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**Research Article**

# AN OPTIMIZATION TECHNIQUE IN ANALYZING THE BURGERS EQUATION

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**ABSTRACT**

This article has explored a hybrid numerical approach in analysis of the Burgers equation with involving steep gradients. The technique is based on a quadratic B-spline finite element method in strong form for space variation. This paper discovers how to find an α-family optimization approach for temporal variations. The proposed method has been shown to be unconditionally stable for $\alpha \geq 0.5$. Yet, the efficiency of the proposed scheme on relatively coarse grids has been demonstrated. The numerical illustrations show that the present method has been seen to be more accurate than the literature and effectively captures the shock behaviours.
**Keywords:** Finite element method, α-family of approximation, advection-diffusion process, Burgers equation, optimization, B-spline.

## 1. INTRODUCTION

Many physical processes encountered in physical environment are represented by differential equations. Most of the processes are used to model physical flows in various fields of sciences such as wave propagation, convection–diffusion processes, biological waves etc. One of those physical models is Burgers equation attracting much attention in dealing with evolution equations constituting different models [1]. Computation of the Burgers equation is an important first step towards developing methods. Under certain conditions, uniqueness and existence of solutions to the Burgers equation were shown and discussed [2].

The Burgers equation is the nonlinear model equation for diffusive waves in fluid dynamics. The corresponding equation has also many applied areas including theory of shock waves, sound waves in a viscous medium, mathematical modeling of turbulent fluid and so on.

Much effort has been spent in solving the Burgers equation for last couple of decades. Since some exact solutions fail for small kinematic viscosity values [3], $\varepsilon < 0.01$, many authors [1,4–20] have suggested various methods such as finite element, finite difference, boundary element in

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numerically analysing the processes represented by the model problem. The Burgers equation were also solved exactly by using Hopf-Cole transformation [21,22].

Model of advection mechanisms, and diffusion transports leading to the Burgers equation is the nonlinear model equation for diffusive waves in fluid dynamics. Yet, the corresponding equation has many applied areas including some wave mechanisms in a viscous medium, modelling of turbulent fluid etc.

This study proposes to establish a Galerkin type finite element method (FEM) in which a strong form of the considered equation is preferred rather than the weak form. Because, the use of the strong form of the FEM in analysing the advection-diffusion processes represented by the Burgers equation has some superiorities comparison to the latter. The weak form of the equation needs more complicated computers codes, more computational time than the first one. Note that the weak form and strong form are mathematically equivalent to each other but computationally it is not the case. As the weak form of the equation requires additional matrices for residual term of the integration, this gives rise to excessive computational time and may therefore lead to loss of accuracy.

It is noticeable that there are few studies [23,24] mainly used quadratic B-splines Galerkin method in solving the Burgers equation. In reference [24], they considered the fist-order splitted direct Crank-Nicolson method. The present article prefers to use the considered equation itself and general time approximation. This is the main difference between the present approach and their approach. In order to find the first approximate solution, an additional equation is required for both the present approach and for theirs. As will be seen in Section 4, the derivative of the initial condition is used here to find one additional condition whilst they used homogeneous condition. Some results have been given to explain the developed procedure in Section 6. Note also that the procedure followed here is fully different from Aksan's work [23].

The proposed technique produces accurate and unconditionally stable results. The computational cost of the technique is acceptable. Even when the advection dominated cases of the Burgers equation are considered, the present approach produces oscillation-free results. The B-spline basis functions cover the spatial domain and for each of the elements of the spatial domain the well approximated solutions can be obtained. In addition to the spatial superiorities, the present time approximation is very suitable for the solution of the dynamical process of the considered equation. Thus, the $\alpha$-family of time approximation is flexible and unconditionally stable for suitable choices of the parameter $\alpha$.

Here a hybrid numerical approach is proposed to solve the Burgers equation. To the best of the authors' knowledge, this approach has not previously been suggested. The present method has been shown to be unconditionally stable for $\alpha \geq 0.5$. Behaviour of many processes arising in various fields of science leads to the Burgers equation problem is considered into the following form

$$u_t + uu_x = \varepsilon u_{xx} \ , \quad a \leq x \leq b \tag{1}$$

with the boundary conditions

$$u(a,t) = f_1(t), \ \ t > 0, \ u(b,t) = f_2(t), \ \ t > 0 \tag{2}$$

and initial condition

$$u(x,0) = g(x), \ a < x < b \tag{3}$$

where $\varepsilon$ is viscosity constant for $\varepsilon > 0$ and $f_1$ , $f_2$ and $g$ are known functions. The subscripts $x$ and $t$ represent differentiations with respect to $x$ and $t$ respectively.

The outline of this paper is as follows. The proposed technique is explained in Section 2. The implementation to the model equation is given in Section 3. The stability of the method is analysed in Section 4. Some numerical illustrations are presented in Section 5. Section 6 consists of some concluding remarks. This article proposes a numerical approach to solve the Burgers equation (1) with a set of boundary and initial conditions given by equations (2) and (3).

## 2. THE NUMERICAL METHOD

### 2.1. Quadratic B Spline Basis Functions

Galerkin finite element method in strong form with quadratic B-spline basis functions is used for spatial approximation to solve equation (1) with given initial and boundary conditions (2) and (3). The selection of these types of basis functions is very effective and has some advantages as we are compared with other basis functions. One of the most important advantages of using quadratic B-splines is the continuity of approximate solution and also first derivatives of the solutions at all-region which include interpolation grid points.

The interval $[a,b]$ was partitioned into $N$ finite elements. Each element has equal length $h$ and element nodes are defined as $a = x_0 < x_1 < \cdots < x_N = b$ where $x_{i+1} = x_i + h$ $(i = 0,1. \ldots, N-1)$. Let $\varphi_i$ be the quadratic B-spline basis functions and it is given [25] as

$$\varphi_i(x) = \frac{1}{h^2}\begin{cases}(x_{i+2}-x)^2 - 3(x_{i+1}-x)^2 + 3(x_i-x)^2, \\ (x_{i+2}-x)^2 - 3(x_{i+1}-x)^2, \\ (x_{i+2}-x)^2, \\ 0,\end{cases} \quad x \in \begin{cases}[x_{i-1}, x_i] \\ [x_i, x_{i+1}] \\ [x_{i+1}, x_{i+2}] \\ otherwise\end{cases} \tag{4}$$

for $i = -1,0, \ldots, N$. The corresponding quadratic B-spline basis functions include the set of splines $\{\varphi_{-1}, \varphi_0, \ldots, \varphi_N\}$ for spatial approximation to the equation (1) and the global approximation function $\tilde{u}_N(x,t)$ can be written as

$$\tilde{u}_N(x,t) = \sum_{i=-1}^{N} \delta_i(t)\varphi_i(x) \tag{5}$$

where $\delta_i(t)$ is the time part of global approximation function $\tilde{u}_N(x,t)$ and will be determined from temporal approximation.

Considering (4) and $\sigma = x - x_i$ with $0 \le \sigma \le 1$, to use local coordinate system to the required computations, the basis functions will be in the following form

$$\varphi_i(x) = \frac{1}{h^2}\begin{cases}h^2 - 2h\sigma + \sigma^2, \\ h^2 + 2h\sigma - \sigma^2, \\ \sigma^2, \\ 0,\end{cases} \quad x \in \begin{cases}[x_{i-1}, x_i] \\ [x_i, x_{i+1}] \\ [x_{i+1}, x_{i+2}] \\ otherwise.\end{cases} \tag{6}$$

Each finite element $[x_i, x_{i+1}]$ is covered by the set of three quadratic B-spline $\{\varphi_{i-1}, \varphi_i, \varphi_{i+1}\}$. Table 1 shows the values of $\varphi_i$ and $\varphi_i{}'$ at the boundaries of element $[x_i, x_{i+1}]$. Local approximation function on the element $[x_i, x_{i+1}]$ can be expressed as

$$\tilde{u}_N(x,t) = \sum_{i=l-1}^{l+1} \delta_i(t)\varphi_i(x). \tag{7}$$

**Table 1**.Values of approximate function and its derivatives at the end points of the element

| $x$ | $x_{i-1}$ | $x_i$ | $x_{i+1}$ | $x_{i+2}$ |
|---|---|---|---|---|
| $\varphi_i$ | 0 | 1 | 1 | 0 |
| $\varphi_i{}'$ | 0 | -2/h | 2/h | 0 |

Values of the locally approximated solution $\tilde{u}_N(x,t)$ and its first derivative at the end points of the interval $[x_i, x_{i+1}]$ are obtained in terms of time dependent quantities $\beta_i(t)$ using Table 1 as follows

$$\tilde{u}_N(x_i,t) = \delta_{i-1} + \delta_i$$
$$\tilde{u}_N(x_{i+1},t) = \delta_i + \delta_{i+1}$$
$$\tilde{u}'_N(x_i,t) = \frac{2}{h}(\delta_{i+1} - \delta_{i-1}) \tag{8}$$

$\tilde{u}'_N(x_{i+1},t)=\frac{2}{h}(\delta_{i+2}-\delta_i)$.

By considering element$[x_i,x_{i+1}]$, equation (1) is multiplied by test function $w$ and integrated over the element. Then one can write

$$\int_{x_l}^{x_{l+1}} w(u_t+uu_x-\varepsilon u_{xx})dx=0. \tag{9}$$

The selection of the test functions is so important and in this study test function $w$ is selected as equal to the B-spline basis functions. This type of selection is called Galerkin approach in the finite element method. Using (7) and local coordinate system (6), equation (9) can thus be rewritten as follows

$$\sum_{j=l-1}^{l+1}\left[\int_0^h \varphi_i\varphi_j d\sigma\right]\frac{d\delta_j^e}{dt}+\sum_{j=l-1}^{l+1}\sum_{k=l-1}^{l+1}\left[\int_0^h \varphi_i\varphi_j{}'\,\varphi_k d\sigma\right]\delta_k^e\delta_j^e-\varepsilon\sum_{j=l-1}^{l+1}\left[\int_0^h \varphi_i\varphi_j'' d\sigma\right]\delta_j^e=0$$

or more compactly

$$M^e\frac{d\delta^{\mathrm{e}}}{dt}+\delta^{\mathrm{e}\mathrm{T}}L^e\delta^{\mathrm{e}}-\varepsilon\mathrm{K}^{\mathrm{e}}\delta^{\mathrm{e}}=0 \tag{10}$$

where

$$\begin{aligned}&\mathrm{M}_{\mathrm{ij}}^{\mathrm{e}}=\int_0^h\varphi_i\varphi_j d\sigma,\\&\mathrm{K}_{\mathrm{ij}}^{\mathrm{e}}=\int_0^h\varphi_i\varphi_j''\,d\sigma,\\&\mathrm{L}_{\mathrm{ijk}}^{\mathrm{e}}=\int_0^h\varphi_i\varphi_j''\,\varphi_k d\sigma,\\&\delta^{\mathrm{e}}=(\delta_{i-1},\delta_i,\delta_{i+1})^{\mathrm{T}}.\end{aligned} \tag{11}$$

In (10), $M^e$ and $K^e$ are $(3\times3)$ matrices are independent of time. A $(3\times3\times3)$ matrix $L$ can then be transformed to a time dependent matrix $R$ by using the following procedure

$$R_{ij}=\sum_{k=l-1}^{l+1}L_{ijk}^e\delta_k^e\,. \tag{12}$$

After assembling process for each element, the system matrix will finally take the following form

$$M^*\,\frac{d\delta}{dt}+R^*\delta-\varepsilon K^*\delta=0 \tag{13}$$

where $M^*$, $R^*$ and $K^*$ are $(N+2)\times(N+2)$ matrices and $\delta=(\delta_{-1},\delta_0,\dots,\delta_{N-1},\delta_N)^{\mathrm{T}}$ is the unknown time approximation vector.

### 2.2. α-family of time approximation

The α-family of approximation is based on finite difference method and can be used for the time integration of system (13). Detailed discussion on the corresponding issue can be found, for instance, in [26], the time discretization procedure of the equation can be written as

$$\begin{aligned}\{\delta\}_{s+1}&=\{\delta\}_s+dt\{\delta\}_{s+\alpha}\\\{\delta\}_{s+\alpha}&=(1-\alpha)\{\dot{\delta}\}_s+\alpha\{\dot{\delta}\}_{s+1}\end{aligned}$$

or

$$dt[(1-\alpha)\{\dot{\delta}\}_s+\alpha\{\dot{\delta}\}_{s+1}]=\{\delta\}_{s+1}-\{\delta\}_s$$

where $0\le\alpha\le1$, $t_{s+1}-t_s=dt$, and $\dot{\delta}$ stands for the time differentiation. Use of the aforementioned procedure makes equation (13)

$$[M^*+\alpha dt\,(R_{s+1}^*-\varepsilon\,K^*)]\{\delta\}_{s+1}=[M^*-dt(1-\alpha)(R_s^*-\varepsilon\,K^*)]\{\delta\}_s \tag{14}$$

where matrices $M^*$ and $K^*$ are independent of time while $R^*$ depends on time.

Note that a hybrid approximation technique has been proposed here to produce the present accurate solutions. The stability of the resulting algebraic system (14) will be dealt with in Section 5.

The selection of the parameter α is one of the reasons affecting the accuracy of the produced solutions. It is noticeable that the time element number plays an important role in the selection of parameter α. As relatively high number of uniform time increment is taken, the acceptance of α to be 0.5 results in stable and accurate solution. Moreover, choice of parameter α is utilized to cope with difficulties in the nonlinearity in the time dependent matrix $R_{s+1}^{*}$ as is the following case

$$\{\delta\}_{s+1} = (1-\alpha)\{\delta\}_{s}^{*} + \alpha\{\delta\}_{s}. \tag{15}$$

By using the recursive relation in (14) and corrector relation in (15), the Burgers equation under consideration of conditions (2) and (3) is solved by computer codes produced MATLAB R2010b.

## 3. IMPLEMENTATION

Time dependent quantities $\{\delta\}_{l}$ must be found out using (14) and (15) to evaluate globally approximated solution (5). The following procedures are implemented to find $\{\delta\}_{l}$:

**(i)** The first approximation $\{\delta\}_{\mathrm{l}}^{0}$ is obtained from the initial condition (3). $(N+1)$ equations are found from the initial condition as well as one additional condition found from Table 1 using first derivative of the approximate function as follows

$$\tilde{u}(x_l,0) = g(x_l) = \{\delta\}_{\mathrm{l-1}}^{0} + \{\delta\}_{\mathrm{l}}^{0}$$
$$\tilde{u}_N'(x_l,0) = g'(x_l) = \frac{2}{h}(\{\delta\}_{\mathrm{l}}^{0} - \{\delta\}_{\mathrm{l-1}}^{0}) \tag{16}$$

Thence the algebraic system will be $(N+2)\times(N+2)$ and is solved using the Thomas algorithm.

**(ii)** To find second approximation $\{\delta\}_{\mathrm{l}}^{1}$, the above approximation $\{\delta\}_{\mathrm{l}}^{0}$ is used at the right-hand side of equation (14). The time dependent matrix $R_1^{*}$ is required correction of $\{\delta\}_{\mathrm{l}}^{0}$. To approximate $R_1^{*}$ first, $\{\delta\}_{\mathrm{l}}^{1} = \{\delta\}_{\mathrm{l}}^{0}$ is chosen and then $\{\delta\}_{\mathrm{l}}^{1^{*}}$ is found. Thus, for the rest of iterations, the coming combination is used about 10 times, to approximate $\{\delta\}_{\mathrm{l}}^{1}$.

$$\{\delta\}_{\mathrm{l}}^{1} = \alpha\{\delta\}_{\mathrm{l}}^{0} + (1-\alpha)\{\delta\}_{\mathrm{l}}^{1^{*}} \tag{17}$$

**(iii)** For the other iterations, $\{\delta\}_{\mathrm{l}}^{\mathrm{I}}$ for $i = 2,\dots,J$ ($J$ is the chosen time element number), case (ii) is applied except that here the refinement is done only about 5 times.

## 4. STABILITY ANALYSIS OF THE HYBRID METHOD

**Theorem:** The proposed method for solving Equation (1) is unconditionally stable for $\alpha \geq 0.5$.

**Proof:** To realize limitations of the computed solution under the consideration of the proposed hybrid approach (14), the stability analysis has been carried out with von Neumann theory taking Fourier growth factor defined by

$$\delta_{\mathrm{s}}^{\mathrm{n}} = \tilde{\delta}^{\mathrm{n}} e^{iskh} \tag{18}$$

where $k$ is mode number and $h$ is the spatial element size selected for recursive approximation (14). To obtain a typical row of (14), values of $\delta_{\mathrm{s+1}}$ and $\delta_{\mathrm{s}}$ in the time dependent matrices $R_{s+1}^{*}$ and $R_{s}^{*}$ are taken to be locally constant and equal to $p$. A typical row of penta-diagonal system (14) can thus be stated as

$$c_1\delta_{\mathrm{s-2}}^{\mathrm{n+1}} + c_2\delta_{\mathrm{s-1}}^{\mathrm{n+1}} + c_3\delta_{\mathrm{s}}^{\mathrm{n+1}} + c_4\delta_{\mathrm{s+1}}^{\mathrm{n+1}} + c_5\delta_{\mathrm{s+2}}^{\mathrm{n+1}}$$
$$= c_6\delta_{\mathrm{s-2}}^{\mathrm{n}} + c_7\delta_{\mathrm{s-1}}^{\mathrm{n}} + c_8\delta_{\mathrm{s}}^{\mathrm{n}} + c_9\delta_{\mathrm{s+1}}^{\mathrm{n}} + c_{10}\delta_{\mathrm{s+2}}^{\mathrm{n}} \tag{19}$$

where

$$c_1 = r_1 - r_2, \quad c_2 = 26r_1 - 2r_2, \quad c_3 = 66r_1 + 6r_2$$
$$c_4 = 26r_1 - 2r_2, \quad c_5 = r_1 - r_2, \quad c_6 = r_1 + r_3$$
$$c_7 = 26r_1 + 2r_3, \; c_8 = 66r_1 - 6r_3, \; c_9 = 26r_1 + 2r_3, \; c_{10} = r_1 + r_3$$
$$r_1 = \frac{h}{30}, \; r_2 = \frac{2\varepsilon\alpha\Delta t}{3h}, \; r_3 = \frac{2\varepsilon(1-\alpha)\Delta t}{3h}. \tag{20}$$

Substitution of (18) into (19) and use of Euler expansion for exponential terms give rise to

$$g\tilde{\delta}^{n+1} = g^{*}\,\tilde{\delta}^{n} \tag{21}$$

where

$$g = (r_1 - r_2)\cos(2kh) + (26r_1 - 2r_2)\cos(kh) + 33r_1 + 3r_2 \tag{22}$$
$$g^{*} = (r_1 + r_3)\cos(2kh) + (26r_1 + 2r_3)\cos(kh) + 33r_1 - 3r_3$$

It can be seen that

$$\tilde{\delta}^{n+1} = \frac{g^{*}}{g}\tilde{\delta}^{n} = z\tilde{\delta}^{n}. \tag{23}$$

Scheme (14) is stable if and only if $|z| < 1$. Thus the following inequality must be fulfilled

$$|z| = \left|\frac{g^{*}}{g}\right| < 1. \tag{24}$$

When $\alpha \geq 0.5$, the inequality $r_3 \leq r_2$ is automatically satisfied. In expression (22), since $r_1$ is depends only parameter $h$ and $r_1 \ll 1$, the absolute value of the term $|r_1 - r_2|$ behaves like $|r_2|$ and the term $|r_1 + r_3|$ behaves like $|r_3|$. Notice that when $\alpha \geq 0.5$, $|33r_1 + 3r_2| > |33r_1 - 3r_3|$. Hence under the condition of being $\alpha \geq 0.5$, $|g| > |g^{*}|$ and (24) is satisfied. Then the proposed hybrid approximation is unconditionally stable under the consideration of the aforementioned cases. The other selections of $\alpha$ values lead to a conditionally stable approximation.

## 5. NUMERICAL EXPERIMENTS

To figure out the effect of Galerkin FEM in strong form and α-family of approximation over numerical solutions of the Burgers equation, let us consider the following two test problems. To produce accurate results by dealing with meaningfully different values of the kinematic viscosity constant $\varepsilon$ is considered.

**Example 1** [1] Let us consider homogeneous problem with initial condition

$$u(x,0) = g(x) = \sin \pi x, \; 0 < x < 1 \tag{25}$$

and homogenous Dirichlet boundary conditions

$$u(0,t) = 0\,, \; t > 0 \tag{26}$$

$$u(1,t) = 0\,, \; t > 0 \tag{27}$$

A smooth exact solution of (1) under the consideration of the cases (25)-(27) given by Cole [22] is

$$u(x,t) = 2\pi\varepsilon \frac{\sum_{n=1}^{\infty} a_n \exp(-n^2\pi^2\varepsilon t) n sin(n\pi x)}{a_0 + \sum_{n=1}^{\infty} a_n \exp(-n^2\pi^2\varepsilon t) cos(n\pi x)} \tag{28}$$

with the Fourier coefficients

$$a_0 = \int_0^1 exp\{-(2\pi\varepsilon)^{-1}[1 - \cos(\pi x)]\}dx,$$

$$a_n = 2\int_0^1 exp\{-(2\pi\varepsilon)^{-1}[1-\cos(\pi x)]\}\cos(n\pi x)dx.$$

In the present example, comparison of the produced results with the results of the literature [1,9,12,26-28] and exact solutions has been carried out in Tables 2-4, for various spatial points at both small and large times. As realized from the tables, the produced results here are more accurate and more economical, even with less number of time elements, comparison to the taken results from the literature.

The computed results revealed that use of far less number of time elements for the proposed method is capable of catching better accuracy than the compared results [1,9,12,26-28]. Figure 1 shows the effect of parameter $\alpha$ under the consideration of $L_2$ error. Now it is time to deal with the smaller kinematic viscosity constants. Comparison of the currently produced solutions has been done with the literature and the exact solution for various values of the physical factors, with challenging kinematic viscosity values as seen in Tables 3-4. The computed results revealed that, even with the use of far less number of time elements, one can find similar or sometimes more accurate results than the literature [1,9,12].

The calculated solutions are depicted at various values of parameters $\varepsilon, dt, h$ at different times in Figures 2-7. The present method is observed to be very effective on capturing the steep behavior of the solution function as seen in Figures 2-3. Figures 8-9 show effects of the optimization factor on the computed results. As seen in Figure 8 under the fixed parameters $\varepsilon = 1$, $h = 0.025$ and $dt = 0.02$, the optimal value of $\alpha$ is equal to $0.525$. Under the consideration of parameters $\varepsilon = 1$, $h = 0.01$ and $dt = 0.05$, the optimal value of $\alpha$ is seen to be $0.550$ (see Figure 9).

The present technique has also been compared with the work of Dag et al. [24] numerically and the current results are seen to be more accurate than theirs. The absolute errors are $1.6E-06$ and $0.6E-05$ (their Table 3), respectively, for the same physical parameters. Their result is taken from their Table 3.

Note that behaviour of the solution of the Burgers equation for various values of viscosity constant $\varepsilon$ were discussed in the literature, e.g. Dag et al. [24], Sari and Gurarslan [1]. Physical behaviour of the nonlinear advection-diffusion process calculated by the proposed technique in terms of the viscosity constant exhibits the expected physical characteristics of the problem.

Comparison of the present method with the differential quadrature based methods [29-31] has also been done in Table 5. Even if their time approximation technique is the RK4 and conditionally stable, the present results are the same as with their results or more accurate than theirs as seen in Table 5.

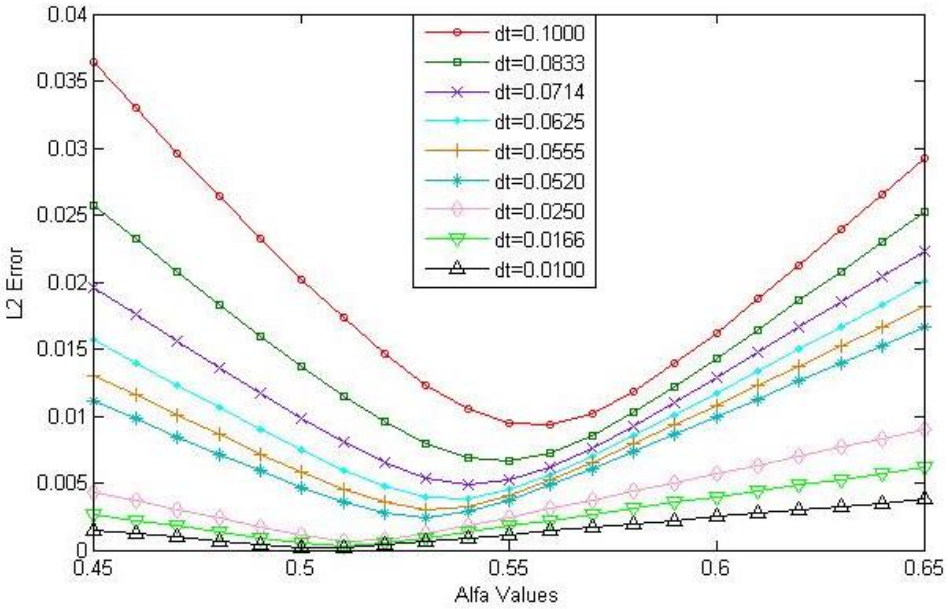


**Figure 1**. Comparison of $L_2$ error norms presented in Table 1

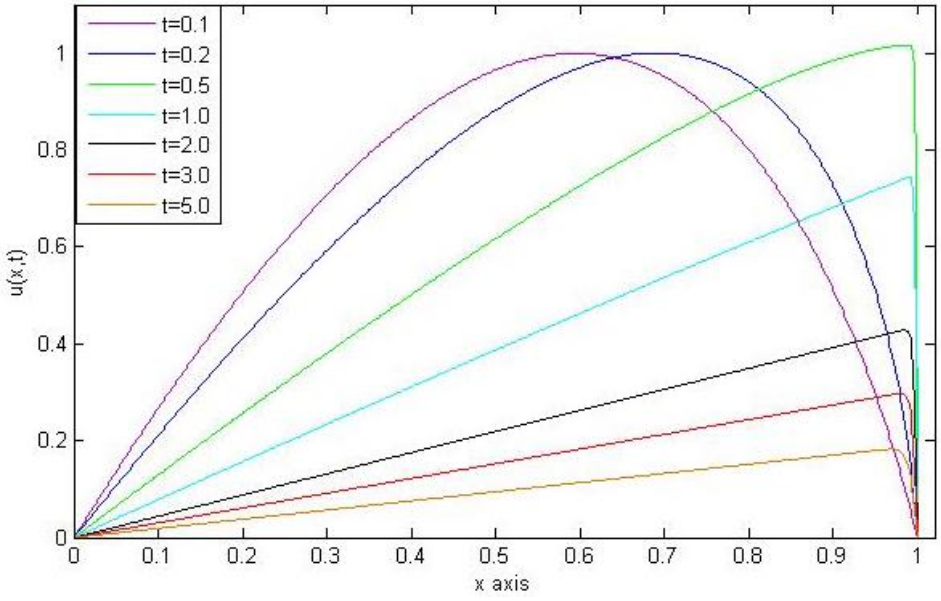


**Figure 2**. Numerical solution of the problem at different times produced for the parameters $\varepsilon = 0.001$, $h = 0.0016$, $\alpha = 0.50$ and $dt = 0.1$

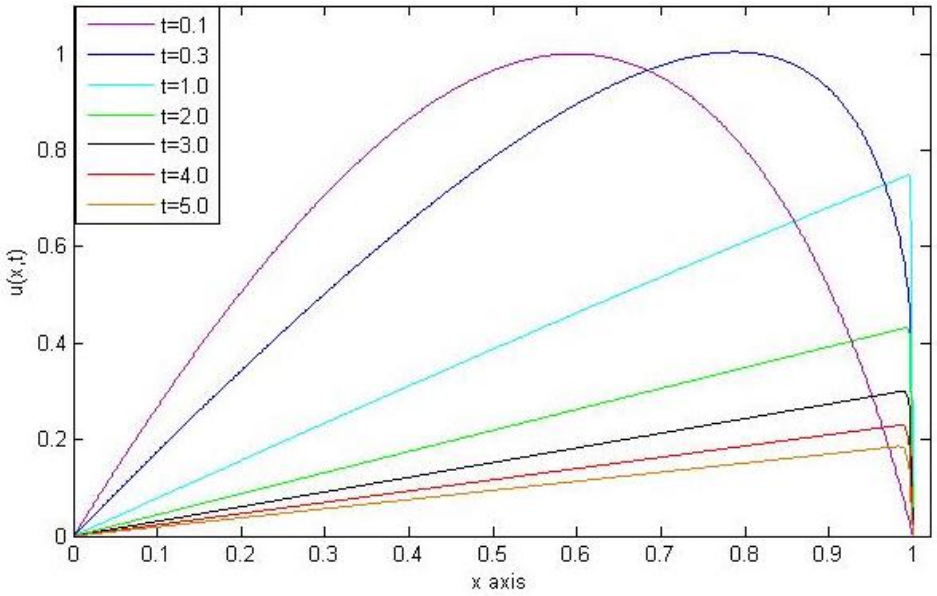


**Figure 3.** Numerical solution of the problem at different times produced for the parameters $\varepsilon = 0.0005$, $h = 0.0014$, $\alpha = 0.50$ and $dt = 0.1$

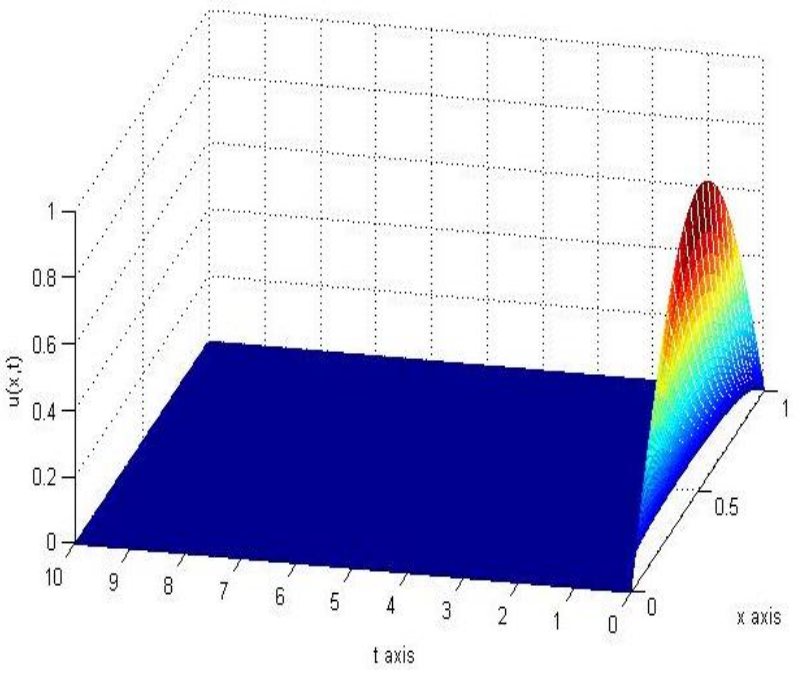


**Figure 4.** Numerical solution of the problem produced for the parameters $\varepsilon = 1$, $h = 0.0125$, $\alpha = 0.50$ and $dt = 0.01$

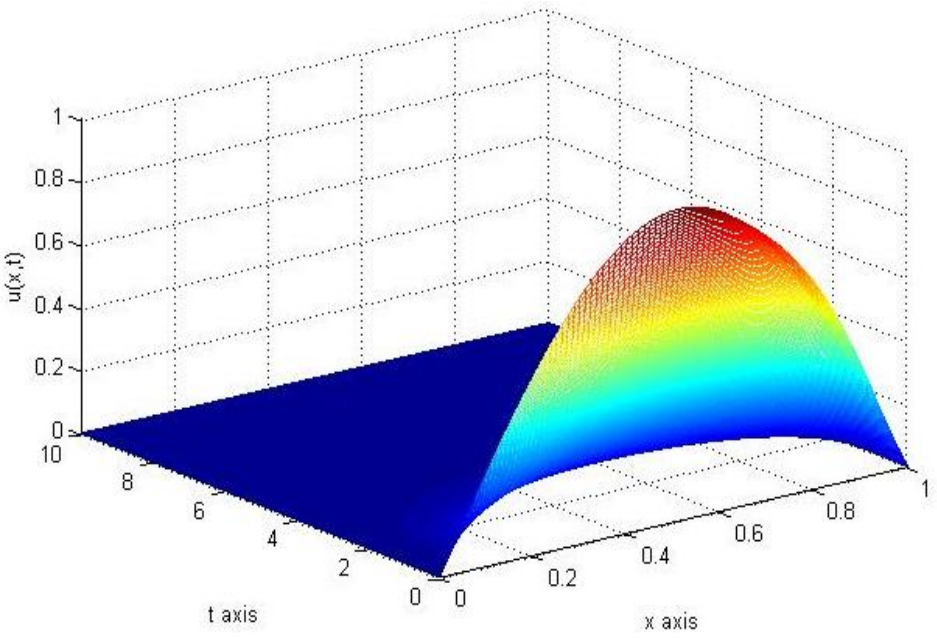


**Figure 5.** Numerical solution of the problem produced for the parameters $\varepsilon = 0.1$, $h = 0.0125$, $\alpha = 0.50$ and $dt = 0.01$

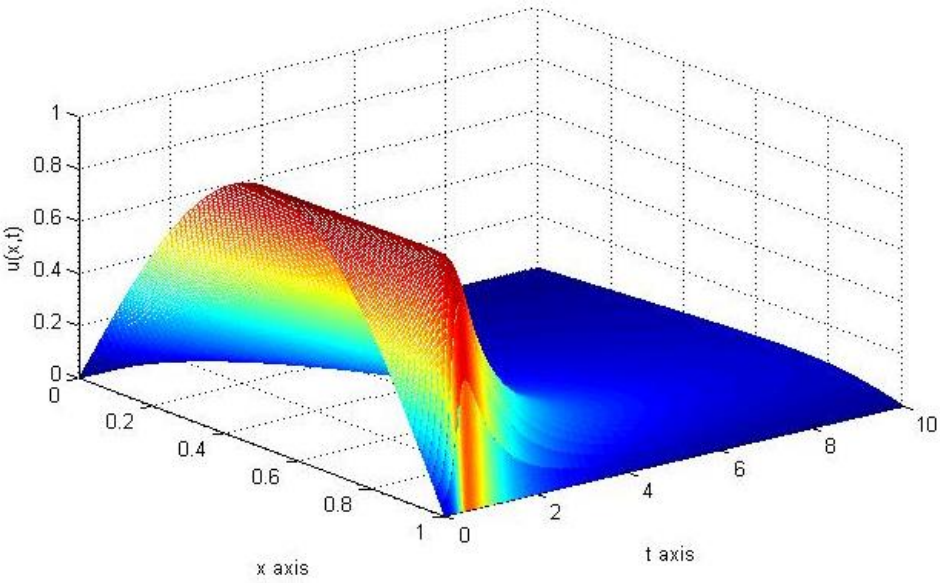


**Figure 6.** Numerical solution of the problem produced by $\varepsilon = 0.01$, $h = 0.0125$, $\alpha = 0.50$ and $dt = 0.01$

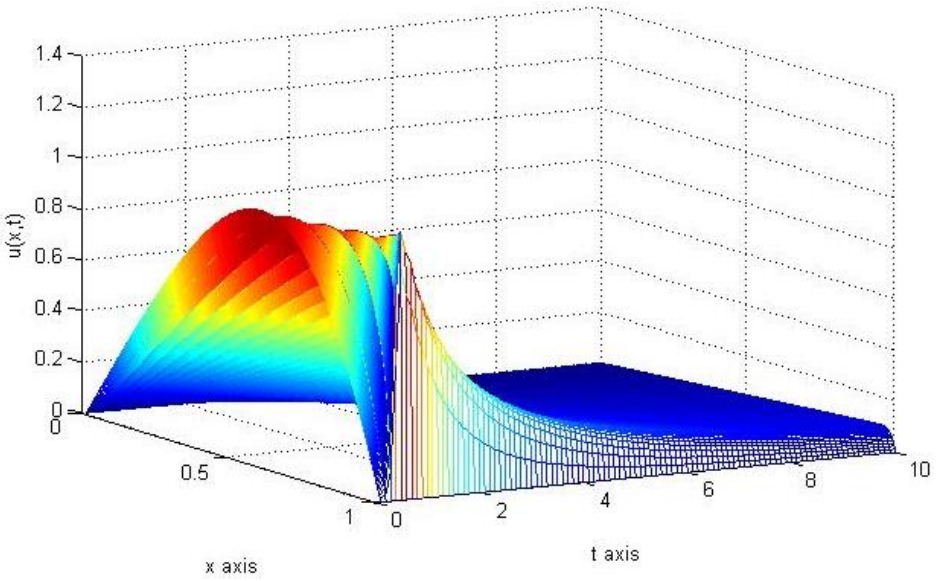


**Figure 7.** Numerical solution of the problem produced by $\varepsilon = 0.001$, $h = 0.0025$, $\alpha = 0.50$ and $dt = 0.1$

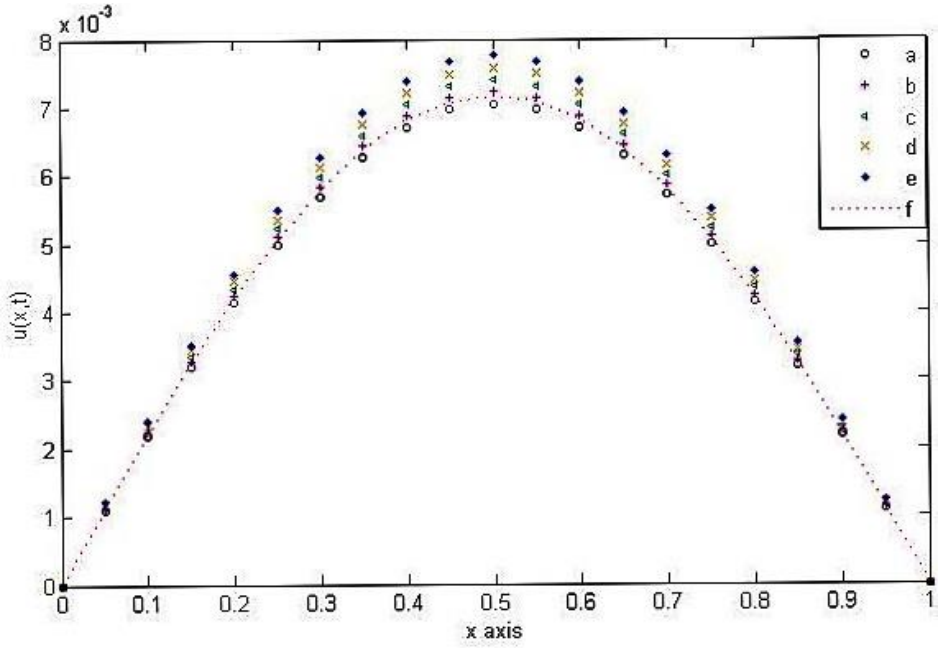


**Figure 8.** Numerical and analytical solutions of the problem at $t = 0.5$ produced by $\varepsilon = 1$, $h = 0.025$ and $dt = 0.02$ where a) $\alpha = 0.500$ b) $\alpha = 0.525$ c) $\alpha = 0.550$ d) $\alpha = 0.575$ e) $\alpha = 0.600$ f) Exact.

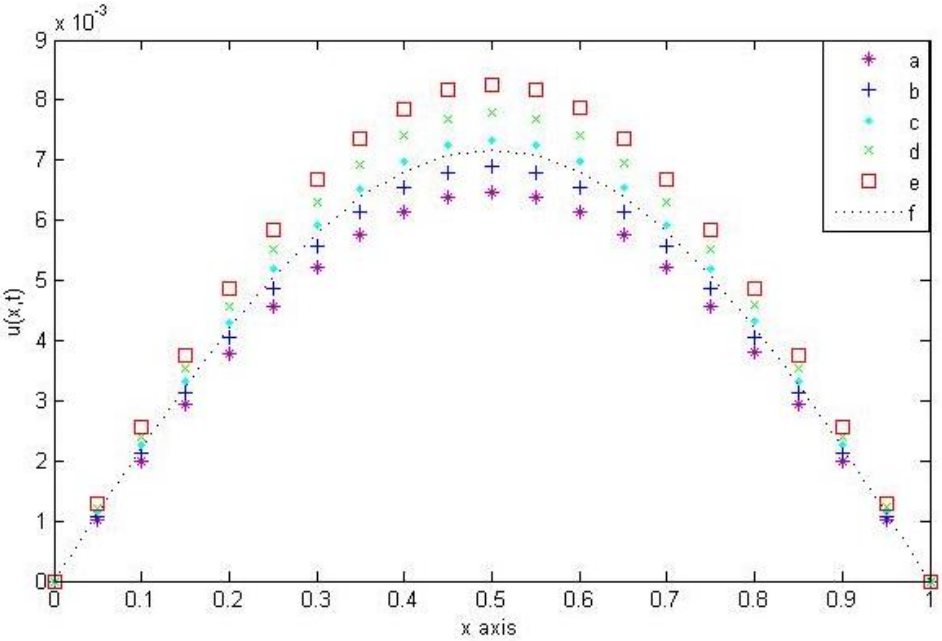


**Figure 9.** Numerical and analytical solutions of the problem at $t = 0.5$ produced by $\varepsilon = 1$, $h = 0.01$ and $dt = 0.05$ where a) $\alpha = 0.500$ b) $\alpha = 0.525$ c) $\alpha = 0.550$ d) $\alpha = 0.575$ e) $\alpha = 0.600$ f) Exact

**Table 2.** Comparison of the results produced with $\varepsilon = 0.05$, $h = 0.01$ for various values of the parameter $\alpha$ and different selection of $dt$.

| $x$ | $t$ | Present α=0.49 $dt = 0.004$ | Present α=0.50 $dt = 0.004$ | Present $\alpha = 0.51$ $dt = 0.004$ | Ref. [12] $dt = 0.001$ | Ref. [1] $dt = 0.001$ | Exact |
|---|---|---|---|---|---|---|---|
| | t=0.50 | 0.12113 | 0.12114 | 0.12115 | 0.12079 | 0.12114 | 0.12114 |
| $x = 0.1$ | t=2.00 | 0.04296 | 0.04296 | 0.04297 | 0.04300 | 0.04295 | 0.04296 |
| | t=4.00 | 0.02310 | 0.02310 | 0.02311 | 0.02324 | 0.02310 | 0.02310 |
| | t=0.50 | 0.36024 | 0.36027 | 0.36030 | 0.36113 | 0.36027 | 0.36027 |
| $x = 0.3$ | t=2.00 | 0.12883 | 0.12884 | 0.12885 | 0.12887 | 0.12882 | 0.12884 |
| | t=4.00 | 0.06930 | 0.06931 | 0.06931 | 0.06935 | 0.06930 | 0.06931 |
| | t=0.50 | 0.58866 | 0.58869 | 0.58873 | 0.59559 | 0.58870 | 0.58870 |
| $x = 0.5$ | t=2.00 | 0.21454 | 0.21456 | 0.21457 | 0.21468 | 0.21455 | 0.21456 |
| | t=4.00 | 0.11549 | 0.11549 | 0.11550 | 0.11550 | 0.11549 | 0.11549 |
| | t=0.50 | 0.79349 | 0.79350 | 0.79351 | 0.81257 | 0.79354 | 0.79349 |
| $x = 0.7$ | t=2.00 | 0.29998 | 0.30000 | 0.30001 | 0.30075 | 0.29999 | 0.30000 |
| | t=4.00 | 0.16121 | 0.16121 | 0.16122 | 0.16125 | 0.16121 | 0.16121 |
| | t=0.50 | 0.93822 | 0.93813 | 0.93805 | 0.97184 | 0.93822 | 0.93811 |
| $x = 0.9$ | t=2.00 | 0.37324 | 0.37327 | 0.37329 | 0.37452 | 0.37328 | 0.37328 |
| | t=4.00 | 0.16604 | 0.16605 | 0.16606 | 0.16515 | 0.16605 | 0.16606 |

**Table 3.** Comparison of the results produced with $\varepsilon = 0.003,\ h = 0.01$ for various values of parameters $\alpha$ and $dt$.

| $x$ | $t$ | Present α=0.49 $dt = 0.0125$ | Present α=0.50 $dt = 0.0125$ | Present α=0.51 $dt = 0.0125$ | Ref. [9] $dt = 0.001$ | Exact |
|---|---|---|---|---|---|---|
| $x = 0.25$ | $t = 1$ | 0.18896 | 0.18900 | 0.18904 | 0.18902 | 0.18901 |
| | $t = 5$ | 0.04697 | 0.04698 | 0.04698 | 0.04698 | 0.04698 |
| | $t = 10$ | 0.02422 | 0.02422 | 0.02422 | 0.02422 | 0.02422 |
| | $t = 15$ | 0.01632 | 0.01632 | 0.01632 | 0.01631 | 0.01631 |
| $x = 0.5$ | $t = 1$ | 0.37613 | 0.37620 | 0.37628 | 0.37623 | 0.37619 |
| | $t = 5$ | 0.09394 | 0.09395 | 0.09396 | 0.09396 | 0.09395 |
| | $t = 10$ | 0.04844 | 0.04844 | 0.04845 | 0.04844 | 0.04843 |
| | $t = 15$ | 0.03263 | 0.03263 | 0.03263 | 0.03263 | 0.03263 |
| $x = 0.75$ | $t = 1$ | 0.55917 | 0.55926 | 0.55935 | 0.55928 | 0.55924 |
| | $t = 5$ | 0.14090 | 0.14091 | 0.14093 | 0.14092 | 0.14095 |
| | $t = 10$ | 0.07260 | 0.07260 | 0.07261 | 0.07261 | 0.07260 |
| | $t = 15$ | 0.04838 | 0.04839 | 0.04839 | 0.04839 | 0.04841 |

**Table 4.** Comparison of maximum error norms of various schemes for $\varepsilon = 0.01, h = 0.0125$.

| $x$ | $t$ | EFDM [28] | EEFDM [28] | TFPM [27] | Present α=0.45 | Present α=0.50 | Present α=0.5 |
|---|---|---|---|---|---|---|---|
| | | | $dt = 0.0001$ | | | $dt = 0.001$ | |
| $x = 0.25$ | $t = 0.4$ | 5.54E-4 | 1.60E-5 | 9.28E-6 | 3.41E-5 | 2.88E-07 | 3.36E-5 |
| | $t = 0.6$ | 3.49E-4 | 1.10E-5 | 1.05E-5 | 2.87E-5 | 2.48E-07 | 2.82E-5 |
| | $t = 0.8$ | 2.46E-4 | 4.40E-6 | 8.46E-6 | 2.35E-5 | 1.99E-07 | 2.31E-5 |
| | $t = 1.0$ | 1.85E-4 | 5.10E-6 | 5.12E-6 | 1.93E-5 | 1.59E-07 | 1.90E-5 |
| | $t = 3.0$ | 2.23E-4 | 2.30E-6 | 6.35E-6 | 5.17E-5 | 3.15E-08 | 5.10E-5 |
| $x = 0.50$ | $t = 0.4$ | 5.22E-4 | 7.50E-6 | 1.44E-5 | 3.60E-5 | 5.78E-08 | 3.61E-5 |
| | $t = 0.6$ | 4.46E-4 | 4.50E-6 | 7.61E-6 | 4.33E-5 | 1.77E-07 | 4.30E-5 |
| | $t = 0.8$ | 3.56E-4 | 3.90E-6 | 2.57E-6 | 4.02E-5 | 2.20E-07 | 3.97E-5 |
| | $t = 1.0$ | 2.96E-4 | 4.00E-6 | 1.14E-5 | 3.50E-5 | 2.10E-07 | 3.46E-5 |
| | $t = 3.0$ | 3.49E-5 | 5.10E-6 | 1.09E-5 | 1.02E-5 | 5.91E-08 | 1.01E-5 |
| $x = 0.75$ | $t = 0.4$ | 1.12E-4 | 1.80E-5 | 4.41E-5 | 4.71E-5 | 1.24E-06 | 4.46E-5 |
| | $t = 0.6$ | 2.05E-4 | 5.00E-6 | 7.80E-5 | 2.00E-5 | 4.32E-07 | 2.08E-5 |
| | $t = 0.8$ | 2.62E-4 | 1.80E-6 | 8.28E-5 | 3.95E-5 | 2.40E-08 | 3.95E-5 |
| | $t = 1.0$ | 2.44E-4 | 5.60E-6 | 7.31E-5 | 4.19E-5 | 1.19E-07 | 4.17E-5 |
| | $t = 3.0$ | 3.21E-5 | 2.10E-6 | 9.26E-6 | 1.55E-5 | 3.78E-07 | 1.47E-5 |

**Table 5.** Comparison of the produced results for $\alpha = 0.50$ and $\varepsilon = 0.1$.

| | | Present N=20 | WDQM [29] N=25 | LDQM [30] N=30 | PDQM [31] N=20 | |
|---|---|---|---|---|---|---|
| $x$ | $t$ | $dt = 0.001$ | $dt = 0.0001$ | $dt = 0.001$ | $dt = 0.01$ | Exact |
| $x = 0.25$ | $t = 0.4$ | 0.30889 | 0.30880 | 0.30889 | 0.30889 | 0.30889 |
| | $t = 0.6$ | 0.24073 | --- | 0.24074 | 0.24074 | 0.24074 |
| | $t = 0.8$ | 0.19567 | 0.19565 | 0.19568 | 0.19568 | 0.19568 |
| | $t = 1.0$ | 0.16256 | 0.16221 | 0.16256 | 0.16256 | 0.16256 |
| | $t = 3.0$ | 0.02719 | 0.02720 | 0.02720 | 0.02720 | 0.02720 |
| $x = 0.50$ | $t = 0.4$ | 0.56963 | 0.56953 | 0.56963 | 0.56963 | 0.56963 |
| | $t = 0.6$ | 0.44720 | --- | 0.44721 | 0.44721 | 0.44721 |
| | $t = 0.8$ | 0.35922 | 0.35922 | 0.35924 | 0.35924 | 0.35924 |
| | $t = 1.0$ | 0.29189 | 0.29190 | 0.29192 | 0.29192 | 0.29192 |
| | $t = 3.0$ | 0.04019 | 0.04020 | 0.04021 | 0.04021 | 0.04021 |
| $x = 0.75$ | $t = 0.4$ | 0.62538 | 0.62554 | 0.62544 | 0.62544 | 0.62544 |
| | $t = 0.6$ | 0.48712 | --- | 0.48722 | 0.48722 | 0.48721 |
| | $t = 0.8$ | 0.37382 | 0.37309 | 0.37392 | 0.37392 | 0.37392 |
| | $t = 1.0$ | 0.28739 | 0.28746 | 0.28747 | 0.28747 | 0.28747 |
| | $t = 3.0$ | 0.02976 | 0.02977 | 0.02977 | 0.02977 | 0.02977 |

**Example 2** [1] Let us take now then Burgers equation (1) with initial condition

$$u(x,0) = g(x) = 4x(1-x), \quad 0 < x < 1 \tag{29}$$

and homogenous boundary conditions

$$u(0,t) = 0, \ t > 0 \tag{30}$$

$$u(1,t) = 0, \ t > 0. \tag{31}$$

The exact solution of (1) under the consideration of the cases (22)-(24) given by Cole [21] as in (28) but with the Fourier coefficients

$$a_0 = \int_0^1 exp\{-x^2(3\varepsilon)^{-1}(3-2x)\}dx,$$

$$a_n = 2\int_0^1 exp\{-x^2(3\varepsilon)^{-1}(3-2x)\}\cos(n\pi x)dx.$$

Table 6 includes comparison of currently produced solutions with exact solution and the literature [16,28] for different values of physical parameters under the consideration of various α values. The computed results for less number of time elements, comparison to some works carried out [16,28], are seen to be more accurate or of the same accuracy with the corresponding literatures. By taking into account various values of the processes of interest, all the presented numerical solutions in Table 7 are compared with exact solution and the literature [8,14]. The numerical behaviour at $t = 0.5$ for challenging values of the kinematic viscosity is exhibited both quantitatively and qualitatively (see Table 6 and Figure 10). The present work has required less number of time elements for more accurate results as compared to the studies done in references [8,14]. As previously stated, the optimum values of the parameter α are affected from the number of the time elements although the corresponding optimum values are seen to be not seriously affected from the number of spatial elements (see Figure 11).

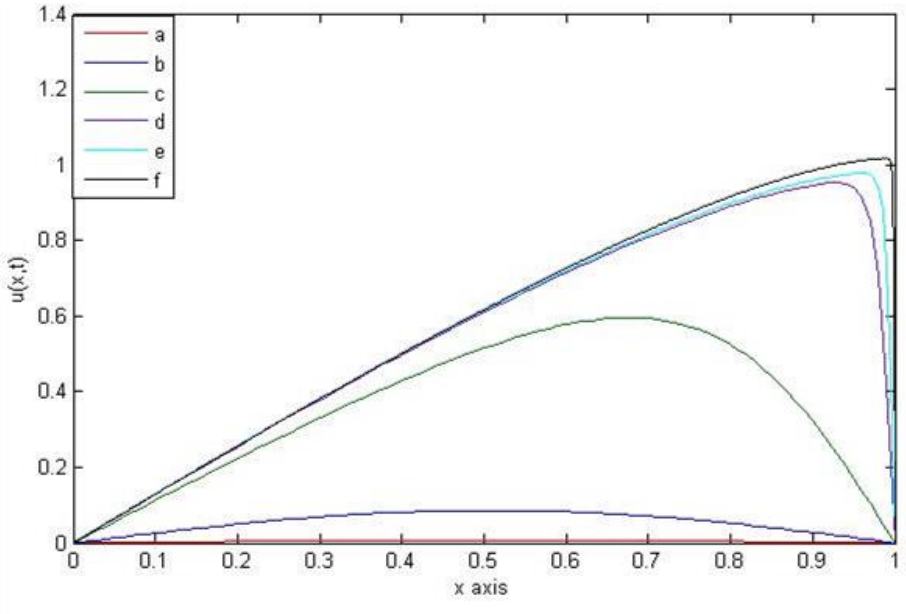


**Figure 10**. Numerical solution of Example 2 at $t = 0.5$ with different kinematic viscosity constants: a) $\varepsilon = 1$ b) $\varepsilon = 0.5$ c) $\varepsilon = 0.1$ d) $\varepsilon = 0.01$ e) $\varepsilon = 0.005$ f) $\varepsilon = 0.001$

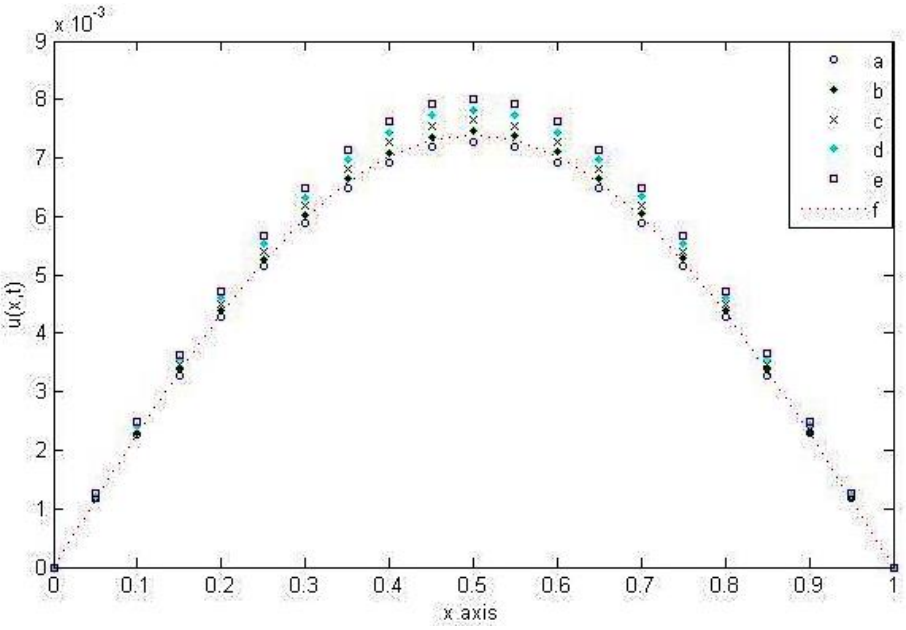


**Figure 11.** Numerical and analytical solutions of the Example 2 at $t = 0.5$ produced by $\varepsilon = 1$, $h = 0.025$ and $dt = 0.02$ where a) $\alpha = 0.500$ b) $\alpha = 0.525$ c) $\alpha = 0.550$ d) $\alpha = 0.575$ e) $\alpha = 0.600$ f) Exact

**Table 6.** Comparison of the results produced with $\varepsilon = 1$ for various values of parameters $\alpha$, $dt$ and $h$ in Example 2

| $x$ | $t$ | Present α=0.49 $h = 0.0125$, $dt = 0.0002$ | Present α=0.50 $h = 0.0125$, $dt = 0.0002$ | Present α=0.51 $h = 0.0125$, $dt = 0.0002$ | Ref. [27] $h = 0.0125$, $dt = 0.0001$ | Ref. [16] $h = 0.250$ $dt = 0.0001$ | Exact |
|---|---|---|---|---|---|---|---|
| x=0.25 | t=0.05 | 0.4262796 | 0.4262850 | 0.4262904 | 0.42629 | 0.4262864 | 0.4262855 |
| | t=0.10 | 0.2614741 | 0.2614793 | 0.2614845 | 0.26149 | 0.2614801 | 0.2614797 |
| | t=0.15 | 0.1614726 | 0.1614772 | 0.1614819 | 0.16148 | 0.1614777 | 0.1614776 |
| | t=0.25 | 0.0610844 | 0.0610873 | 0.0610903 | 0.06109 | 0.0610875 | 0.0610875 |
| x=0.50 | t=0.05 | 0.6280775 | 0.6280833 | 0.6280892 | 0.62809 | 0.6280846 | 0.6280837 |
| | t=0.10 | 0.3834143 | 0.3834218 | 0.3834294 | 0.38343 | 0.3834228 | 0.3834224 |
| | t=0.15 | 0.2340479 | 0.2340548 | 0.2340617 | 0.23406 | 0.2340554 | 0.2340553 |
| | t=0.25 | 0.0872281 | 0.0872324 | 0.0872366 | 0.08724 | 0.0872327 | 0.0872327 |
| x=0.75 | t=0.05 | 0.4652483 | 0.4652521 | 0.4652558 | 0.46526 | 0.4652528 | 0.4652526 |
| | t=0.10 | 0.2815666 | 0.2815721 | 0.2815775 | 0.28158 | 0.2815727 | 0.2815726 |
| | t=0.15 | 0.1697327 | 0.1697378 | 0.1697429 | 0.16974 | 0.1697383 | 0.1697382 |
| | t=0.25 | 0.0622865 | 0.0622896 | 0.0622927 | 0.06229 | 0.0622898 | 0.0622898 |

**Table 7.** Comparison of the results produced with $\varepsilon = 0.1, h = 0.0125$ for various values of parameters $\alpha$ and $dt$ in Example 2

| $x$ | $t$ | Present α=0.49 $dt = 0.001$ | Present α=0.50 $dt = 0.001$ | Present α=0.51 $dt = 0.001$ | Ref. [14] $dt = 0.0001$ | Ref. [8] $dt = 0.0001$ | Exact |
|---|---|---|---|---|---|---|---|
| $x = 0.25$ | $t = 0.4$ | 0.31752 | 0.31752 | 0.31753 | 0.32091 | 0.31749 | 0.31752 |
| | $t = 0.6$ | 0.24613 | 0.24614 | 0.24614 | 0.24910 | 0.24612 | 0.24614 |
| | $t = 0.8$ | 0.19955 | 0.19956 | 0.19956 | 0.20211 | 0.19954 | 0.19956 |
| | $t = 1.0$ | 0.16560 | 0.16560 | 0.16560 | 0.16782 | 0.16559 | 0.16560 |
| | $t = 3.0$ | 0.02776 | 0.02776 | 0.02776 | 0.02828 | 0.02776 | 0.02776 |
| $x = 0.50$ | $t = 0.4$ | 0.58453 | 0.58454 | 0.58454 | 0.58788 | 0.58448 | 0.58454 |
| | $t = 0.6$ | 0.45797 | 0.45798 | 0.45798 | 0.46174 | 0.45793 | 0.45798 |
| | $t = 0.8$ | 0.36739 | 0.36740 | 0.36740 | 0.37111 | 0.36736 | 0.36740 |
| | $t = 1.0$ | 0.29834 | 0.29834 | 0.29835 | 0.30183 | 0.29831 | 0.29834 |
| | $t = 3.0$ | 0.04106 | 0.04106 | 0.04107 | 0.04185 | 0.04106 | 0.04106 |
| $x = 0.75$ | $t = 0.4$ | 0.64562 | 0.64561 | 0.64561 | 0.65054 | 0.64547 | 0.64562 |
| | $t = 0.6$ | 0.50267 | 0.50267 | 0.50268 | 0.50825 | 0.50255 | 0.50268 |
| | $t = 0.8$ | 0.38533 | 0.38533 | 0.38534 | 0.39068 | 0.38523 | 0.38534 |
| | $t = 1.0$ | 0.29585 | 0.29586 | 0.29586 | 0.30057 | 0.29578 | 0.29586 |
| | $t = 3.0$ | 0.03044 | 0.03044 | 0.03044 | 0.03106 | 0.03044 | 0.03044 |

## 5. CONCLUSIONS AND RECOMMENDATION

This paper has proposed a hybrid numerical technique to deal with the Burgers equation. The designed technique has then been shown to be unconditionally stable for parameter $\alpha \geq 0.5$. The technique has been illustrated to have great potentiality in analysing the Burgers equation with less number of elements used in both time and space even for challenging cases of the problem. The computed results have also been realized to have the same level of accuracy or to be better

than existing results in the literature, even with the use of a modest time approximation. Note that the suggested approach is seen to be a very good alternative to achieve a high degree of accuracy while analysing the advection-diffusion processes. In addition to both time and space discretization, the choice of parameter α is observed to be effective on capturing the behavior of the problem. The presented results have validated all cases of α-optimization. Advantages of the current algorithm are clearly seen especially from the steep behavior of the produced results. As is the case for all versions of the finite element based methods, storage drawbacks of the proposed methods are expected to come out in two/three dimensional cases of the current models for very large domains. Future studies can focus on designing of the current technique to physical processes represented by more involved time-dependent models.

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