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Research Article

DIFFERENTIAL INVARIANTS FOR A CURVE FAMILY IN $GL(n,\mathbb{R})$

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ABSTRACT

In this paper, we obtain generators of differential invariants for a curve family in $GL(n,\mathbb{R})$. Then we define $GL(n,\mathbb{R})$ −equivalence of the curve families and develop a point of view for equivalence problem. Using these generators, we give a solution to the problem.

Keywords: Affine differential geometry, affine differential invariants, affine equivalence.

1. INTRODUCTION

The notion of affine differential geometry arose from Felix Klein's Erlangen Program in 1872. According to this program, affine differential geometry consists of properties which are invariant under the affine transformations. In affine differential geometry, studies have been done about affine invariants and generators of affine invariants. Based on this, solution of the equivalence problem has been studied also.

Differential geometry of curves has been studied for many years. It's been studied in many aspects in the groups $SL(n,\mathbb{R}), EA(n,\mathbb{R}), GL(n,\mathbb{R})$ which are the subgroups of the affine group. In some of these studies, invariants such as arc-length, curvature have been examined. In [1] centro-affine invariants, arc length and curvature functions, of a curve in affine n −space are obtained. In addition, several authors studied the affine curves and their invariants in several works [2-6]. Also, affine surfaces studied in [7-9].

Invariants of n curves and equivalence of n curves in $SL(n,\mathbb{R})$ are given in [10]. In $SAff(n,\mathbb{R})$, the equivalence problem of two curves is studied in [11]. In this study, the system of differential invariants for three curves in $GL(n,\mathbb{R})$ is studied and by using this system, equivalence of two curve families which consist of three curves is given. Also, it is shown that the system of differential invariants of this curve family is minimal. It should be noted that we study the problem under a fixed parametrization of the curves.

2. GENERATING SYSTEM OF DIFFERENTIAL INVARIANTS

For three curves x_1, x_2, x_3 a differential polynomial of these curves is given by $P\{x_1,x_2,x_3\} = P(x_1,x_2,x_3,x_1',x_2',x_3',\ldots,x_1^{(m)},x_2^{(m)},x_3^{(m)})$ for some natural number m.

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Function $f < x_1, x_2, x_3 >= \frac{P_1\{x_1,x_2,x_3\}}{P_2\{x_1,x_2,x_3\}}$ such that $P_2\{x_1, x_2, x_3\} \neq 0$ is called a differential rational function. The set of all differential rational functions is denoted by $\mathbb{R} < x_1, x_2, x_3 >$.

For an element $g \in GL(n, \mathbb{R})$, if $f < gx_1, gx_2, gx_3 >= f < x_1, x_2, x_3 >$ then the fuction f is called a $GL(n, \mathbb{R})$ −invariant differential rational function. The set of all $GL(n, \mathbb{R})$ −invariant differential functions is denoted by $\mathbb{R} < x_1, x_2, x_3 >^G$. $\mathbb{R} < x_1, x_2, x_3 >^G$ is a differential subfield and a sub $\mathbb{R}$ −algebra of $\mathbb{R} < x_1, x_2, x_3 >$.

The following lemma is the standard bracket syzygy in classical invariant theory.

Lemma 2.1. For vectors $x_0, x_1, \ldots, x_n, y_2, \ldots, y_n$ in $\mathbb{R}^n$, the following equation holds:

$$[x_1\, x_2 \ldots x_n][x_0\, y_2 \ldots y_n] - [x_0\, x_2 \ldots x_n][x_1\, y_2 \ldots y_n] - \ldots - [x_1\, x_2 \ldots x_0][x_n\, y_2 \ldots y_n] = 0.$$

Definition 2.2. A curve x_1 in $\mathbb{R}^n$ is called $GL(n, \mathbb{R})$ −regular if $[x_1\, x_1{}' \ldots x_1^{(n-1)}] \neq 0$.

Theorem 2.3. Let x_1, x_2, x_3 be curves in $\mathbb{R}^n$ such that x_1 is $GL(n, \mathbb{R})$ −regular, then the generating system of $\mathbb{R} < x_1, x_2, x_3 >^G$ is as follows

$$\frac{\left[x_1 \ldots x_1^{(i-1)}\, x_1^{(n)}\, x_1^{(i+1)} \ldots x_1^{(n-1)}\right]}{\left[x_1\, x_1{}' \ldots x_1^{(n-1)}\right]}, \frac{\left[x_1 \ldots x_1^{(i-1)}\, x_2\, x_1^{(i+1)} \ldots x_1^{(n-1)}\right]}{\left[x_1\, x_1{}' \ldots x_1^{(n-1)}\right]},$$
$$\frac{[x_1 \ldots x_1^{(i-1)}\, x_3\, x_1^{(i+1)} \ldots x_1^{(n-1)}]}{[x_1\, x_1{}' \ldots x_1^{(n-1)}]}, i = 0,1, \ldots, n-1.$$

Proof. For the group $G = GL(n, \mathbb{R})$, generators of the set $\mathbb{R}(x_\tau, \tau \in \Delta)^G$ with respect to a family of vectors $\{x_\tau, \tau \in \Delta\}$ are

$$\frac{[x_0 \ldots x_{(i-1)}\, x_\tau\, x_{(i+1)} \ldots x_{n-1}]}{[x_0 \ldots x_{n-1}]}, \qquad i = 0,1, \ldots, n-1, \qquad \tau \in \Delta \backslash \{0,1, \ldots, n-1\},$$

where $\Delta = \mathbb{N} \cup \{0\}$. [12]

Substituting the vectors $x_1, x_2, x_3, x_1{}', x_2{}', x_3{}', \ldots, x_1^{(K)}, x_2^{(K)}, x_3^{(K)}, \ldots$ for the vectors x_τ in the above generators, we obtain generators of $\mathbb{R}(x_1, x_2, x_3, x_1{}', x_2{}', x_3{}', \ldots, x_1^{(K)}, x_2^{(K)}, x_3^{(K)}, \ldots)^G$ as

$$\begin{aligned} &\frac{\left[x_1 \ldots x_1^{(i-1)}\, x_1^{(\tau)}\, x_1^{(i+1)} \ldots x_1^{(n-1)}\right]}{\left[x_1\, x_1{}' \ldots x_1^{(n-1)}\right]}, \; i = 0, \ldots, n-1, \tau \in \Delta \backslash \{0,1, \ldots, n-1\} \\ &\frac{\left[x_1 \ldots x_1^{(i-1)}\, x_2^{(\tau)}\, x_1^{(i+1)} \ldots x_1^{(n-1)}\right]}{\left[x_1\, x_1{}' \ldots x_1^{(n-1)}\right]}, \; \tau \geq 0 \\ &\frac{[x_1 \ldots x_1^{(i-1)}\, x_3^{(\tau)}\, x_1^{(i+1)} \ldots x_1^{(n-1)}]}{[x_1\, x_1{}' \ldots x_1^{(n-1)}]}, \; \tau \geq 0. \end{aligned} \tag{2.1}$$

Firstly, we will show by induction on τ that the functions $\frac{\left[x_1 \ldots x_1^{(i-1)}\, x_1^{(\tau)}\, x_1^{(i+1)} \ldots x_1^{(n-1)}\right]}{\left[x_1\, x_1{}' \ldots x_1^{(n-1)}\right]}$, $\tau \geq n$ are generated by $\frac{\left[x_1 \ldots x_1^{(i-1)}\, x_1^{(n)}\, x_1^{(i+1)} \ldots x_1^{(n-1)}\right]}{\left[x_1\, x_1{}' \ldots x_1^{(n-1)}\right]}$, $i = 0,1, \ldots, n-1$.

Let $\tau = n$. Thus the obtained function is in the generating system.

Let $\tau > n$. For $\tau - 1$, assume that (2.1) is the generating system. Then, $\frac{[x_1 \ldots x_1^{(i-1)}\, x_1^{(\tau-1)}\, x_1^{(i+1)} \ldots x_1^{(n-1)}]}{[x_1\, x_1{}' \ldots x_1^{(n-1)}]}$, $i = 0,1, \ldots, n-1$ is generated by (2.1).

On the other hand, differentiating the determinant $[x_1 \ldots x_1^{(i-1)}\, x_1^{(\tau-1)}\, x_1^{(i+1)} \ldots x_1^{(n-1)}]$, we obtain

$$\left[x_1 \ldots x_1^{(i-1)}\, x_1^{(\tau)}\, x_1^{(i+1)} \ldots x_1^{(n-1)}\right] = [x_1 \ldots x_1^{(i-1)}\, x_1^{(\tau-1)}\, x_1^{(i+1)} \ldots x_1^{(n-1)}]'$$
$$-[x_1 \ldots x_1^{(i-2)}\, x_1^{(i)}\, x_1^{(\tau-1)}\, x_1^{(i+1)} \ldots x_1^{(n-1)}] - [x_1 \ldots x_1^{(i-1)}\, x_1^{(\tau-1)}\, x_1^{(i+1)} \ldots x_1^{(n-2)}\, x_1^{(n)}].$$

Dividing both sides of the above equation by $[x_1\, x_1' \dots x_1^{(n-1)}]$ yields

$$\frac{\left[x_1 \dots x_1^{(i-1)}\, x_1^{(\tau)}\, x_1^{(i+1)} \dots x_1^{(n-1)}\right]}{\left[x_1\, x_1' \dots x_1^{(n-1)}\right]} = \frac{[x_1 \dots x_1^{(i-1)}\, x_1^{(\tau-1)}\, x_1^{(i+1)} \dots x_1^{(n-1)}]'}{\left[x_1\, x_1' \dots x_1^{(n-1)}\right]}$$
$$- \frac{[x_1 \dots x_1^{(i-2)}\, x_1^{(i)}\, x_1^{(\tau-1)}\, x_1^{(i+1)} \dots x_1^{(n-1)}]}{[x_1\, x_1' \dots x_1^{(n-1)}]} - \frac{[x_1 \dots x_1^{(i-1)}\, x_1^{(\tau-1)}\, x_1^{(i+1)} \dots x_1^{(n-2)}\, x_1^{(n)}]}{[x_1\, x_1' \dots x_1^{(n-1)}]}. \tag{2.2}$$

For the first term of the right hand side of (2.2), we get

$$\frac{[x_1 \dots x_1^{(\tau-1)} \dots x_1^{(n-1)}]'}{[x_1 \dots x_1^{(n-1)}]}$$
$$= \left(\frac{[x_1 \dots x_1^{(\tau-1)} \dots x_1^{(n-1)}]}{[x_1 \dots x_1^{(n-1)}]}\right)' + \frac{[x_1 \dots x_1^{(\tau-1)} \dots x_1^{(n-1)}]}{[x_1 \dots x_1^{(n-1)}]} \frac{[x_1 \dots x_1^{(n-2)}\, x_1^{(n)}]}{[x_1 \dots x_1^{(n-1)}]}.$$

Since all terms of the right hand side of the above equation are generated by the generators, also $\frac{[x_1 \dots x_1^{(\tau-1)} \dots x_1^{(n-1)}]'}{[x_1 \dots x_1^{(n-1)}]}$ is genereted.

One can see that the second term of the right hand side of (2.2) is generated by the induction hypothesis.

Finally, for the third term of the right hand side of (2.2), if we put $x_1 = x_1, x_2 = x_1', \dots, x_n = x_1^{(n-1)}, x_0 = x_1^{(n)}, y_2 = x_1, \dots, y_{(i+1)} = x_1^{(i-1)}, y_{(i+2)} = x_1^{(\tau-1)}, y_{(i+3)} = x_1^{(i+1)}, \dots, y_n = x_1^{(n-2)}$ in Lemma 2.1., then we have

$$\frac{\left[x_1^{(n)}\, x_1\, \dots\, x_1^{(\tau-1)}\, \dots\, x_1^{(n-2)}\right]}{\left[x_1\, x_1'\, \dots\, x_1^{(n-1)}\right]} = \frac{\left[x_1\, \dots\, x_1^{(i-1)}\, x_1^{(n)}\, x_1^{(i+1)}\, \dots\, x_1^{(n-2)}\right]}{\left[x_1\, x_1'\, \dots\, x_1^{(n-1)}\right]} \frac{\left[x_1^{(i)}\, x_1\, \dots\, x_1^{(\tau-1)}\, \dots\, x_1^{(n-2)}\right]}{\left[x_1\, x_1'\, \dots\, x_1^{(n-1)}\right]}$$
$$+ \frac{[x_1 \dots x_1^{(n-2)}\, x_1^{(n)}]}{[x_1\, x_1' \dots x_1^{(n-1)}]} \frac{[x_1^{(n-1)}\, x_1 \dots x_1^{(\tau-1)} \dots x_1^{(n-2)}]}{[x_1\, x_1' \dots x_1^{(n-1)}]}.$$

Fort he right hand side of the above equation, it is easy to see that $\frac{[x_1 \dots x_1^{(i-1)}\, x_1^{(n)}\, x_1^{(i+1)} \dots x_1^{(n-2)}]}{[x_1\, x_1' \dots x_1^{(n-1)}]}$ and $\frac{[x_1 \dots x_1^{(n-2)}\, x_1^{(n)}]}{[x_1\, x_1' \dots x_1^{(n-1)}]}$ are in the generating system; $\frac{[x_1^{(i)}\, x_1 \dots x_1^{(\tau-1)} \dots x_1^{(n-2)}]}{[x_1\, x_1' \dots x_1^{(n-1)}]}$ and $\frac{[x_1^{(n-1)}\, x_1 \dots x_1^{(\tau-1)} \dots x_1^{(n-2)}]}{[x_1\, x_1' \dots x_1^{(n-1)}]}$ are generated by the induction hypothesis. Thus the left hand side of the equation is generated by (2.1).

Now we will show by induction on τ that the function $\frac{[x_1 \dots x_1^{(i-1)}\, x_2^{(\tau)}\, x_1^{(i+1)} \dots x_1^{(n-1)}]}{[x_1\, x_1' \dots x_1^{(n-1)}]}, \tau \geq 0$ is generated by the system.

For $\tau = 0$, $\frac{[x_1 \dots x_1^{(i-1)}\, x_2\, x_1^{(i+1)} \dots x_1^{(n-1)}]}{[x_1\, x_1' \dots x_1^{(n-1)}]}, i = 0,1,\dots,n$ is in the generating system.

Let $\tau = n$, and let us assume, as the induction hypothesis, that $\frac{[x_1 \dots x_1^{(i-1)}\, x_2^{(n)}\, x_1^{(i+1)} \dots x_1^{(n-1)}]}{[x_1\, x_1' \dots x_1^{(n-1)}]}, i = 0,1,\dots,n$ is generated by the system.

Let $\tau = n+1$. By the equation

$$\left(\frac{\left[x_1 \dots x_1^{(i-1)}\, x_2^{(n)}\, x_1^{(i+1)} \dots x_1^{(n-1)}\right]}{\left[x_1\, x_1' \dots x_1^{(n-1)}\right]}\right)' = \frac{[x_1 \dots x_1^{(i-1)}\, x_2^{(n)}\, x_1^{(i+1)} \dots x_1^{(n-1)}]'}{\left[x_1\, x_1' \dots x_1^{(n-1)}\right]}$$
$$- \frac{[x_1\, x_1' \dots x_1^{(n-2)}\, x_1^{(n)}]}{[x_1\, x_1' \dots x_1^{(n-1)}]} \frac{[x_1 \dots x_1^{(i-1)}\, x_2^{(n)}\, x_1^{(i+1)} \dots x_1^{(n-1)}]}{[x_1\, x_1' \dots x_1^{(n-1)}]}, \tag{2.3}$$

it is easilly seen that the function $\frac{[x_1 \dots x_1^{(i-1)}\, x_2^{(n)}\, x_1^{(i+1)} \dots x_1^{(n-1)}]'}{[x_1\, x_1' \dots x_1^{(n-1)}]}$ is generated by the system.

Differentiating the determinant $[x_1 \dots x_1^{(i-1)}\, x_2^{(n)}\, x_1^{(i+1)} \dots x_1^{(n-1)}]$ yields

$$[x_1 \dots x_1^{(i-1)}\, x_2^{(n)}\, x_1^{(i+1)} \dots x_1^{(n-1)}]' = [x_1 \dots x_1^{(i-2)}\, x_1^{(i)}\, x_2^{(n)}\, x_1^{(i+1)} \dots x_1^{(n-1)}]$$
$$+[x_1 \dots x_1^{(i-1)}\, x_2^{(n+1)}\, x_1^{(i+1)} \dots x_1^{(n-1)}] + [x_1 \dots x_1^{(i-1)}\, x_2^{(n)}\, x_1^{(i+1)} \dots x_1^{(n-2)}\, x_1^{(n)}].$$

Dividing both side of the above equation by $[x_1\, x_1' \dots x_1^{(n-1)}]$ gives

$$\frac{[x_1 \dots x_1^{(i-1)}\, x_2^{(n)}\, x_1^{(i+1)} \dots x_1^{(n-1)}]'}{\left[x_1\, x_1' \dots x_1^{(n-1)}\right]} = \frac{\left[x_1 \dots x_1^{(i-2)}\, x_1^{(i)}\, x_2^{(n)}\, x_1^{(i+1)} \dots x_1^{(n-1)}\right]}{\left[x_1\, x_1' \dots x_1^{(n-1)}\right]}$$
$$+\frac{[x_1 \dots x_1^{(i-1)}\, x_2^{(n+1)}\, x_1^{(i+1)} \dots x_1^{(n-1)}]}{[x_1\, x_1' \dots x_1^{(n-1)}]} + \frac{[x_1 \dots x_1^{(i-1)}\, x_2^{(n)}\, x_1^{(i+1)} \dots x_1^{(n-2)}\, x_1^{(n)}]}{[x_1\, x_1' \dots x_1^{(n-1)}]}. \tag{2.4}$$

We know that the left hand side of (2.4) is generated by the system. Also first term of the right hand side of the equation is generated by the induction hypothesis. Now, we are going to show that the last term of (2.4) is generated by the system. If we put $x_1 = x_1, x_2 = x_1', \dots, x_n = x_1^{(n-1)}, x_0 = x_1^{(n)}, y_2 = x_1, \dots, y_{(i+1)} = x_1^{(i-1)}, y_{(i+2)} = x_2^{(n)}, y_{(i+3)} = x_1^{(i+1)}, \dots, y_n = x_1^{(n-2)}$ in Lemma 2.1., then we obtain

$$\left[x_1\, x_1' \dots x_1^{(n-1)}\right]\left[x_1^{(n)}\, x_1 \dots x_1^{(i-1)}\, x_2^{(n)}\, x_1^{(i+1)} \dots x_1^{(n-2)}\right]$$
$$-\left[x_1 \dots x_1^{(i-1)}\, x_2^{(n)}\, x_1^{(i+1)} \dots x_1^{(n-1)}\right]\left[x_1^{(i)}\, x_1 \dots x_1^{(i-1)}\, x_2^{(n)}\, x_1^{(i+1)} \dots x_1^{(n-2)}\right]$$
$$-[x_1 \dots x_1^{(n-2)}\, x_1^{(n)}][x_1^{(n-1)}\, x_1 \dots x_1^{(i-1)}\, x_2^{(n)}\, x_1^{(i+1)} \dots x_1^{(n-2)}] = 0.$$

Dividing both sides of the above equation by $[x_1\, x_1' \dots x_1^{(n-1)}]^2$, we have

$$\frac{\left[x_1^{(n)}\, x_1 \dots x_1^{(i-1)}\, x_2^{(n)}\, x_1^{(i+1)} \dots x_1^{(n-2)}\right]}{\left[x_1\, x_1' \dots x_1^{(n-1)}\right]}$$
$$= \frac{[x_1 \dots x_1^{(i-1)}\, x_2^{(n)}\, x_1^{(i+1)} \dots x_1^{(n-1)}]}{[x_1\, x_1' \dots x_1^{(n-1)}]}\frac{[x_1^{(i)}\, x_1 \dots x_1^{(i-1)}\, x_2^{(n)}\, x_1^{(i+1)} \dots x_1^{(n-2)}]}{[x_1\, x_1' \dots x_1^{(n-1)}]}$$
$$+\frac{[x_1 \dots x_1^{(n-2)}\, x_1^{(n)}]}{[x_1\, x_1' \dots x_1^{(n-1)}]}\frac{[x_1^{(n-1)}\, x_1 \dots x_1^{(i-1)}\, x_2^{(n)}\, x_1^{(i+1)} \dots x_1^{(n-2)}]}{[x_1\, x_1' \dots x_1^{(n-1)}]}.$$

Since all terms of the right hand side of the above equation are generated by the system, $\frac{[x_1^{(n)}\, x_1 \dots x_1^{(i-1)}\, x_2^{(n)}\, x_1^{(i+1)} \dots x_1^{(n-2)}]}{[x_1\, x_1' \dots x_1^{(n-1)}]}$ is also generated.

Therefore, by (2.4), we obtain that $\frac{[x_1 \dots x_1^{(i-1)}\, x_2^{(n+1)}\, x_1^{(i+1)} \dots x_1^{(n-1)}]}{[x_1\, x_1' \dots x_1^{(n-1)}]}$ is generated by the system.

Finally, we will show by induction on τ that the function $\frac{[x_1 \dots x_1^{(i-1)}\, x_3^{(\tau)}\, x_1^{(i+1)} \dots x_1^{(n-1)}]}{[x_1\, x_1' \dots x_1^{(n-1)}]}, \tau \geq 0$ is generated by (2.1).

For $\tau = 0$, $\frac{[x_1 \dots x_1^{(i-1)}\, x_3\, x_1^{(i+1)} \dots x_1^{(n-1)}]}{[x_1\, x_1' \dots x_1^{(n-1)}]}$ is in the generating system.

Let $\tau = n$, and let us assume that $\frac{[x_1 \dots x_1^{(i-1)}\, x_3^{(n)}\, x_1^{(i+1)} \dots x_1^{(n-1)}]}{[x_1\, x_1' \dots x_1^{(n-1)}]}$ is generated by (2.1).

Let $\tau = n + 1$. By the equation

$$\left(\frac{[x_1 \dots x_1^{(i-1)}\, x_3^{(n)}\, x_1^{(i+1)} \dots x_1^{(n-1)}]}{[x_1\, x_1' \dots x_1^{(n-1)}]}\right)' = \frac{[x_1 \dots x_1^{(i-1)}\, x_3^{(n)}\, x_1^{(i+1)} \dots x_1^{(n-1)}]'}{[x_1\, x_1' \dots x_1^{(n-1)}]}$$
$$-\frac{[x_1\, x_1' \dots x_1^{(n-2)}\, x_1^{(n)}]}{[x_1\, x_1' \dots x_1^{(n-1)}]}\frac{[x_1 \dots x_1^{(i-1)}\, x_3^{(n)}\, x_1^{(i+1)} \dots x_1^{(n-1)}]}{[x_1\, x_1' \dots x_1^{(n-1)}]}, \tag{2.5}$$

we see that $\frac{[x_1 \dots x_1^{(i-1)}\, x_3^{(n)}\, x_1^{(i+1)} \dots x_1^{(n-1)}]'}{[x_1\, x_1{}' \dots x_1^{(n-1)}]}$ is generated by the system.

Same as before, we obtain

$$\frac{[x_1 \dots x_1^{(i-1)}\, x_3^{(n)}\, x_1^{(i+1)} \dots x_1^{(n-1)}]'}{[x_1\, x_1{}' \dots x_1^{(n-1)}]} = \frac{[x_1 \dots x_1^{(i-2)}\, x_1^{(i)}\, x_3^{(n)}\, x_1^{(i+1)} \dots x_1^{(n-1)}]}{[x_1\, x_1{}' \dots x_1^{(n-1)}]}$$
$$+ \frac{[x_1 \dots x_1^{(i-1)}\, x_3^{(n+1)}\, x_1^{(i+1)} \dots x_1^{(n-1)}]}{[x_1\, x_1{}' \dots x_1^{(n-1)}]} + \frac{[x_1 \dots x_1^{(i-1)}\, x_3^{(n)}\, x_1^{(i+1)} \dots x_1^{(n-2)}\, x_1^{(n)}]}{[x_1\, x_1{}' \dots x_1^{(n-1)}]}. \tag{2.6}$$

In (2.6), we know that the left hand side, first term of the right hand side and the last term of the right hand side of the equation is generated by the system. Thus, $\frac{[x_1 \dots x_1^{(i-1)}\, x_3^{(n+1)}\, x_1^{(i+1)} \dots x_1^{(n-1)}]}{[x_1\, x_1{}' \dots x_1^{(n-1)}]}$ is generated by (2.1).

This completes the proof.

■

3. SOLUTION OF EQUIVALENCE PROBLEM

Definition 3.1. Let $\{x_1, x_2, x_3\}$ and $\{y_1, y_2, y_3\}$ be two curve families such that for $i = 1,2,3$ $x_i, y_i \colon I \subset \mathbb{R} \to \mathbb{R}^n$. If there exists an element $g \in GL(n, \mathbb{R})$ such that $g x_i(t) = y_i(t)$ for all $t \in I$ and $i = 1,2,3$, then the curve families $\{x_1, x_2, x_3\}$ and $\{y_1, y_2, y_3\}$ are said to be $GL(n, \mathbb{R})$ −equivalent. $GL(n, \mathbb{R})$ −equivalence is denoted by $\{x_1, x_2, x_3\} \approx^G \{y_1, y_2, y_3\}$.

Theorem 3.2. Let $G = GL(n, \mathbb{R})$ and $\{x_1, x_2, x_3\}$ and $\{y_1, y_2, y_3\}$ be two curve families such that x_1 and y_1 are $GL(n, \mathbb{R})$ −regular. For $i = 0,1, \dots, n-1$, if

$$\frac{[x_1 \dots x_1^{(i-1)}\, x_1^{(n)}\, x_1^{(i+1)} \dots x_1^{(n-1)}]}{[x_1\, x_1{}' \dots x_1^{(n-1)}]} = \frac{[y_1 \dots y_1^{(i-1)}\, y_1^{(n)}\, y_1^{(i+1)} \dots y_1^{(n-1)}]}{[y_1\, y_1{}' \dots y_1^{(n-1)}]},$$
$$\frac{[x_1 \dots x_1^{(i-1)}\, x_2\, x_1^{(i+1)} \dots x_1^{(n-1)}]}{[x_1\, x_1{}' \dots x_1^{(n-1)}]} = \frac{[y_1 \dots y_1^{(i-1)}\, y_2\, y_1^{(i+1)} \dots y_1^{(n-1)}]}{[y_1\, y_1{}' \dots y_1^{(n-1)}]},$$
$$\frac{[x_1 \dots x_1^{(i-1)}\, x_3\, x_1^{(i+1)} \dots x_1^{(n-1)}]}{[x_1\, x_1{}' \dots x_1^{(n-1)}]} = \frac{[y_1 \dots y_1^{(i-1)}\, y_3\, y_1^{(i+1)} \dots y_1^{(n-1)}]}{[y_1\, y_1{}' \dots y_1^{(n-1)}]},$$

then $\{x_1, x_2, x_3\} \approx^G \{ y_1, y_2, y_3\}$.

Proof. Let us form the following matrices:

$$A_{x_1} = \begin{bmatrix} x_{11}(t) & \vdots & x_{11}^{(n-1)}(t) \\ \vdots & \vdots & \vdots \\ x_{1n}(t) & & x_{1n}^{(n-1)}(t) \end{bmatrix}, \; A'_{x_1} = \begin{bmatrix} x'_{11}(t) & \vdots & x_{11}^{(n)}(t) \\ \vdots & \vdots & \vdots \\ x'_{1n}(t) & & x_{1n}^{(n)}(t) \end{bmatrix}.$$

The inverse $A_{x_1}^{-1}$ of the matrix A_{x_1} exists because the curve x_1 is regular. Let $A_{x_1}^{-1} A'_{x_1} = C$, thus we have $A'_{x_1} = A_{x_1} C$. With a simple calculation we obtain matrix C as

$$C = \begin{bmatrix} 0 & & 0 & c_{1n} \\ 1 & & 0 & c_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & & 1 & c_{nn} \end{bmatrix},$$

where

$$c_{1n} = \frac{[x_1^{(n)}\, x_1{}' \dots x_1^{(n-1)}]}{[x_1\, x_1{}' \dots x_1^{(n-1)}]}, \; c_{2n} = \frac{[x_1\, x_1^{(n)}\, x_1{}'' \dots x_1^{(n-1)}]}{[x_1\, x_1{}' \dots x_1^{(n-1)}]}, \; \dots, \; c_{nn} = \frac{[x_1\, x_1{}' \dots x_1^{(n-2)}\, x_1^{(n)}]}{[x_1\, x_1{}' \dots x_1^{(n-1)}]}.$$

By the equations given in the theorem, similarly we have $A'_{y_1} = A_{y_1}C$ giving $A_{x_1}^{-1}A_{x_1}{}' = A_{y_1}^{-1}A_{y_1}{}'$. Calculating

$$(A_{y_1}A_{x_1}^{-1})' = A_{y_1}{}'A_{x_1}^{-1} + A_{y_1}(A_{x_1}^{-1})' = A_{y_1}{}'A_{x_1}^{-1} + A_{y_1}(-A_{x_1}^{-1}A_{x_1}{}'A_{x_1}^{-1})$$
$$= A_{y_1}(A_{y_1}^{-1}A_{y_1}{}' - A_{x_1}^{-1}A_{x_1}{}')A_{x_1}^{-1} = 0,$$

it is obtained that $A_{y_1}A_{x_1}^{-1} = g$, where g is a constant matrix. Since det $g = \det(A_{y_1}A_{x_1}^{-1}) = \det A_{y_1} \det A_{x_1}^{-1} \neq 0$, $g \in GL(n,\mathbb{R})$. Thus, $A_{y_1} = gA_{x_1}$and

$$\begin{bmatrix} y_{11}(t) & & y_{11}^{(n-1)}(t) \\ \vdots & \vdots & \vdots \\ y_{1n}(t) & & y_{1n}^{(n-1)}(t) \end{bmatrix} = \begin{bmatrix} g_{11} & & g_{1n} \\ \vdots & \vdots & \vdots \\ g_{n1} & & g_{nn} \end{bmatrix} \begin{bmatrix} x_{11}(t) & & x_{11}^{(n-1)}(t) \\ \vdots & \vdots & \vdots \\ x_{1n}(t) & & x_{1n}^{(n-1)}(t) \end{bmatrix}$$

giving

$$\begin{bmatrix} y_{11}(t) \\ \vdots \\ y_{1n}(t) \end{bmatrix} = \begin{bmatrix} g_{11} & & g_{1n} \\ \vdots & \vdots & \vdots \\ g_{n1} & & g_{nn} \end{bmatrix} \begin{bmatrix} x_{11}(t) \\ \vdots \\ x_{1n}(t) \end{bmatrix}.$$

The last equation means that for all $t \in I$,

$$y_1(t) = gx_1(t). \tag{3.1}$$

Consider the matrices

$$D_{x_2} = \begin{bmatrix} x_{21} \\ \vdots \\ x_{2n} \end{bmatrix}, D_{x_3} = \begin{bmatrix} x_{31} \\ \vdots \\ x_{3n} \end{bmatrix}.$$

Set $A_{x_1}^{-1}D_{x_2} = H$ giving $D_{x_2} = A_{x_1}H$. We now find the matrix H which satisfies the last equation. The equation

$$\begin{bmatrix} x_{21} \\ \vdots \\ x_{2n} \end{bmatrix} = \begin{bmatrix} x_{11} & & x_{11}^{(n-1)} \\ \vdots & \vdots & \vdots \\ x_{1n} & & x_{1n}^{(n-1)} \end{bmatrix} \begin{bmatrix} h_{11} \\ \vdots \\ h_{1n} \end{bmatrix}$$

let us form the system of differential equations

$$\begin{aligned} &x_{11}h_{11} + x_{11}{}'h_{12} + +x_{11}^{(n-1)}h_{1n} = x_{21} \\ &x_{12}h_{11} + x_{12}{}'h_{12} + +x_{12}^{(n-1)}h_{1n} = x_{22} \\ &\qquad\vdots \\ &x_{1n}h_{11} + x_{1n}{}'h_{12} + +x_{1n}^{(n-1)}h_{1n} = x_{2n}. \end{aligned} \tag{3.2}$$

We obtain the solution of (3.2) as

$$h_{11} = \frac{[x_2\, x_1{}' \ldots x_1^{(n-1)}]}{[x_1\, x_1{}' \ldots x_1^{(n-1)}]},\ h_{12} = \frac{[x_1\, x_2 \ldots x_1^{(n-1)}]}{[x_1\, x_1{}' \ldots x_1^{(n-1)}]}, \ldots, h_{1n} = \frac{[x_1\, x_1{}' \ldots x_1^{(n-2)}\, x_2]}{[x_1\, x_1{}' \ldots x_1^{(n-1)}]}.$$

Similarly, let us form the matrix $A_{y_1}^{-1}D_{y_2}$. By the equations given in the theorem, we obtain $A_{y_1}^{-1}D_{y_2} = A_{x_1}^{-1}D_{x_2}$. Also recall that $A_{y_1} = gA_{x_1}$, then it is obtained

$$A_{x_1}^{-1}D_{x_2} = (gA_{x_1})^{-1}D_{y_2} = A_{x_1}^{-1}g^{-1}D_{y_2}$$

giving $D_{x_2} = g^{-1}D_{y_2}$ and $D_{y_2} = gD_{x_2}$. Thus we have for all $t \in I$

$$y_2(t) = gx_2(t). \tag{3.3}$$

On the other hand, for D_{x_3}, similarly put $A_{x_1}^{-1}D_{x_3} = K$. Hence we get $D_{x_3} = A_{x_1}K$, and the matrix equation

$$\begin{bmatrix} x_{31} \\ \vdots \\ x_{3n} \end{bmatrix} = \begin{bmatrix} x_{11} & \vdots & x_{11}^{(n-1)} \\ \vdots & & \vdots \\ x_{1n} & & x_{1n}^{(n-1)} \end{bmatrix} \begin{bmatrix} k_{11} \\ \vdots \\ k_{1n} \end{bmatrix}$$

giving the system of differential equations

$$\begin{aligned} &x_{11}k_{11} + x_{11}'k_{12} + +x_{11}^{(n-1)}k_{1n} = x_{31} \\ &x_{12}k_{11} + x_{12}'k_{12} + +x_{12}^{(n-1)}k_{1n} = x_{32} \\ &\qquad\vdots \\ &x_{1n}k_{11} + x_{1n}'k_{12} + +x_{1n}^{(n-1)}k_{1n} = x_{3n}. \end{aligned} \tag{3.4}$$

The solution of (3.4) is

$$k_{11} = \frac{[x_3\, x_1' \dots x_1^{(n-1)}]}{[x_1\, x_1' \dots x_1^{(n-1)}]},\ k_{12} = \frac{[x_1\, x_3 \dots x_1^{(n-1)}]}{[x_1\, x_1' \dots x_1^{(n-1)}]}, \dots, k_{1n} = \frac{[x_1\, x_1' \dots x_1^{(n-2)}\, x_3]}{[x_1\, x_1' \dots x_1^{(n-1)}]}.$$

Same as before, let us form the matrix $A_{y_1}^{-1}D_{y_3}$. By the equations given in the theorem, we obtain $A_{y_1}^{-1}D_{y_3} = A_{x_1}^{-1}D_{x_3}$. Also, recall that $A_{y_1} = gA_{x_1}$, then it is obtained

$$A_{x_1}^{-1}D_{x_3} = (gA_{x_1})^{-1}D_{y_3} = A_{x_1}^{-1}g^{-1}D_{y_3}$$

giving $D_{x_3} = g^{-1}D_{y_3}$ and $D_{y_3} = gD_{x_3}$. Thus we hav efor all $t \in I$

$$y_3(t) = gx_3(t). \tag{3.5}$$

Finally, by (3.1), (3.3) and (3.5), for the same $g \in GL(n, \mathbb{R})$ we have

$$\begin{gathered} y_1(t) = gx_1(t) \\ y_2(t) = gx_2(t) \\ y_3(t) = gx_3(t),\ \forall t \in I \end{gathered}$$

which means $\{x_1, x_2, x_3\} \approx^G \{y_1, y_2, y_3\}$.

■

Theorem 3.3. Let $G = GL(n, \mathbb{R})$ and $f_i(t), f_{2i}(t), f_{3i}(t)$ be C^∞ −functions for all $i = 0,1,\dots,n-1$, $t \in I \subset \mathbb{R}$. There exists a curve family $\{x_1, x_2, x_3\}$ such that x_1 is $GL(n, \mathbb{R})$ −regular, which satisfies the following equations:

$$\begin{gathered} \frac{[x_1 \dots x_1^{(i-1)}\, x_1^{(n)}\, x_1^{(i+1)} \dots x_1^{(n-1)}]}{[x_1\, x_1' \dots x_1^{(n-1)}]} = f_i(t) \\ \frac{[x_1 \dots x_1^{(i-1)}\, x_2\, x_1^{(i+1)} \dots x_1^{(n-1)}]}{[x_1\, x_1' \dots x_1^{(n-1)}]} = f_{2i}(t) \\ \frac{[x_1 \dots x_1^{(i-1)}\, x_3\, x_1^{(i+1)} \dots x_1^{(n-1)}]}{[x_1\, x_1' \dots x_1^{(n-1)}]} = f_{3i}(t),\ i = 0,1,\dots,n-1. \end{gathered}$$

Proof. Consider the matrix A_{x_1} and put $A_{x_1}^{-1}A_{x_1}' = B$. Then we have $A_{x_1}' = A_{x_1}B$, where the matrix B has the following form:

$$B = \begin{bmatrix} 0 & & 0 & f_0(t) \\ 1 & & 0 & f_1(t) \\ \vdots & \vdots & \vdots & \vdots \\ 0 & & 1 & f_{n-1}(t) \end{bmatrix}.$$

The matrix equation

$$\begin{bmatrix} x'_{11} & x''_{11} & \dots & x_{11}^{(n)} \\ x'_{12} & x''_{12} & \dots & x_{12}^{(n)} \\ \vdots & \vdots & \vdots & \vdots \\ x'_{1n} & x''_{1n} & \dots & x_{1n}^{(n)} \end{bmatrix} = \begin{bmatrix} x_{11} & x'_{11} & \dots & x_{11}^{(n-1)} \\ x_{12} & x'_{12} & \dots & x_{12}^{(n-1)} \\ \vdots & \vdots & \vdots & \vdots \\ x_{1n} & x'_{1n} & \dots & x_{1n}^{(n-1)} \end{bmatrix} \begin{bmatrix} 0 & \dots & 0 & f_0(t) \\ 1 & \dots & 0 & f_1(t) \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 1 & f_{n-1}(t) \end{bmatrix}$$

giving the system of differential equations

$$\begin{aligned} & x_{11}(t)f_0(t) + x'_{11}(t)f_1(t) + \dots + x_{11}^{(n-1)}(t)f_{n-1}(t) = x_{11}^{(n)}(t) \\ & x_{12}(t)f_0(t) + x'_{12}(t)f_1(t) + \dots + x_{12}^{(n-1)}(t)f_{n-1}(t) = x_{12}^{(n)}(t) \quad \vdots \\ & x_{1n}(t)f_0(t) + x'_{1n}(t)f_1(t) + \dots + x_{1n}^{(n-1)}(t)f_{n-1}(t) = x_{1n}^{(n)}(t). \end{aligned} \tag{3.6}$$

In (3.6), if we put $y(t) = \begin{bmatrix} x_{11}(t) \\ x_{12}(t) \\ \vdots \\ x_{1n}(t). \end{bmatrix}$, then we rewrite (3.6) in the form of

$$f_0(t)y(t) + f_1(t)y'(t) + \dots + f_{n-1}(t)y^{(n-1)}(t) - y^{(n)}(t) = 0. \tag{3.7}$$

The differential equation (3.7) has at least one solution since the fuctions $f_i(t)$, $i = 0,1,\dots,n-1$ are all C^∞ −functions. Let the solution be $x_1(t) = y(t)$. Thus the curve $x_1(t)$ satisfies the first condition of the theorem.

Let

$$A_2 = \begin{bmatrix} x_{11} & x_{11}' & & x_{11}^{(n-2)} & x_{21} \\ x_{12} & x_{12}' & & x_{12}^{(n-2)} & x_{22} \\ \vdots & \vdots & \vdots & \vdots & \\ x_{1n} & x_{1n}' & & x_{1n}^{(n-2)} & x_{2n} \end{bmatrix}, A_3 = \begin{bmatrix} x_{11} & x_{11}' & & x_{11}^{(n-2)} & x_{31} \\ x_{12} & x_{12}' & & x_{12}^{(n-2)} & x_{32} \\ \vdots & \vdots & \vdots & \vdots & \\ x_{1n} & x_{1n}' & & x_{1n}^{(n-2)} & x_{3n} \end{bmatrix},$$

and let $A_{x_1}^{-1}A_2 = M$ and $A_{x_1}^{-1}A_3 = N$. We obtain $A_2 = A_{x_1}M$ and $A_3 = A_{x_1}N$, where

$$M = \begin{bmatrix} 1 & 0 & & 0 & f_{20}(t) \\ 0 & 1 & & 0 & f_{21}(t) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & & 1 & f_{2(n-2)}(t) \\ 0 & 0 & & 0 & f_{2(n-1)}(t) \end{bmatrix}, N = \begin{bmatrix} 1 & 0 & & 0 & f_{30}(t) \\ 0 & 1 & & 0 & f_{31}(t) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & & 1 & f_{3(n-2)}(t) \\ 0 & 0 & & 0 & f_{3(n-1)}(t) \end{bmatrix}.$$

The above matrix equations lead to the following systems of differential equations repectively:

$$\begin{aligned} & x_{11}(t)f_{20}(t) + x'_{11}(t)f_{21}(t) + \dots + x_{11}^{(n-1)}(t)f_{2(n-1)}(t) = x_{21}(t) \\ & x_{12}(t)f_{20}(t) + x'_{12}(t)f_{21}(t) + \dots + x_{12}^{(n-1)}(t)f_{2(n-1)}(t) = x_{22}(t) \\ & \qquad \vdots \\ & x_{1n}(t)f_{20}(t) + x'_{1n}(t)f_{21}(t) + \dots + x_{1n}^{(n-1)}(t)f_{2(n-1)}(t) = x_{2n}(t) \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} & x_{11}(t)f_{30}(t) + x'_{11}(t)f_{31}(t) + \dots + x_{11}^{(n-1)}(t)f_{3(n-1)}(t) = x_{31}(t) \\ & x_{12}(t)f_{30}(t) + x'_{12}(t)f_{31}(t) + \dots + x_{12}^{(n-1)}(t)f_{3(n-1)}(t) = x_{32}(t) \\ & \qquad \vdots \\ & x_{1n}(t)f_{30}(t) + x'_{1n}(t)f_{31}(t) + \dots + x_{1n}^{(n-1)}(t)f_{3(n-1)}(t) = x_{3n}(t). \end{aligned} \tag{3.9}$$

The systems (3.8) and (3.9) has at least one solution since the fuctions $f_{2i}(t), \mathrm{f}_{3\mathrm{i}}(\mathrm{t})$ $i = 0,1,\dots,n-1$ are all C^∞ −functions. Let the solution of (3.8) be $x_2(t) = y_2(t)$ and let the solution of (3.9) be $x_3(t) = y_3(t)$. Thus the curve $x_2(t)$ satisfies the second condition of the theorem and $x_3(t)$ satisfies the third condition of the theorem. This completes the proof.

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