Sigma J Eng & Nat Sci 36 (2), 2018, 361-372

Sigma Journal of Engineering and Natural Sciences
Sigma Mühendislik ve Fen Bilimleri Dergisi

**Research Article**

# DERIVATIVES WITH RESPECT TO COMPLETE AND VERTICAL LIFTS OF THE CHEEGER-GROMOLL METRIC ${}^{CG}g$ ON COTANGENT BUNDLE

**Haşim ÇAYIR***

*Department of Mathematics, Giresun University, GİRESUN;* ORCID:0000-0003-0348-8665

**Received: 17.04.2017 Accepted: 12.02.2018**

**ABSTRACT**

In this paper, we define the Cheeger-Gromoll metric in the cotangent bundle $T^*M^n$, which is completely determined by its actionon complete lifts of vector fields. Later, we obtain the covariant and Lie derivatives applied to Cheeger-Gromoll metrics with respect to the complete and vertical lifts of vector and kovector fields, respectively.
**Keywords:** Covarient derivative, lie derivative, Cheeger-Gromoll metric, complete lift, vertical lift.

## 1. INTRODUCTION

Cheeger-Gromoll metric was defined by Cheeger and Gromoll in [2] and the explicit formula for this metric was given by Musso and Tricerri in [12]. The Levi-Civita connection of ${}^{CG}g$ and its Riemannian curvature tensor are calculated by Sekizawa in [17] and corrected by Gudmundsson and Kappos in [9]. In [16] Salimov and Kazimova studied geodesics of the Cheeger-Gromoll metric on the tangent bundle. The similar metric in theoritical physics has been obtained by Tamm (Nobel Laureate in Physics for the year 1958, see [18]). The geometry of the tangent bundle equipped with Cheeger-Gromoll metric is well known and intensively studied (see for example [8, 9, 11, 15, 16]). In [1] Ağca and Salimov investigate curvature properties and geodesics on the cotangent bundle with respect to the Cheeger-Gromoll metric.

The tangent bundles of differentiable manifolds are very important in many areas of mathematics and physics. Cotangent bundle is dual of the tangent bundle. Because of this duality, some of the geometric results are similar to each other. The most significant difference between them is construction of lifts (see [19] for more details). In this paper, we define the Cheeger-Gromoll metric in the cotangent bundle $T^*M^n$, which is completely determined by its action on complete lifts of vector fields. Later, we obtain the covariant and Lie derivatives applied to Cheeger-Gromoll metrics with respect to the complete and vertical lifts of vector and kovector fields, respectively.

[*] Corresponding Author: e-mail: hasim.cayir@giresun.edu.tr, tel: (454) 310 14 44

Let $(M^n, g)$ be $n-$ dimensiona l Riemannian manifold $T^*M^n$ be the cotangent bundle of $M^n$ and $\pi$ the natural projection $T^*M^n \to M^n$. A system of local coordinates $(U, x^i), i = 1,...,n$ in $M^n$ induces on $T^*M^n$ a system of local coordinates $(\pi^{-1}(U), x^i, x^{\bar{i}} = p_i)$, $\bar{i} := n + i = n+1,...,2n,$ where $x^{\bar{i}} = p_i$ are the components of the covector $p$ in each cotangent space $T_x^*M^n$, $x \in U$ with respect to the natural coframe $\{dx^i\}$, $i = 1,...,n$.

We denote $\Im_s^r(M^n)$ the set of all tensor fields of type $(r, s)$ on $M^n$ and by $\Im_s^r(T^*M^n)$ the corresponding set on the cotangent bundle $T^*M^n$. During this paper, manifolds, tensor fields and connections are always supposed to be differentiable of class $C^\infty$.

The local expressions a vector and a covector (1-form) field $X \in \Im_0^1(M^n)$ and $\omega \in \Im_1^0(M^n)$ are $X = X^i \frac{\partial}{\partial x^i}$ and $\omega = \omega_i dx^i$ in $U \subset M^n$, respectively. Then the complete and horizontal lifts ${}^C X, {}^H X \in \Im_0^1(T^*M^n)$ of $X \in \Im_0^1(M^n)$ and the vertical lift ${}^V \omega \in \Im_0^1(T^*M^n)$ of $\omega \in \Im_1^0(M^n)$ are given, with respect to the natural frame $\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^{\bar{i}}}\}$, by

$$X^C = X^i \frac{\partial}{\partial x^i} - \sum_i p_h \partial_i X^h \frac{\partial}{\partial x^{\bar{i}}}, \tag{1.1}$$

$$X^H = X^i \frac{\partial}{\partial x^i} + \sum_i p_h \Gamma_{ij}^h X^j \frac{\partial}{\partial x^{\bar{i}}}, \tag{1.2}$$

$$\omega^V = \sum_i \omega_i \frac{\partial}{\partial x^{\bar{i}}} \tag{1.3}$$

where $\Gamma_{ij}^h$ are the components of the Levi-Civita connection $\nabla_g$ on $M^n$ [13](see [19] for more details).

**Definition 1.1** A Cheeger-Gromoll metric ${}^{CG}g$ is defined on $T^*M^n$ by the following three equations [1, 13]

$${}^{CG}g({}^H X, {}^H Y) = {}^V (g(X, Y)) = g(X, Y) \circ \pi, \tag{1.4}$$

$${}^{CG}g({}^V \omega, {}^H Y) = 0, \tag{1.5}$$

$$^{CG}g(^V\omega, ^V\theta) = \frac{1}{1+r^2}(g^{-1}(\omega,\theta) + g^{-1}(\omega, p)g^{-1}(\theta, p)) \qquad (1.6)$$

f or any $X, Y \in \Im_0^1(M^n)$ and $\omega, \theta \in \Im_1^0(M^n)$, where $r^2 = g^{-1}(p, p) = g^{ij} p_i p_j$
.

Since any tensor field of type $(0,2)$ on $T^*M^n$ is completely determined by its action on vector fields of type $^H X$ and $^V\omega$, it follows that $^{CG}g$ is completely determined by its equations (1.4), (1.5) and (1.6).

We know see, from (1.1) and (1.2), that the complete lift $^C X$ of $X \in \Im_0^1(M^n)$ is expressed by

$$^C X = ^H X - ^V(p(\nabla X)), \qquad (1.7)$$

where $p(\nabla X) = p_i(\nabla_h X^i)dx^h$. Using (1.4), (1.5), (1.6) and (1.7), we have

$$^{CG}g(X^C, Y^C) = (g(X, Y))^V$$
$$+\frac{1}{1+r^2}(g^{-1}(p(\nabla X), p(\nabla Y)) + g^{-1}(p(\nabla X), p).g^{-1}(p(\nabla Y), p), \qquad (1.8)$$

where $g^{-1}(p(\nabla X), p(\nabla Y)) = g^{ij}(p_t \nabla_i X^t)(p_k \nabla_j Y^k)$,

$g^{-1}(p(\nabla X), p) = g^{ij} p_i (p(\nabla X))_j$.

Since the tensor field $^{CG}g \in \Im_2^0(T^*M^n)$ is completely determined also by its action on vector fields type $^C X$ and $^C Y$ (see [19], p.237), we have an alternative characterization of $^{CG}g$ on $T^*M^n$ : $^{CG}g$ is completely determined by the condition (1.8). Similarly, we get the following results

$$\begin{aligned} ^{CG}g(X^C, \omega^V) &= ^{CG}g(X^H - (p(\nabla X))^V, \omega^V) \qquad (1.9) \\ &= -^{CG}g(p(\nabla X))^V, \omega^V) \\ &= -\frac{1}{1+r^2}(g^{-1}(p(\nabla X), \omega) + g^{-1}(p(\nabla X), p)g^{-1}(\omega, p)) \end{aligned}$$

$$\begin{aligned} ^{CG}g(\omega^V, X^C) &= ^{CG}g(\omega^V, X^H - (p(\nabla X))^V) \qquad (1.10) \\ &= ^{CG}g(\omega^V, X^H) - ^{CG}g(\omega^V, (p(\nabla X))^V) \\ &= -\frac{1}{1+r^2}(g^{-1}(\omega, p(\nabla X)) + g^{-1}(\omega, p)g^{-1}(p(\nabla X), p)) \end{aligned}$$

$$^{CG}g(\omega^V, \theta^V) = \frac{1}{1+r^2}(g^{-1}(\omega,\theta) + g^{-1}(\omega, p)g^{-1}(\theta, p)) \qquad (1.11)$$

## 2. MAIN RESULTS

**Definition 2.1** Let $M^n$ be an $n-$dimensional diferentiable manifold. Differantial transformation of algebra $T(M^n)$, defined by

$$D = \nabla_X : T(M^n) \to T(M^n), X \in \Im_0^1(M^n)$$

is called as covariant derivation with respect to vector field $X$ if

$$\nabla_{fX+gY} t = f\nabla_X t + g\nabla_Y t, \tag{2.1}$$

$$\nabla_X f = Xf,$$

where $\forall f, g \in \Im_0^0(M^n),\ \forall X, Y \in \Im_0^1(M^n),\ \forall t \in \Im(M^n)$ (see [10], p.123).

On the other hand, a transformation defined by

$$\nabla : \Im_0^1(M^n) \times \Im_0^1(M^n) \to \Im_0^1(M^n)$$

is called as an affin connection (see for details [10, 14]).

**Proposition 2.1** Covarient differentiation with respect to the complete lift $\nabla^C$ of a symetric affine connection $\nabla$ in $M^n$ to $T^*(M^n)$ has the following properties:

$$\nabla^C_{\omega^V} \theta^V = 0, \nabla^C_{\omega^V} Y^C = -\gamma(\omega o(\nabla Y)) = -(p(\omega o(\nabla Y)))^V, \nabla^C_{X^C} \theta^V = (\nabla_X \theta)^V,$$

$$\nabla^C_{X^C} Y^C = (\nabla_X Y)^C + \gamma(\nabla(\nabla_X Y + \nabla_Y X)) - \gamma(\nabla_X \nabla Y + \nabla_Y \nabla X)$$

$$= (\nabla_X Y)^C + (p((\nabla(\nabla_X Y + \nabla_Y X)) - (\nabla_X \nabla Y + \nabla_Y \nabla X)))^V$$

for $X, Y \in \Im_0^1(M^n),\ \theta, \omega \in \Im_1^0(M^n)$ [19].

**Proposition 2.2** Covarient differentiation with respect to the horizontal lift $\nabla^H$ of a symetric affine connection $\nabla$ in $M^n$ to $T^*(M^n)$ satisfies

$$\nabla^H_{X^H} Y^H = (\nabla_X Y)^H, \nabla^H_{\theta^V} \omega^V = 0,$$

$$\nabla^H_{X^H} \omega^V = (\nabla_X \omega)^V, \nabla^H_{\theta^V} Y^H = 0,$$

for any $X, Y \in \Im_0^1(M^n),\ \theta, \omega \in \Im_1^0(M^n)$ [19].

**Theorem 2.1** Let ${}^{CG}g$ be the Cheeger-Gromoll metric, is defined by (1.8),(1.9),(1.10) and the complete lift $\nabla^C$ of symetric affine connection $\nabla$ in $M^n$ to $T^*(M^n)$. From proposition (1) and proposition (2), we get the following results

$i)$ $(\nabla^C_{\omega^V} {}^{CG}g)(\theta^V, \xi^V) = 0,$

$ii)$ $(\nabla^C_{X^C} {}^{CG}g)(\theta^V, \xi^V) = (\nabla_X (\frac{1}{1+r^2}(g^{-1}(\theta, \xi) + g^{-1}(\theta, p) g^{-1}(\xi, p))))^V$

$$-(\frac{1}{1+r^2}(g^{-1}((\nabla_X\theta),\xi)+g^{-1}((\nabla_X\theta),p)g^{-1}(\xi,p)))^V$$
$$-(\frac{1}{1+r^2}(g^{-1}(\theta,(\nabla_X\xi))+g^{-1}(\theta,p)g^{-1}((\nabla_X\xi),p)))^V,$$

*iii)* $(\nabla^C_{\omega^V}{}^{CG}g)(\theta^V,Z^C)=\frac{1}{1+r^2}(g^{-1}(\theta,p(\omega(\nabla Z)))+g^{-1}(\theta,p)g^{-1}(p(\omega(\nabla Z)),p)),$

*iv)* $(\nabla^C_{X^C}{}^{CG}g)(\theta^V,Z^C)=-(\nabla_X(\frac{1}{1+r^2}(g^{-1}(\theta,p(\nabla Z))+g^{-1}(\theta,p)g^{-1}(p(\nabla Z),p))))^V$

$$+\frac{1}{1+r^2}(g^{-1}(\nabla_X\theta,p(\nabla Z))+g^{-1}(\nabla_X\theta,p)g^{-1}(p(\nabla Z),p)$$
$$+\frac{1}{1+r^2}(g^{-1}(\theta,p(\nabla(\nabla_X Z)))+g^{-1}(\theta,p)g^{-1}(p(\nabla(\nabla_X Z))))$$
$$-\frac{1}{1+r^2}(g^{-1}(\theta,(p(\nabla(\nabla_X Z+\nabla_Z X)-(\nabla_X(\nabla Z)+\nabla_Z(\nabla_X))))$$
$$-\frac{1}{1+r^2}(g^{-1}(\theta,p)g^{-1}(p(\nabla(\nabla_X Z+\nabla_Z X)-(\nabla_X(\nabla Z)+\nabla_Z(\nabla_X))),p)),$$

*v)* $(\nabla^C_{\omega^V}{}^{CG}g)(Y^C,\xi^V)=\frac{1}{1+r^2}(g^{-1}(p(\omega(\nabla Y)),\xi)+g^{-1}(p(\omega(\nabla Y),p)g^{-1}(\xi,p)),$

*vi)* $(\nabla^C_{X^C}{}^{CG}g)(Y^C,\xi^V)=-(\nabla_X\frac{1}{1+r^2}(g^{-1}(p(\nabla Y),\xi)+g^{-1}(p(\nabla Y),p)g^{-1}(\xi,p)))$

$$+\frac{1}{1+r^2}(g^{-1}(p(\nabla(\nabla_X Y)),\xi)+g^{-1}(p(\nabla(\nabla_X X)),p)g^{-1}(\xi,p))$$
$$-\frac{1}{1+r^2}(g^{-1}(p(\nabla(\nabla_X Y+\nabla_Y X)-(\nabla_X(\nabla Y)+\nabla_Y(\nabla X))),\xi))$$
$$-\frac{1}{1+r^2}(g^{-1}(p(\nabla(\nabla_X Y+\nabla_Y X)-(\nabla_X(\nabla Y)+\nabla_Y(\nabla X))),p)g^{-1}(\xi,p))$$
$$+\frac{1}{1+r^2}(g^{-1}(p(\nabla Y),\nabla_X\xi)+g^{-1}(p(\nabla Y),p)g^{-1}(\nabla_X\xi,p)),$$

*vii)* $(\nabla^C_{\omega^V}{}^{CG}g)(Y^C,Z^C)=-\frac{1}{1+r^2}(g^{-1}(p(\omega(\nabla Y),p(\nabla Z))+g^{-1}(p(\omega(\nabla Y)),p)g^{-1}(p(\nabla Z),p))$

$$-\frac{1}{1+r^2}(g^{-1}(p(\nabla Y),p(\omega(\nabla Z))+g^{-1}(p(\nabla Y),p)g^{-1}(p(\omega(\nabla Z)),p)),$$

where the complete and horizontal lifts $X^C\in\Im^1_0(T^*M^n)$ of $X\in\Im^1_0(M^n)$ and the vertical lift $\omega^V\in\Im^1_0(T^*M^n)$ of $\omega\in\Im^0_1(M^n)$ defined by (1.1) and (1.3), respectively.

**Proof.** $i)\ (\nabla^{C}_{\omega^V}{}^{CG}g)(\theta^V,\xi^V)=\nabla^{C}_{\omega^V}{}^{CG}g(\theta^V,\xi^V)-{}^{CG}g(\nabla^{C}_{\omega^V}\theta^V,\xi^V)-{}^{CG}g(\theta^V,\nabla^{C}_{\omega^V}\xi^V)$

$$=\omega^V(\frac{1}{1+r^2}(g^{-1}(\theta,\xi)+g^{-1}(\theta,p)g^{-1}(\xi,p)))^V$$
$$=0$$

$ii)\ (\nabla^{C}_{X^C}{}^{CG}g)(\theta^V,\xi^V)=\nabla^{C}_{X^C}{}^{CG}g(\theta^V,\xi^V)-{}^{CG}g(\nabla^{C}_{X^C}\theta^V,\xi^V)-{}^{CG}g(\theta^V,\nabla^{C}_{X^C}\xi^V)$

$$=(\nabla_X(\frac{1}{1+r^2}(g^{-1}(\theta,\xi)+g^{-1}(\theta,p)g^{-1}(\xi,p))))^V$$
$$-(\frac{1}{1+r^2}(g^{-1}((\nabla_X\theta),\xi)+g^{-1}((\nabla_X\theta),p)g^{-1}(\xi,p)))^V$$
$$-(\frac{1}{1+r^2}(g^{-1}(\theta,(\nabla_X\xi))+g^{-1}(\theta,p)g^{-1}((\nabla_X\xi),p)))^V$$

$iii)\ (\nabla^{C}_{\omega^V}{}^{CG}g)(\theta^V,Z^C)=\nabla^{C}_{\omega^V}{}^{CG}g(\theta^V,Z^C)-{}^{CG}g(\nabla^{C}_{\omega^V}\theta^V,Z^C)-{}^{CG}g(\theta^V,\nabla^{C}_{\omega^V}Z^C)$

$$=\nabla^{C}_{\omega^V}(-\frac{1}{1+r^2}(g^{-1}(\theta,p(\nabla Z))+g^{-1}(\theta,p)g^{-1}(p(\nabla Z),p))$$
$$-{}^{CG}g(\theta^V,-(p(\omega(\nabla Z)))^V)$$
$$=-\omega^V(\frac{1}{1+r^2}(g^{-1}(\theta,p(\nabla Z))+g^{-1}(\theta,p)g^{-1}(p(\nabla Z),p)))^V$$
$$+\frac{1}{1+r^2}(g^{-1}(\theta,p(\omega(\nabla Z)))+g^{-1}(\theta,p)g^{-1}(p(\omega(\nabla Z)),p))$$
$$=\frac{1}{1+r^2}(g^{-1}(\theta,p(\omega(\nabla Z)))+g^{-1}(\theta,p)g^{-1}(p(\omega(\nabla Z)),p))$$

$iv)\ (\nabla^{C}_{X^C}{}^{CG}g)(\theta^V,Z^C)=\nabla^{C}_{X^C}{}^{CG}g(\theta^V,Z^C)-{}^{CG}g(\nabla^{C}_{X^C}\theta^V,Z^C)-{}^{CG}g(\theta^V,\nabla^{C}_{X^C}Z^C)$

$$=-(\nabla_X(\frac{1}{1+r^2}(g^{-1}(\theta,p(\nabla Z))+g^{-1}(\theta,p)g^{-1}(p(\nabla Z),p))))^V$$
$$+\frac{1}{1+r^2}(g^{-1}(\nabla_X\theta,p(\nabla Z))+g^{-1}(\nabla_X\theta,p)g^{-1}(p(\nabla Z),p)$$
$$+\frac{1}{1+r^2}(g^{-1}(\theta,p(\nabla(\nabla_X Z)))+g^{-1}(\theta,p)g^{-1}(p(\nabla(\nabla_X Z))))$$
$$-\frac{1}{1+r^2}(g^{-1}(\theta,(p(\nabla(\nabla_X Z+\nabla_Z X)-(\nabla_X(\nabla Z)+\nabla_Z(\nabla_X))))$$
$$-\frac{1}{1+r^2}(g^{-1}(\theta,p)g^{-1}(p(\nabla(\nabla_X Z+\nabla_Z X)-(\nabla_X(\nabla Z)+\nabla_Z(\nabla_X))),p))$$

$v)\ (\nabla^{C}_{\omega^V}{}^{CG}g)(Y^C,\xi^V)=\nabla^{C}_{\omega^V}{}^{CG}g(Y^C,\xi^V)-{}^{CG}g(\nabla^{C}_{\omega^V}Y^C,\xi^V)-{}^{CG}g(Y^C,\nabla^{C}_{\omega^V}\xi^V)$

$$= -\omega^V (\frac{1}{1+r^2}(g^{-1}(p(\nabla Y),\xi) + g^{-1}(p(\nabla Y),p)g^{-1}(\xi,p)))^V$$

$$+\frac{1}{1+r^2}(g^{-1}(p(\omega(\nabla Y)),\xi) + g^{-1}(p(\omega(\nabla Y),p)g^{-1}(\xi,p))$$

$$= \frac{1}{1+r^2}(g^{-1}(p(\omega(\nabla Y)),\xi) + g^{-1}(p(\omega(\nabla Y),p)g^{-1}(\xi,p))$$

*vi*) $(\nabla^C_{X^C}{}^{CG}g)(Y^C,\xi^V) = \nabla^C_{X^C}{}^{CG}g(Y^C,\xi^V) - {}^{CG}g(\nabla^C_{X^C}Y^C,\xi^V) - {}^{CG}g(Y^C,\nabla^C_{X^C}\xi^V)$

$$= -(\nabla_X \frac{1}{1+r^2}(g^{-1}(p(\nabla Y),\xi) + g^{-1}(p(\nabla Y),p)g^{-1}(\xi,p)))$$

$$+\frac{1}{1+r^2}(g^{-1}(p(\nabla(\nabla_X Y)),\xi) + g^{-1}(p(\nabla(\nabla_X X)),p)g^{-1}(\xi,p))$$

$$-\frac{1}{1+r^2}(g^{-1}(p(\nabla(\nabla_X Y + \nabla_Y X) - (\nabla_X(\nabla Y) + \nabla_Y(\nabla X))),\xi))$$

$$-\frac{1}{1+r^2}(g^{-1}(p(\nabla(\nabla_X Y + \nabla_Y X) - (\nabla_X(\nabla Y) + \nabla_Y(\nabla X))),p)g^{-1}(\xi,p))$$

$$+\frac{1}{1+r^2}(g^{-1}(p(\nabla Y),\nabla_X\xi) + g^{-1}(p(\nabla Y),p)g^{-1}(\nabla_X\xi,p))$$

*vii*) $(\nabla^C_{\omega^V}{}^{CG}g)(Y^C,Z^C) = \nabla^C_{\omega^V}{}^{CG}g(Y^C,Z^C) - {}^{CG}g(\nabla^C_{\omega^V}Y^C,Z^C) - {}^{CG}g(Y^C,\nabla^C_{\omega^V}Z^C)$

$$= (\frac{1}{1+r^2}(g^{-1}(p(\nabla Y),p(\nabla Z)) + g^{-1}(p(\nabla Y),p)g^{-1}(p(\nabla Z),p)))^V$$

$$+\nabla^C_{\omega^V}(g(Y,Z))^V + {}^{CG}g((p(\omega(\nabla Y)))^V,Z^C) + {}^{CG}g(Y^C,(p(\omega(\nabla Z)))^V)$$

$$= -\frac{1}{1+r^2}(g^{-1}(p(\omega(\nabla Y),p(\nabla Z)) + g^{-1}(p(\omega(\nabla Y)),p)g^{-1}(p(\nabla Z),p))$$

$$-\frac{1}{1+r^2}(g^{-1}(p(\nabla Y),p(\omega(\nabla Z)) + g^{-1}(p(\nabla Y),p)g^{-1}(p(\omega(\nabla Z)),p))$$

**Definition 2.2** Let $M^n$ be an $n-$dimensional diferentiable manifold. Differential transformation $D = L_X$ is called as Lie derivation with respect to vector field $X \in \Im^1_0(M^n)$ if

$$L_X f = Xf, \forall f \in \Im^0_0(M^n), \tag{2.2}$$

$$L_X Y = [X,Y], \forall X,Y \in \Im^1_0(M^n).$$

$[X,Y]$ is called by Lie bracked. The Lie derivative $L_X F$ of a tensor field $F$ of type $(1,1)$ with respect to a vector field $X$ is defined by [3, 4, 5, 6, 7, 19].

$$(L_X F)Y = [X, FY] - F[X, Y]. \tag{2.3}$$

**Proposition 2.3** If $X, Y \in \Im_0^1(M)$, $\omega, \theta \in \Im_0^1(M)$ and $F, G \in \Im_1^1(M)$, then

$$\left[\omega^V, \theta^V\right] = 0, \left[\omega^V, \gamma F\right] = \left(\omega o F\right)^V$$
$$\left[\gamma F, \gamma G\right] = \gamma\left[F, G\right], \left[X^C, \omega^V\right] = \left(L_X \omega\right)^V$$
$$\left[X^C, \gamma F\right] = \gamma(L_X F), \left[X^C, Y^C\right] = \left[X, Y\right]^C,$$

where $\omega o F$ is a $1-$form defined by $\left(\omega o F\right)\left(Z\right) = \omega\left(FZ\right)$ for any $Z \in \Im_0^1(M)$ and $L_X$ the operator of Lie derivation with respect to $X$ [19].

**Proposition 2.4** *If* $X \in \Im_0^1(M)$, $\omega \in \Im_1^0(M)$ *and* $F, G \in \Im_1^1(M)$*, then [19]*

$$F^C \omega^V = \left(\omega o F\right)^V, F^C \gamma G = \gamma(GF),$$
$$F^C X^C = (FX)^C + \gamma(L_X F).$$

**Theorem 2.2** Let ${}^{CG}g$ be Cheeger-Gromoll metric, is deffined by (1.8), (1.9), (1.10), and $L_X$ the operator Lie derivation with respect to $X$. From proposition (3) and proposition (4), we get the following results

$i)$ $(L_{\omega^V}{}^{CG}g)(\theta^V, \xi^V) = 0,$

$$ii) \quad (L_{X^C}{}^{CG}g)(\theta^V, \xi^V) = (L_X \frac{1}{1+r^2}(g^{-1}(\theta, \xi) + g^{-1}(\theta, p)g^{-1}(\xi, p)))^V$$
$$-(\frac{1}{1+r^2}(g^{-1}(L_X\theta, \xi) + g^{-1}(L_X\theta, p)g^{-1}(\xi, p))$$
$$-\frac{1}{1+r^2}({}^{CG}g(\theta, L_X\xi) + g^{-1}(\theta, p)g^{-1}(L_X\xi), p)),$$

$$iii) \quad (L_{\omega^V}{}^{CG}g)(\theta^V, Z^C) = \frac{1}{1+r^2}(g^{-1}(\theta, L_Z\omega) + g^{-1}(\theta, p)g^{-1}(L_Z\omega, p)),$$

$$iv) \quad (L_{X^C}{}^{CG}g)(\theta^V, Z^C) = -(L_X \frac{1}{1+r^2}(g^{-1}(\theta, p(\nabla Z)) + g^{-1}(\theta, p)g^{-1}(p(\nabla Z), p)))^V$$
$$+\frac{1}{1+r^2}(g^{-1}(L_X\theta, p(\nabla Z)) + g^{-1}(L_X\theta, p)g^{-1}(p(\nabla Z), p))$$
$$+\frac{1}{1+r^2}(g^{-1}(\theta, p(\nabla(L_X Z))) + g^{-1}(\theta, p)g^{-1}(p(\nabla(L_X Z)), p))$$

$$v) \quad (L_{\omega^V}{}^{CG}g)(Y^C, \xi^V) = \frac{1}{1+r^2}(g^{-1}(L_Y\omega, \xi) + g^{-1}(L_Y\omega, p)g^{-1}(\xi, p)),$$

$$vi)\ \ (L_{X^C}{}^{CG}g)(Y^C,\xi^V)=-(L_X\frac{1}{1+r^2}(g^{-1}(p(\nabla Y),\xi)+g^{-1}(p(\nabla Y),p)g^{-1}(\xi,p)))^V$$
$$+\frac{1}{1+r^2}(g^{-1}(p(\nabla(L_XY)),\xi)+g^{-1}(p(\nabla(L_XY)),p)g^{-1}(\xi,p))$$
$$+\frac{1}{1+r^2}(g^{-1}(p(\nabla Y),L_X\xi)+g^{-1}(p(\nabla Y),p)g^{-1}(L_X\xi,p)),$$

$$vii)\ \ (L_{\omega^V}{}^{CG}g)(Y^C,Z^C)=-\frac{1}{1+r^2}(g^{-1}(L_Y\omega,p(\nabla Z))+g^{-1}(L_Y\omega,p)g^{-1}(p(\nabla Z),p))$$
$$-\frac{1}{1+r^2}(g^{-1}(p(\nabla Y),L_Z\omega)+g^{-1}(p(\nabla Y),p)g^{-1}(L_Z\omega,p))$$

$$vii)\ \ (L_{X^C}{}^{CG}g)(Y^C,Z^C)=((L_Xg)(Y,Z))^V$$
$$+(L_X\frac{1}{1+r^2}(g^{-1}(p(\nabla Y),p(\nabla Z))+g^{-1}(p(\nabla Y),p)g^{-1}(p(\nabla Z),p)))^V$$
$$-\frac{1}{1+r^2}(g^{-1}(p(\nabla(L_XY)),p(\nabla Z))+g^{-1}(p(\nabla(L_XY)),p)g^{-1}(p(\nabla Z),p))$$
$$-\frac{1}{1+r^2}(g^{-1}(p(\nabla Y),p(\nabla(L_XZ)))+g^{-1}(p(\nabla Y),p)g^{-1}(p(\nabla(L_XZ)),p)),$$

where the complete lifts $X^C\in\Im_0^1(T^*M^n)$ of $X\in\Im_0^1(M^n)$ and the vertical lift $\omega^V\in\Im_0^1(T^*M^n)$ of $\omega\in\Im_1^0(M^n)$ defined by (1.1) and (1.3), respectively.

*Proof.*

$$i)\ \ (L_{\omega^V}{}^{CG}g)(\theta^V,\xi^V)=L_{\omega^V}{}^{CG}g(\theta^V,\xi^V)-{}^{CG}g(L_{\omega^V}\theta^V,\xi^V)-{}^{CG}g(\theta^V,L_{\omega^V}\xi^V)$$
$$=L_{\omega^V}(\frac{1}{1+r^2}(g^{-1}(\theta,\xi)+g^{-1}(\theta,p)g^{-1}(\xi,p)))^V$$
$$=0$$

$$ii)\ \ (L_{X^C}{}^{CG}g)(\theta^V,\xi^V)=L_{X^C}{}^{CG}g(\theta^V,\xi^V)-{}^{CG}g(L_{X^C}\theta^V,\xi^V)-{}^{CG}g(\theta^V,L_{X^C}\xi^V)$$
$$=L_{X^C}(\frac{1}{1+r^2}(g^{-1}(\theta,\xi)+g^{-1}(\theta,p)g^{-1}(\xi,p)))^V$$
$$-{}^{CG}g((L_X\theta)^V,\xi^V)-{}^{CG}g(\theta^V,(L_X\xi)^V)$$
$$=(L_X\frac{1}{1+r^2}(g^{-1}(\theta,\xi)+g^{-1}(\theta,p)g^{-1}(\xi,p)))^V$$
$$-(\frac{1}{1+r^2}(g^{-1}(L_X\theta,\xi)+g^{-1}(L_X\theta,p)g^{-1}(\xi,p))$$

$$-\frac{1}{1+r^2}({}^{CG}g(\theta, L_X\xi)+g^{-1}(\theta,p)g^{-1}(L_X\xi),p))$$

$$iii)\ \ (L_{\omega^V}{}^{CG}g)(\theta^V,Z^C)=L_{\omega^V}{}^{CG}g(\theta^V,Z^C)-{}^{CG}g(L_{\omega^V}\theta^V,Z^C)-{}^{CG}g(\theta^V,L_{\omega^V}Z^C)$$

$$=-\omega^V(\frac{1}{1+r^2}(g^{-1}(\theta,p(\nabla Z))+g^{-1}(\theta,p)g^{-1}(p(\nabla Z),p)))^V$$

$$+\frac{1}{1+r^2}(g^{-1}(\theta,L_Z\omega)+g^{-1}(\theta,p)g^{-1}(L_Z\omega,p))$$

$$=\frac{1}{1+r^2}(g^{-1}(\theta,L_Z\omega)+g^{-1}(\theta,p)g^{-1}(L_Z\omega,p))$$

$$iv)\ \ (L_{X^C}{}^{CG}g)(\theta^V,Z^C)=L_{X^C}{}^{CG}g(\theta^V,Z^C)-{}^{CG}g(L_{X^C}\theta^V,Z^C)-{}^{CG}g(\theta^V,L_{X^C}Z^C)$$

$$=-L_{X^C}(\frac{1}{1+r^2}(g^{-1}(\theta,p(\nabla Z))+g^{-1}(\theta,p)g^{-1}(p(\nabla Z),p)))^V$$

$$-{}^{CG}g((L_X\theta)^V,Z^C)-{}^{CG}g(\theta^V,(L_XZ)^C)$$

$$=-(L_X\frac{1}{1+r^2}(g^{-1}(\theta,p(\nabla Z))+g^{-1}(\theta,p)g^{-1}(p(\nabla Z),p)))^V$$

$$+\frac{1}{1+r^2}(g^{-1}(L_X\theta,p(\nabla Z))+g^{-1}(L_X\theta,p)g^{-1}(p(\nabla Z),p))$$

$$+\frac{1}{1+r^2}(g^{-1}(\theta,p(\nabla(L_XZ)))+g^{-1}(\theta,p)g^{-1}(p(\nabla(L_XZ)),p))$$

$$v)\ \ (L_{\omega^V}{}^{CG}g)(Y^C,\xi^V)=L_{\omega^V}{}^{CG}g(Y^C,\xi^V)-{}^{CG}g(L_{\omega^V}Y^C,\xi^V)-{}^{CG}g(Y^C,L_{\omega^V}\xi^V)$$

$$=-\omega^V(\frac{1}{1+r^2}(g^{-1}(p(\nabla Y),\xi)+g^{-1}(p(\nabla Y),p)g^{-1}(\xi,p)))^V$$

$$+\frac{1}{1+r^2}(g^{-1}(L_Y\omega,\xi)+g^{-1}(L_Y\omega,p)g^{-1}(\xi,p))$$

$$=\frac{1}{1+r^2}(g^{-1}(L_Y\omega,\xi)+g^{-1}(L_Y\omega,p)g^{-1}(\xi,p))$$

$$vi)\ \ (L_{X^C}{}^{CG}g)(Y^C,\xi^V)=L_{X^C}{}^{CG}g(Y^C,\xi^V)-{}^{CG}g(L_{X^C}Y^C,\xi^V)-{}^{CG}g(Y^C,L_{X^C}\xi^V)$$

$$=-L_{X^C}(\frac{1}{1+r^2}(g^{-1}(p(\nabla Y),\xi)+g^{-1}(p(\nabla Y),p)g^{-1}(\xi,p)))^V$$

$$-{}^{CG}g((L_XY)^C,\xi^V)-{}^{CG}g(Y^C,(L_X\xi)^V)$$

$$=-(L_X\frac{1}{1+r^2}(g^{-1}(p(\nabla Y),\xi)+g^{-1}(p(\nabla Y),p)g^{-1}(\xi,p)))^V$$

$$+\frac{1}{1+r^2}(g^{-1}(p(\nabla(L_X Y)),\xi)+g^{-1}(p(\nabla(L_X Y)),p)g^{-1}(\xi,p))$$

$$+\frac{1}{1+r^2}(g^{-1}(p(\nabla Y),L_X\xi)+g^{-1}(p(\nabla Y),p)g^{-1}(L_X\xi,p))$$

*vii)* $(L_{\omega^V}{}^{CG}g)(Y^C,Z^C)=L_{\omega^V}{}^{CG}g(Y^C,Z^C)-{}^{CG}g(L_{\omega^V}Y^C,Z^C)-{}^{CG}g(Y^C,L_{\omega^V}Z^C)$

$$=L_{\omega^V}(\frac{1}{1+r^2}(g^{-1}(p(\nabla Y),p(\nabla Z))+g^{-1}(p(\nabla Y),p)g^{-1}(p(\nabla Z),p)))^V$$

$$+L_{\omega^V}(g(Y,Z))^V+{}^{CG}g((L_Y\omega)^V,Z^C)+{}^{CG}g(Y^C,L_Z\omega)^V)$$

$$=-\frac{1}{1+r^2}(g^{-1}(L_Y\omega,p(\nabla Z))+g^{-1}(L_Y\omega,p)g^{-1}(p(\nabla Z),p))$$

$$-\frac{1}{1+r^2}(g^{-1}(p(\nabla Y),L_Z\omega)+g^{-1}(p(\nabla Y),p)g^{-1}(L_Z\omega,p))$$

*viii)* $(L_{X^C}{}^{CG}g)(Y^C,Z^C)=L_{X^C}{}^{CG}g(Y^C,Z^C)-{}^{CG}g(L_{X^C}Y^C,Z^C)-{}^{CG}g(Y^C,L_{X^C}Z^C)$

$$=L_{X^C}(\frac{1}{1+r^2}(g^{-1}(p(\nabla Y),p(\nabla Z))+g^{-1}(p(\nabla Y),p)g^{-1}(p(\nabla Z),p))$$

$$+L_{X^C}(g(Y,Z))^V-{}^{CG}g((L_X Y)^C,Z^C)-{}^{CG}g(Y^C,(L_X Z)^C)$$

$$=((L_X g)(Y,Z))^V$$

$$+(L_X\frac{1}{1+r^2}(g^{-1}(p(\nabla Y),p(\nabla Z))+g^{-1}(p(\nabla Y),p)g^{-1}(p(\nabla Z),p)))^V$$

$$-\frac{1}{1+r^2}(g^{-1}(p(\nabla(L_X Y)),p(\nabla Z))+g^{-1}(p(\nabla(L_X Y)),p)g^{-1}(p(\nabla Z),p))$$

$$-\frac{1}{1+r^2}(g^{-1}(p(\nabla Y),p(\nabla(L_X Z)))+g^{-1}(p(\nabla Y),p)g^{-1}(p(\nabla(L_X Z)),p))$$

## 3. CONCLUSION

In this paper, firstly, we define the Cheeger-Gromoll metric in the cotangent bundle $T^*M^n$, which is completely determined by its action on complete lifts of vector fields. Later, we obtain the covariant and Lie derivatives applied to Cheeger-Gromoll metrics with respect to the complete and vertical lifts of vector and kovector fields, respectively.

## REFERENCES


[1] F. Ağca, A.A. Salimov, Some notes concerning Cheenger-Gromoll metrics. Hacet. J. Math. Stat., 42 **(**2013), no.5, 533-549.

[2] J. Cheege, D. Gromoll, On the structure of complete manifolds of nonnegative curvature. Ann. of Math. 96 **(**1972), 413-443.

[3] H. Çayır, Lie derivatives of almost contact structure and almost paracontact structure with respect to $X^V$ and $X^H$ on tangent bundle $T(M)$. Proceedings of the Institute of Mathematics and Mechanics. 42 **(**2016**),** no.1, 38-49.

[4] H. Çayır, Tachibana and Vishnevskii Operators Applied to $X^V$ and $X^C$ in Almost Paracontact Structure on Tangent Bundle $T(M)$. Ordu Üniversitesi Bilim ve Teknoloji Dergisi, 6 (2016), no.1, 67-82.

[5] H. Çayır, Tachibana and Vishnevskii Operators Applied to $X^V$ and $X^H$ in Almost Paracontact Structure on Tangent Bundle $T(M)$. New Trends in Mathematical Sciences, 4 (2016), no.3, 105-115.

[6] H. Çayır, G. Köseoğlu, Lie Derivatives of Almost Contact Structure and Almost Paracontact Structure With Respect to $X^C$ and $X^V$ on Tangent Bundle $T(M)$. New Trends in Mathematical Sciences, 4 (2016), no.1, 153-159.

[7] H. Çayır, K. Akdağ, Some notes on almost paracomplex structures associated with the diagonal lifts and operators on cotangent bundle. New Trends in Mathematical Sciences. 4 (2016), no.4, 42-50.

[8] A. Gezer, M. Altunbas, Some notes concerning Riemannian metrics of Cheeger Gromoll type.- J. Math. Anal. Appl. 396 (2012), 119-132.

[9] S. Gudmundsson, E. Kappos, On the geometry of the tangent bunles with the Chegeer-Gromoll metric.- Tokyo J. Math. 25 (2002), no.1, 75-83.

[10] S. Kobayashi and K. Nomizu, Foundations of Differential Geometry-Volume I. John Wiley & Sons, Inc, New York, 1963.

[11] M. I. Muntenau, Chegeer Gromoll type metrics on the tangent bundle.- Sci. Ann. Univ. Agric. Sci. Vet. Med., 49 **(**2006), No.2, 257-268.

[12] E. Musso and F. Tricerri, Riemannian metric on tangent bundles.- Ann. Math. Pura. Appl. 150 **(**1988**),** no.4, 1-19.

[13] F. Ocak, Para-Nordenian Structures on the Cotangent Bundle with Respect to the Cheeger-Gromoll Metric.- Proceedings of the Institute of Mathematics and Mechanics, 41 **(**2015), no.2, 63-69.

[14] A. A. Salimov,Tensor Operators and Their applications, Nova Science Publ., New York, 2013.

[15] A. A. Salimov, A. Gezer and M. Iscan, On para-Kahler-Norden structures on the tangent bundles.- Ann. Polon. Math., 103 (2012), no.3, 247-261.

[16] A. A. Salimov, S. Kazimova, Geodesics of the Cheeger-Gromoll metric.- Turk. J. Math., 32 (2008).

[17] M. Sekizawa, Curvatures of tangent bundles with Cheeger-Gromoll metric.- Tokyo J. Math., 14 (1991), 407-417.

[18] I. E. Tamn, Sobrainie nauchnyh trudov (Collection of scientific works), II, (Russian) Nauka, Moscow, 1975. 448 pp.

[19] K. Yano, S .Ishihara, Tangent and Cotangent Bundles Differential geometry. Pure and Applied Mathematics. Mercel Dekker, Inc, New York, 1973.