big)ds \\
&= F(c,d)(t)
\end{aligned}$$
$$\Rightarrow F(a,b)(t) \le F(c,d)(t) \text{ for all } t \in [0,T],$$

and

$$\begin{aligned}
F(b,a)(t) &= v(t) + \int_0^T M(t,s) f\big(s, b(s), a(s)\big)ds \\
&= v(t) + \int_0^T M(t,s) f\big(s, g(b)(s), g(a)(s)\big)ds \\
&\le v(t) + \int_0^T M(t,s) f\big(s, g(d)(s), g(c)(s)\big)ds \\
&= v(t) + \int_0^T M(t,s) f\big(s, d(s), c(s)\big)ds \\
&= F(d,c)(t)
\end{aligned}$$
$$\Rightarrow F(b,a)(t) \le F(d,c)(t) \text{ for all } t \in [0,T].$$

Hence, F and g are G −edge preserving.
On the other side, we have

$$\begin{aligned}
d\big(F(a,b), F(c,d)\big) &= |F(a,b)(t) - F(c,d)(t)| \\
&\le \left| \int_0^T M(t,s)\big[f\big(s, a(s), b(s)\big) - f\big(s, c(s), d(s)\big)\big] \right| ds \\
&\le \int_0^T M(t,s) \big| f\big(s, a(s), b(s)\big) - f\big(s, c(s), d(s)\big) \big| \,\mathrm{ds}
\end{aligned}$$

$$\leq \frac{1}{T}\int_0^T M(t,s)\ln(1+k_1|a(s)-c(s)|+k_2|b(s)-d(s)|)\,ds$$
$$\leq \frac{1}{T} sup_{t\in[0,T]} \int_0^T M(t,s)ds \ln(1+k_1|a(s)-c(s)|+k_2|b(s)-d(s)|)$$
$$\leq \frac{1}{T}\ln(1+k_1|a(s)-c(s)|+k_2|b(s)-d(s)|)$$
$$\leq \ln(1+k_1|a(s)-c(s)|+k_2|b(s)-d(s)|)$$
$$= \ln\big(1+k_1 d(a,c)+k_2 d(b,d)\big)$$
$$= \frac{\ln\big(1+k_1 d(a,c)+k_2 d(b,d)\big)}{k_1 d(a,c)+k_2 d(b,d)}\big(k_1 d(a,c)+k_2 d(b,d)\big)$$

where $k_1 = k_2 = \frac{1}{2}$.

Put $\beta(z) = {\ln(1+z)}/{z}$. Thus, we have

$$d\big(F(a,b),F(c,d)\big) \leq \beta\left(2^{-1}\times\big(d(ga,gc)+d(gb,gd)\big)\right)\left(2^{-1}\times\big(d(ga,gc)+d(gb,gd)\big)\right).$$

This proves that the pair F and g is a generalized Banach type contraction appearing in Theorem 2.1.

Next, assumption (4) in Theorem 3.1 implies that there exists $(a_0,b_0)\in C\times C$ such that

$$a_0(t) \leq v(t) + \int_0^T M(t,s) f\big(s,a_0(s),b_0(s)\big)ds,$$
$$b_0(t) \geq v(t) + \int_0^T M(t,s) f\big(s,b_0(s),a_0(s)\big)ds,$$

where $t\in[0,T]$. Yet, $g(a_0)(t) = (a_0)(t)$ and $g(b_0)(t) = (b_0)(t)$. Thus,

$$g(a_0) \leq F(a_0,b_0) \text{ and } F(b_0,a_0) \leq g(b_0)$$
$$\Rightarrow \big(ga_0, F(a_0,b_0)\big), \big(gb_0, F(b_0,a_0)\big) \in E(G)$$
$$\Rightarrow (C\times C)_{Fg} \neq \emptyset.$$

Moreover, $F(C\times C)\subseteq g(C)$, and (F,g) is compatible.

Farther, assumption (1) in Theorem 3.1 implies that *(i)* and $(iii)_1$ in Theorem 2.1 is satisfied. Besides, $(iii)_2$ in Theorem 2.1 holds by the fact that (C,d,G) has a property A. Therefore, all assumptions of Theorem 2.1 are fulfilled.

4. CONCLUSION

In the current paper, new generalized forms of Banach and Kannan type contractions have been defined. Existence results for coupled common fixed point and coupled coincidence point have been established for such contractions in a more general setting of metric space endowed with a graph instead of partially ordered set. We have proved Theorem 2.1 and Theorem 2.3 by imposing a strong invariance criterion that $F(C\times C)\subseteq g(C)$. *It would be a good subject for future study if this invariance criterion can be relaxed. Further, the uniqueness of coupled coincidence point may also be investigated.*

Within the future scope of the idea, reader may indicate the coupled coincidence point of the following problems.

- Let $(\mathrm{C},\preccurlyeq)$ be a partially ordered set, and suppose there is a metric d on C such that (C,d) is a complete metric space. Assume that $\mathrm{F},\mathrm{G}:\mathrm{C}\times\mathrm{C}\to\mathrm{C}$ are two generalized compatible mappings such that F is G −increasing w.r.t $\preccurlyeq$, G is continuous and has the mixed monotone property, and there exists two elements $\alpha_0,\theta_0\in\mathrm{C}$ with $\mathrm{G}(\alpha_0,\theta_0)\preccurlyeq\mathrm{F}(\alpha_0,\theta_0)$ and $\mathrm{G}(\theta_0,\alpha_0)\succcurlyeq\mathrm{F}(\theta_0,\alpha_0)$. Suppose that there exists $\beta\in\mathcal{L}$ such that

$$\mathrm{d}\big(\mathrm{F}(\alpha,\theta),\mathrm{F}(\gamma,\delta)\big) \le \beta\big(2^{-1}\times\big\{\mathrm{d}\big(\mathrm{G}(\alpha,\theta),\mathrm{G}(\gamma,\delta)\big)+\mathrm{d}\big(\mathrm{G}(\theta,\alpha),\mathrm{G}(\delta,\gamma)\big)\big\}\big)$$
$$\times\big(2^{-1}\times\big\{\mathrm{d}\big(\mathrm{G}(\alpha,\theta),\mathrm{G}(\gamma,\delta)\big)+\mathrm{d}\big(\mathrm{G}(\theta,\alpha),\mathrm{G}(\delta,\gamma)\big)\big\}\big),$$

for all $\alpha,\theta,\gamma,\delta\in \mathrm{C}$ *with* $\mathrm{G}(\alpha,\theta)\preccurlyeq \mathrm{G}(\gamma,\delta)$ *and* $\mathrm{G}(\theta,\alpha)\preccurlyeq \mathrm{G}(\delta,\gamma)$. *Assume that for any* $\alpha,\theta\in \mathrm{C}$, *there exists* $\gamma,\delta\in \mathrm{C}$ *such that* $\mathrm{F}(\alpha,\theta)=\mathrm{G}(\gamma,\delta)$, $\mathrm{F}(\theta,\alpha)=\mathrm{G}(\delta,\gamma)$. *Suppose that also* F *is continuous or (i)* if a non-decreasing sequence $\{\alpha_n\}\to\alpha$, then $\alpha_n\preccurlyeq\alpha$ for all $n\in\mathbb{N}$, *(ii)* if a non-increasing sequence $\{\theta_n\}\to\theta$, then $\theta_n\succcurlyeq\theta$ for all $n\in\mathbb{N}$. Then F and G have a coupled coincidence point in C.

- With similar thought, using the notion of generalized compatibility of a pair of mappings identified by $\mathrm{F},\mathrm{G}:\mathrm{C}\times\mathrm{C}\to\mathrm{C}$ *with* generalized *Kannan type contraction mappings in partially ordered metric space, some coincidence point results can be attained.*

Moreover, reader could established common fixed point theorems for new generalized (φ,f) −contraction type mappings where φ is the altering distance function via C −class functions (see, [21]) in metric spaces as follows. The family of all altering distance functions stand for Φ.

- Let (C,d) is a metric space, $\mathrm{F}:\mathrm{C}\times\mathrm{C}\to\mathrm{C}$ is said to be a generalized (φ,f) −contraction type mappings. Assume that there exists $\varphi\in\Phi$ and $\mathrm{f}\in\mathrm{C}$ such that

$$\mathrm{d}\big(\mathrm{F}(\mathrm{a},\mathrm{b}),\mathrm{F}(\mathrm{c},\mathrm{d})\big)\le f\begin{pmatrix} 2^{-1}\times\big(\mathrm{d}(\mathrm{ga},\mathrm{gc})+\mathrm{d}(\mathrm{gb},\mathrm{gd})\big),\\ \varphi\left(2^{-1}\times\big(\mathrm{d}(\mathrm{ga},\mathrm{gc})+\mathrm{d}(\mathrm{gb},\mathrm{gd})\big)\right)\end{pmatrix}. \tag{34}$$

In the above definition, taking $f(s,t)=s\beta(s)$, then (34) reduces to the contraction (1).

- Let (C,d) is a metric space, $\mathrm{F}:\mathrm{C}\times\mathrm{C}\to\mathrm{C}$ is said to be a generalized (φ,f) −contraction type mappings. Assume that there exists $\varphi\in\Phi$ and $\mathrm{f}\in\mathrm{C}$ such that

$$d\big(F(a,b),F(c,d)\big)\le f\begin{pmatrix} 4^{-1}\times\begin{pmatrix} d\big(ga,F(a,b)\big)+d\big(gb,F(b,a)\big)\\ +d\big(gc,F(c,d)\big)+d\big(gd,F(d,c)\big)\end{pmatrix},\\ \varphi\left(4^{-1}\times\begin{pmatrix} d\big(ga,F(a,b)\big)+d\big(gb,F(b,a)\big)\\ +d\big(gc,F(c,d)\big)+d\big(gd,F(d,c)\big)\end{pmatrix}\right)\end{pmatrix}. \tag{35}$$

In the above definition, taking $f(s,t)=s\beta(s)$, then (35) reduces to the contraction (15).

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