Sigma J Eng & Nat Sci 35 (2), 2017, 333-346

Sigma Journal of Engineering and Natural Sciences
Sigma Mühendislik ve Fen Bilimleri Dergisi

Research Article

COMPARISON BETWEEN NUMERICAL AND ANALYTICAL SOLUTIONS FOR THE RECEDING CONTACT PROBLEM

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Received: 30.11.2016 Revised: 14.01.2017 Accepted: 25.02.2017

ABSTRACT

A finite element calculation has been utilized to investigate the plane symmetric double receding contact problem for a rigid stamp and two elastic layers. Elastic layers have different elastic constants and heights. The external load is applied to the upper elastic layer by means of a rigid stamp and the lower elastic layer is bonded to a rigid support. The external load is applied to the upper elastic layer by means of a rigid stamp and the lower elastic layer is bonded to a rigid support. The problem is solved under the assumptions that the contact between two elastic layers, and between the rigid stamp and the upper elastic layer are frictionless, the effect of gravity force is neglected. Numerical simulations are realized by the world wide code ANYS software based on FEM. The model provides dimensionless expressions for the contact areas and contact pressures. This paper presents comparison with numerical solutions and analytical solutions. Calculated contact areas and contact pressures may be used for the optimal design of layer system as well as together with analytical solutions.
Keywords: Contact mechanics, contact area, contact stress, FEM.

1. INTRODUCTION

Boundary value problems, including contact are of great significant in industrial practice in mechanical and civil engineering. Metal forming processes, drilling problems, bearings, crash analysis of cars, car tires or cooling of electronic devices are the range of application. Other applications are related to biomechanics where human joints, implants or teeth are of consideration. Due to this variety contact problems are today combined either with large elastic or inelastic deformations including time dependent responses. Thermal coupling might have to be considered, see the cooling of electronic devices, the heat removal within nuclear power plant vessels or thermal insulation of astronautic vehicles. Even stability behavior has to be linked to contact, like wrinkling arising in metal forming problems Wriggers (1995).

Many known researchers in the past have investigated contact problems for the technical importance. There are many contact problems in the area of the civil engineering. It is so difficult to experience problem that is not contact present. Spite of the fact that more than one century has passed since the basic work of Hertz (1881) the contact problem is still of great interest. Hertz (1896) determined the distribution of stress throughout the contact area that appear when two

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bodies with curved surfaces are pressed against each other. Many contact problems usually involving simple geometries with infinite dimensions have been solved analytically since then. Many of these problems can be found in Gladwell (1980), Johnson (1985), Ozsahin et al. (2007), Comez et al. (2004) and Oner and Birinci (2014). The receding contact problem has been studied for more than four decades by many researchers both numerically and analytically. The latest numerical studies on this topic were based on either finite element method Francavilla and Zienkiewicz (1975) and Jing and Liao(1990). Long and Wang (2013) investigated effects of surface tension on axisymmetric Hertzian contact problem. Yang (2013) studied solutions of dissimilar material contact problems. Chidlow and Teodorescu (2013) examined the frictionless two-dimensional contact problem of an inhomogeneously elastic material under a rigid punch. The periodic contact problem of the plane theory of elasticity with taking friction, wear and adhesion into account was examined by Soldatenkov (2013). Li et al. (2014) studied the fundamental contact solutions of a magneto-electro-elastic half-space indented by a smooth and rigid half-infinite punch. Gun and Gao (2014) presented a quadratic boundary element formulation for continuously non-homogeneous, isotropic and linear elastic functionally graded material contact problems with friction.

On the main of the confinement analytical approaches and the development of computer has led to improve of the numerical methods for the solving contact problems. One of the numerical method currently used to solve contact mechanics problems is the Finite Element Method (FEM). In the area of the finite element method the references to the papers of Chan and Tuba (1971), Fredricksson (1976), Okamoto and Nakazawa (1979) and Oden and Pires (1984), Bathe and Chaudhary (1985) and Klarbring and Björkman (1992). Schwarzer et al. (1995) compare a finite element method with an analytical mode1 to describe the stress distribution in layered materials under spherical non-Hertzian load. Zhu (1995) studied a finite element–mathematical programming method for elastoplastic contact problems with friction. Papadopoulos and Solberg (1998) investigated a novel Lagrange multiplier–based formulation for the finite element solution of the quasistatic two-body contact problem in the presence of finite motions and deformations. The mortar finite element method for contact problems was examined by Belgacem et al. (1998). Guyot et al. (2000) presented coupling of finite elements and boundary elements methods for study of the frictional contact problem. A residual type a posteriori error estimator for finite element approximations of a frictional contact problem for linearized elastic materials was analyzed by Bostan and Han (2006). Solberg et al. (2007) studied a family of simple two-pass dual formulations for the finite element solution of contact problems. Oysu (2007) investigated finite element and boundary element contact stress analysis with remeshing technique. Zhang et al. (2012) reported a finite element model for 2D elastic-plastic contact analysis of multiple cosserat materials. The comparative studies of numerical solution and analytical solution of the contact problem is conducted by Birinci et al. (2015).

This paper is concerned with the analysis of FEM for receding contact problems. In the present study, the plane symmetric double receding contact problem of a rigid stamp and two infinite elastic layers with different elastic constants and heights is investigated. The external load is applied to the upper elastic layer by means of a rigid stamp and the lower layer is bonded to a rigid support, shown in Fig. 1. It is assumed that the contact surfaces are frictionless, the effect of gravity force is neglected Comez et al. (2004). The problem is developed based on the FEM ANSYS (2013) software. . The numerical results for the contact areas and contact stresses are obtained for various quantities and shown in the figures and tables. The numerical results are verified by comparison with analytical results in literature. Since then the finite element method and the techniques to contain the contact limitations were further developed. Different methods, like the penalty method, the barrier method, the Lagrange multiplier method and the augmented Lagrangian method were developed Laursen (2003) and Wriggers and Nackenhorst (2006). Finite Element Method can deal with elastic-plastic frictional contact problems Blushan (1996).

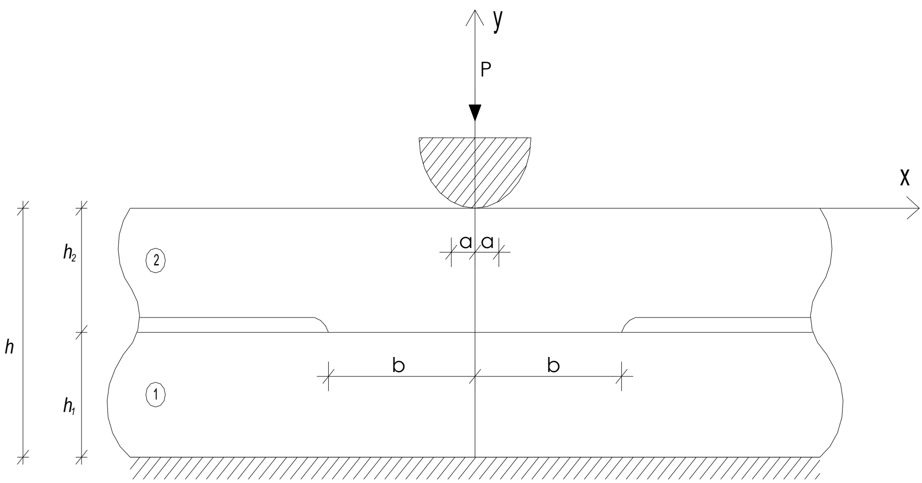

Figure 1. Geometry of the contact problem

2. FINITE ELEMENT METHOD CONTACT PROBLEMS

In many problems cases start where the status parts of the boundary of one body come across with those of another part of the boundary of the same or another body. This kind of problems are usually named contact problems. Contact between bodies is one of the methods of load transferring. The technique of that kind of load transferring depends on a nature of interaction between two or more contact surfaces. The information of such technique has a great practical feature. The direct investigation of the contact event and measuring of the certain values is so difficult. Contact problem is complex because of fact that the behavior of the elements which create contact stresses, depends on property of materials.

Contact problems are very difficult to model by finite elements. But finite element methods have been used for many years to solve contact problems. The popularity of FEM has by degrees risen in recent years. This is widely due to development in the contact mechanics, that has made it possible to simulate problems with ever increasing difficulty. In the fact the FEM was used for linear problems in strength analyses and safety. Currently, the method has also found a place in industry for decreasing costs in the design and development cycles of products.

Figure 2 shows a ordinary condition that one body is being pressed into a second body. In Fig. 2(a) the two objects are not in contact and the boundary conditions are specified by zero traction conditions for both bodies. In Fig. 2(b) the two objects are in contact along a part of the boundary segment and here conditions must be inserted to ensure that penetration does not occur and traction is consistent Zienkiewicz (2005).

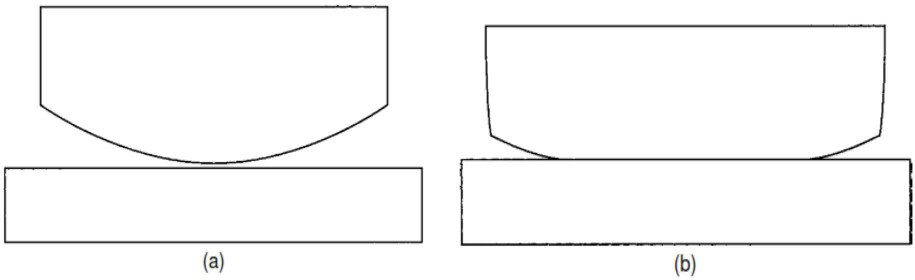

Figure 2. Contact between two bodies: (a) no contact condition; (b) contact state

An illustration of the process for a 2-dimensional case is given in Figure 3. In this figure a part of a slave boundary and part of a master boundary are shown.

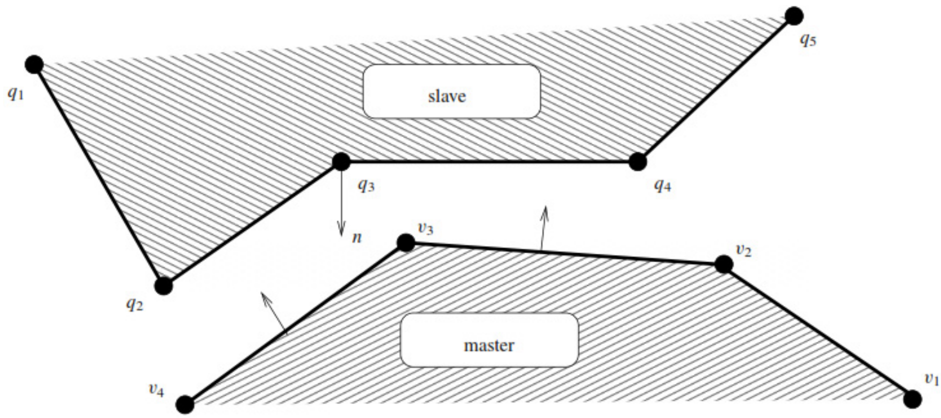

Figure 3. Model of contact

The finite element method is in general used to for scientific computing in engineering. Without going into all the details, they present here just the algorithm for contact modeling. After finite element separation in the context of small displacements, the global set of equilibrium equations of two contacting elastic bodies can be written as

$$KU = F + R \tag{1}$$

where K denotes the stiffness matrix. U is the displacement vector and F external known forces vector. R is the contact reactions vector. As U and R are both unknown, Eq. (1) cannot be directly solved. In the popular penalty method is directly managed. Accordingly, K is modified by introducing contact elements and the global set of equations is solved at each iteration. The resulting numerical algorithms are not very reliable. Their idea is to determine iteratively the contact reactions vector R in a reduced system which only concerns the contact nodes. Then, vector U can be computed in the whole structure, using contact reactions as external loading. Consequently, the global set of equations is just solved once. Another advantage is that the stiffness matrix is not changed as opposite to the penalty method or to the Lagrange multiplier method Renaud and Feng (2003).

3. FINITE ELEMENT MODEL

A 2-D model of receding contact problem was built by using APDL language embedded in the finite element software ANSYS. Through contact analysis, the changes could be showed in stress, strain, penetration, sliding and distance. Furthermore, the simulation results revealed that the computational values were consistent with theoretical values. The all showed that the model and boundary conditions were correct and it would provide a scientific basis for optimum design of contact problem under loads

A commercial ANSYS 11 package was used to solve the contact problem. The problem is considered as a two-dimensional contact problem and the material of the layers are assumed elastic and isotropic. The physical system under consideration exhibits symmetry in geometry, material properties and loading. Taking advantage of symmetry, only one half of the geometry of the problem is to be modeled. The finite element method numerical solution requires as an input some material and geometrical properties. In the analyses, geometric properties are taken as $L = 20m$ (length of the layer in x direction), $h_2 = 1m$ (thickness of the lower layer in y direction), $P = 12000N$ load and material properties are taken as $E_2 = 3 \cdot 10^6 Pa$, $\nu_2 = \nu_1 = 0.25$. Other parameters are chosen such that $h_1/h_2, R/h_2, E_2/E_1$ and $\mu_2/(P/h_2)$ ratios are compatible with dimensionless values which are obtained analytical solution. The geometry and the applied load are shown schematically in Fig. 4.

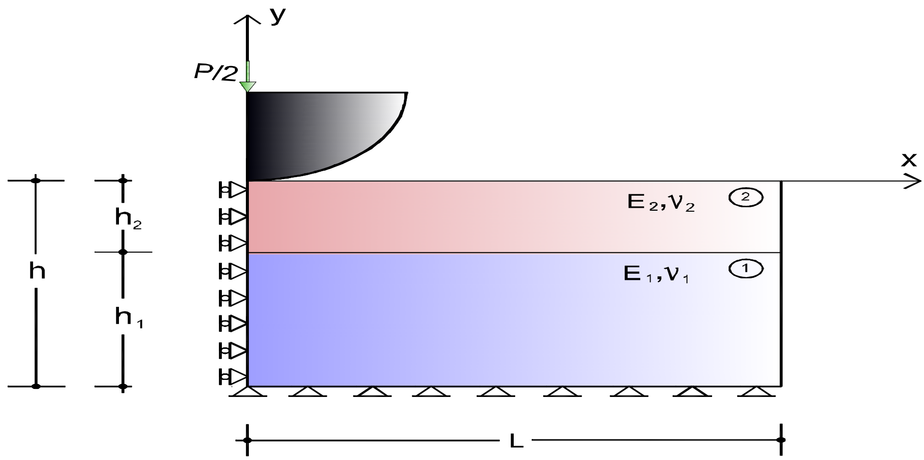

Figure 4. The geometry for the analysis

In contact problem included two boundaries, it is natural that take one boundary as contact surface and take the other one as target surface. Surface-surface contact is very suitable for those problems. Typical surface-surface contact's analysis steps mainly include: (1) Build 2D geometry model; (2) Identify material properties; (3) Mesh; (4) Identify contact pairs, define target surface and define contact surface; (5) Apply the necessary boundary conditions and load steps; (6) Define solution options; (7) Solve contact problems; (8) Review analyze results.

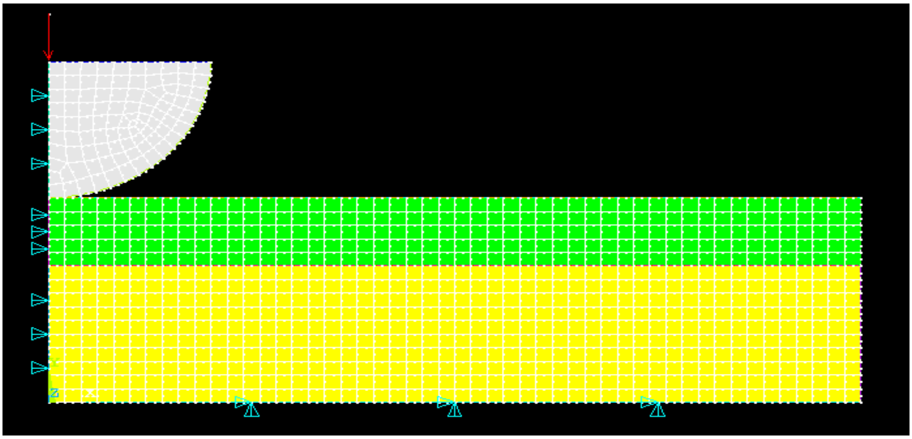

Figure 5. The finite element mesh

Finite Element model of the problem before analysis as modeled using ANSYS are shown in Fig. 5.The program ANSYS is used in the finite element analysis (FEA) modeling. The mesh is generated using two dimensional solid 8-node PLANE 183. PLANE183 is defined by eight nodes having two degrees of freedom at each node: translations in the nodal x and y directions. In addition this element has the capability, plasticity, elasticity, creep, swelling, stress stiffening, large deflection, and large strain capabilities. Mesh size and configuration are important parts of modeling with precise mesh refinement being necessary in regions of high-stress intensity. The preliminary finite element mesh consisted of 225 eight-node quadrilateral elements comprising a total of 714 nodes. The contact area is meshed by surface-to-surface CONTA172 and TARGE169 contact elements. CONTA172 is used to represent the mechanical contact analysis. The target surface, defined by TARGE169, was used to represent 2D target surfaces for the associated contact elements CONTA172 (Fig. 6). Frictionless surface-to-surface contact elements are used to model the interaction between the contact surfaces, and the augmented Lagrangian method is used as the contact algorithm. Deformation shape after analysis by using these elements is shown in Fig. 7.

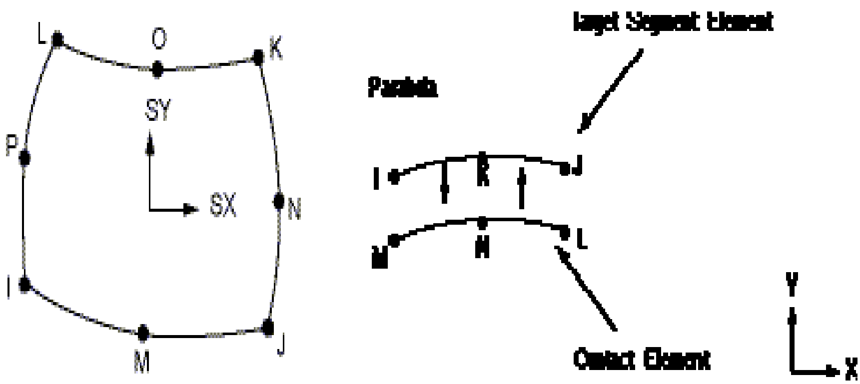

Figure 6. PLANE 183, CONTA 172 and TARGE 169 elements

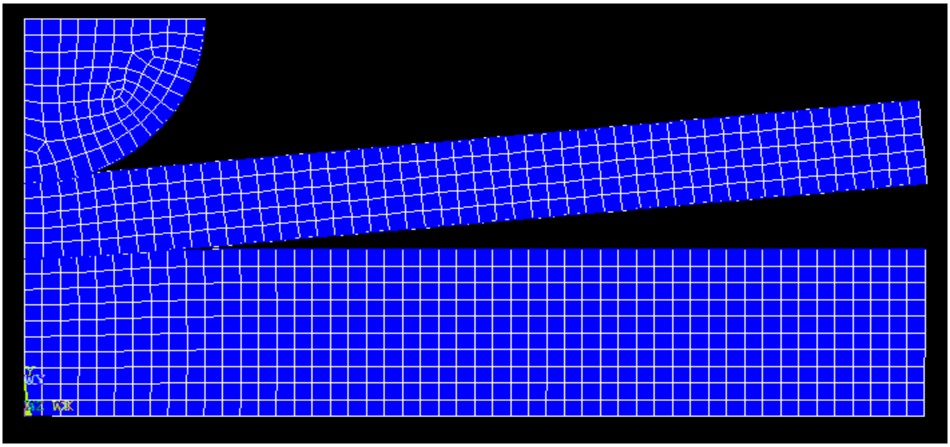

Figure 7. Deformed geometry for the preliminary analysis

4. COMPARISON WITH ANALYTICAL SOLUTIONS

The most convincing way to verify the FEA results is to compare them with the known analytical solutions. Finite element analysis is shortly reported for the analytical results in literature Comez et al. (2004). Contact areas and contact stress are found for various values running finite element analysis with the aid of ANSYS (2013). The results from the running ANSYS codes for various dimensionless quantities such as $h_1/h_2, \mu_2/\mu_1, R/h_2$ and $\mu_2/(P/h_2)$ are shown in Figs. 8-12 and Tables 1-6.The presents results compared with already available in the literature which is studied by Comez et al. (2004). As shown in the results given below, the results cohere well with the known theoretical results which indicate the combination finite element method used here appears to be a good approximation method to solve actual contact problems. The comparison of results with those in literature and with the finite element software ANSYS are found in good agreement.

Fig. 8 shows comparing the contact area between two elastic layers (b/h_2) with different $\mu_2/(P/h_2)$ and (R/h_2) for fixed values $(h_1/h_2 = 2, \mu_2/\mu_1 = 1, \kappa_1 = \kappa_2 = 2)$. For constant $h_1/h_2 = 2, \mu_2/\mu_1 = 1 \; and \; \kappa_1 = \kappa_2 = 2$, comparing the contact area between circular rigid stamp and elastic layers (a/h_2) with various values $\mu_2/(P/h_2)$ and (R/h_2) is shown in Fig. 9. Fig. 10 and 11 show comparing the contact pressure distribution between two elastic layers$P_1(x)$ and under the circular rigid stamp $P_2(x)$ for various values μ_2/μ_1. Fig. 12 shows comparing the contact areas (a/h_2) and (b/h_2) with μ_2/μ_1 for circular rigid stamp profile.

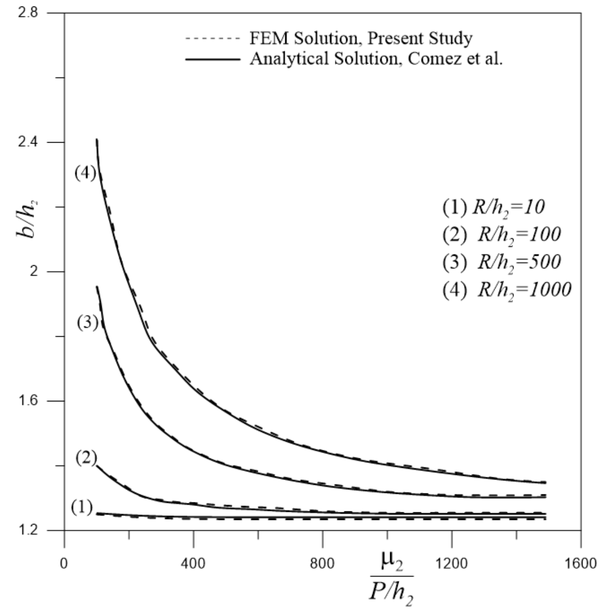

Figure 8. Comparisons of analytical solution with FEM solution for the contact area between two elastic layers (b/h_2) with $\mu_2/(P/h_2)$ for various values R/h_2 $(h_1/h_2 = 2, \mu_2/\mu_1 = 1, \kappa_1 = \kappa_2 = 2)$

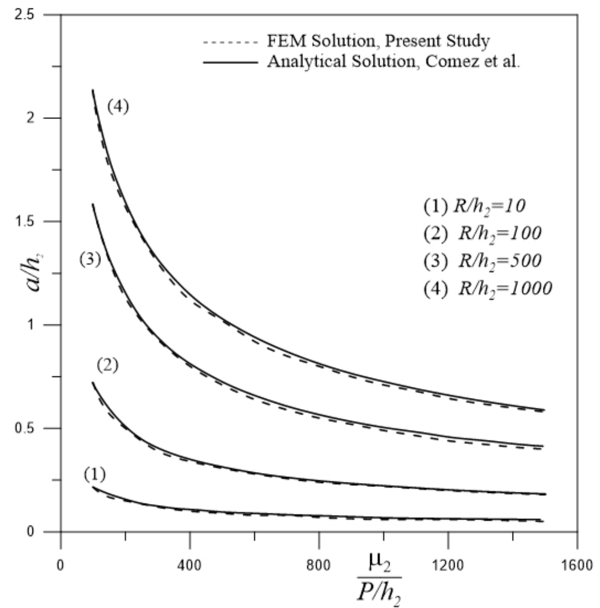

Figure 9. Comparisons of analytical solution with FEM solution for the contact are between circular rigid stamp and elastic layers (a/h_2) with $\mu_2/(P/h_2)$ for various values R/h_2 $(h_1/h_2 = 2, \mu_2/\mu_1 = 1, \kappa_1 = \kappa_2 = 2)$

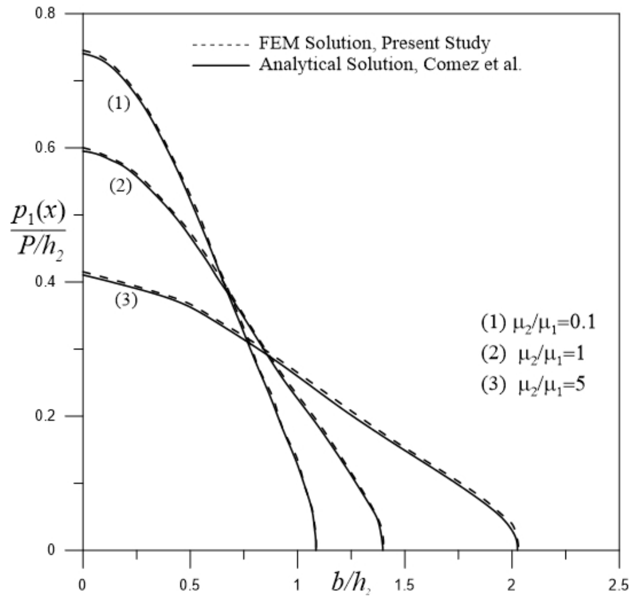

Figure 10. Comparisons of analytical solution with FEM solution for contact stress distribution between two elastic layers for various value of μ_2/μ_1
$(h_1/h_2 = 2, \kappa_1 = \kappa_2 = 2, R/h_2 = 500, \mu_2/(P/h_2) = 500)$

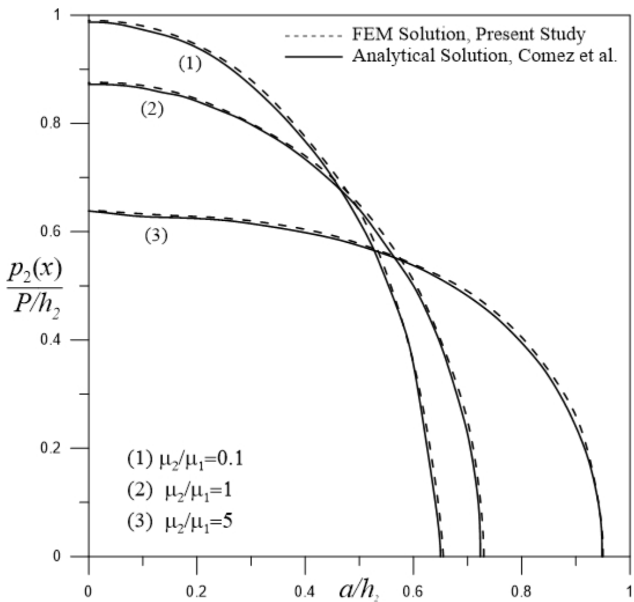

Figure 11. Comparisons of analytical solution with FEM solution for contact stress distribution under circular rigid stamp for various value of μ_2/μ_1
$(h_1/h_2 = 2, \kappa_1 = \kappa_2 = 2, R/h_2 = 1000, \mu_2/(P/h_2) = 500)$

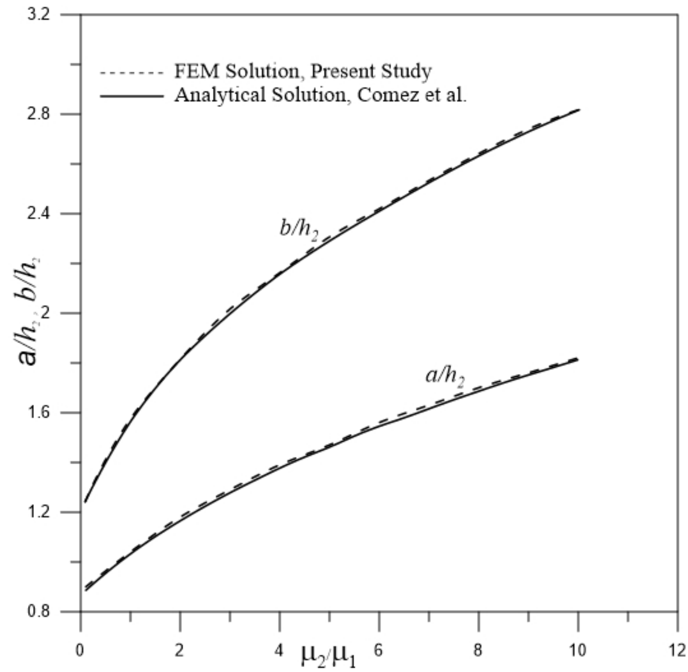

Figure 12. Comparisons of analytical solution with FEM solution for variations of the contact areas (a/h_2) and (b/h_2) with μ_2/μ_1 for circular rigid stamp $(h_1/h_2 = 2, \kappa_1 = \kappa_2 = 2, R/h_2 = 1000, \mu_2/(P/h_2) = 500)$

Table 1 shows comparing the contact areas between two elastic layers (b/h_2) and between circular rigid stamp and elastic layers (a/h_2) with different (μ_2/μ_1) for fixed values $(h_1/h_2 = 2, R/h_2 = 1000, \mu_2/(P/h_2) = 500, \kappa_1 = \kappa_2 = 2)$. For constant $(h_1/h_2 = 2, \mu_2/\mu_1 = 1, \kappa_1 = \kappa_2 = 2)$, comparing the contact area between circular rigid stamp and elastic layers (a/h_2) with various values $\mu_2/(P/h_2)$ and (R/h_2) is shown in Table 2. Table 3 shows comparing the contact area between two elastic layers (b/h_2) with different $\mu_2/(P/h_2)$ and (R/h_2) for fixed values $(h_1/h_2 = 2, \mu_2/\mu_1 = 1, \kappa_1 = \kappa_2 = 2)$. It is seen from all tables that contact areas $(a/h_2, b/h_2)$ obtained from finite element results present work are close analytical results Comez et al. (2004). The differences between them are less than % 3.04.

Table 1. Comparisons of analytical solution with FEM solution for variations of the contact areas (a/h_2) and (b/h_2) with μ_2/μ_1 for circular rigid stamp $(h_1/h_2 = 2, \kappa_1 = \kappa_2 = 2, R/h_2 = 1000, \mu_2/(P/h_2) = 500)$

PARAMETER	$\mu_2/\mu_1 = 1$		$\mu_2/\mu_1 = 2$		$\mu_2/\mu_1 = 4$		$\mu_2/\mu_1 = 8$	
	a/h_2	b/h_2	a/h_2	b/h_2	a/h_2	b/h_2	a/h_2	b/h_2
Comez et al.	1.0333	1.5666	1.1655	1.8097	1.3788	2.1553	1.6860	2.6331
Present work	1.05	1.55	1.15	1.75	1.35	2.15	1.7	2.60
Error (%)	1.62	1.06	1.33	3.30	2.09	0.25	0.83	1.26

Table 2. Comparisons of analytical solution with FEM solution for the contact area between circular rigid stamp and elastic layers (a/h_2) $(h_1/h_2 = 2, \mu_2/\mu_1 = 1, \kappa_1 = \kappa_2 = 2)$

“	$\frac{\mu_2}{P/h_2} = 250$			$\frac{\mu_2}{P/h_2} = 500$			$\frac{\mu_2}{P/h_2} = 1000$		
$\frac{R}{h_2} \rightarrow$	100	500	1000	100	500	1000	100	500	1000
Comez et al.	0.4437	0.3154	0.2184	1.0260	0.7244	0.5061	1.4350	1.0295	0.7244
Present work	1.45	0.325	2.225	1.025	0.725	0.50	1.425	1.025	0.725
Error (%)	1.42	3.04	3.02	0.10	0.08	1.21	0.7	0.44	0.08

Table 3. Comparisons of analytical solution with FEM solution for the contact area between two elastic layers (b/h_2) $(h_1/h_2 = 2, \mu_2/\mu_1 = 1, \kappa_1 = \kappa_2 = 2)$

PARAMETER	$\frac{\mu_2}{P/h_2} = 250$			$\frac{\mu_2}{P/h_2} = 500$			$\frac{\mu_2}{P/h_2} = 1000$		
$\frac{R}{h_2}$	100	500	1000	100	500	1000	100	500	1000
Comez et al.	1.3035	1.2703	1.2538	1.5641	1.4048	1.3179	1.8393	1.5662	1.4028
Present work	1.300	1.275	1.250	1.55	1.4	1.325	1.85	1.55	1.40
Error (%)	0.27	0.37	0.30	0.90	0.34	0.54	0.58	1.03	0.20

The compare of dimensionless contact pressures and contact areas for analytical and numerical results by means of root mean square error (RMSE) between layer and stamp contact surface and between layers contact surface are given in Tables 4-6. It is seen from Tables 1-6 and all figures that dimensionless contact pressures distributions and contact areas obtained from finite element solution and analytical solution agree well.

Table 4. RMSE for the dimensionless contact areas

FIGURES	Fig. 2	Fig. 3
GRAPH (1)	$1.71 * 10^{-5}$	$0.40 * 10^{-5}$
GRAPH (2)	$4.47 * 10^{-5}$	$13.58 * 10^{-5}$
GRAPH (3)	$7.68 * 10^{-5}$	$22.65 * 10^{-5}$
GRAPH (4)	$11.38 * 10^{-5}$	$34.63 * 10^{-5}$

Table 5. RMSE for the dimensionless contact stresses

FIGURES	Fig. 4	Fig. 5
GRAPH (1)	$2.66 * 10^{-5}$	$7.15 * 10^{-5}$
GRAPH (2)	$2.06 * 10^{-5}$	$10.51 * 10^{-5}$
GRAPH (3)	$2.82 * 10^{-5}$	$5.85 * 10^{-5}$

Table 6. RMSE for the dimensionless contact areas

FIGURE	Fig. 6
(a/h_2)	$5.08 * 10^{-5}$
(b/h_2)	$7.45 * 10^{-5}$

5. CONCLUSION

The main purpose of this paper is to present a comparative study of the finite element method (FEM) and the analytical method in contact problems. The results have shown that finite element method is in good agreement with the analytical method.

This paper presents a finite element method for calculating contact areas and contact stresses for receding contact problem in a rigid stamp and two elastic layers. The results of the had FEM are included to show that the method is very efficient and accurate for calculating contact areas and contact stresses for receding contact problem. From the FEA analysis is concluded that the exactly results of contact stresses and areas can be theoretically estimated. The engineer should always get in the mind that materials can contain cracks which are very large collections of mechanical stresses. From the obtained results can be seen that the numerical results give very exactly data if their compare with the analytically obtained results. Thus the numerical approach can be used for the solving of crack problem with very well accuracy of results. The numerical results had the deviate from the analytics results in the range very low.

Acknowledgements

Murat YAYLACI was supported by a scholarship from Recep Tayyip Erdoğan University Scientific Research Project (Grant no: 2015.53005.109.03.03) during his visit to the University of Stuttgart, Germany.

REFERENCES

[1] ANSYS (2013), Swanson Analysis Systems Inc., Houston, PA, USA.

[2] Bathe, K. J. and Chaudhary, A. (1985), "A solution method for planar and axisymmetric contact problems", International Journal for Numerical Methods in Engineering, 21, 65-88.

[3] Belgacem, F. B. Hild, P. and Laborde, P. (1998), "The mortar finite element method for contact problems", Math. Comput. 28(4–8), 263–271.

[4] Bhushan, B. 1996, "Contact Mechanics of Rough Surfaces in Tribology: Single Asperity Contact", Appl. Mech. Rev., 49, pp. 275–298.

[5] Birinci, A. Adıyaman, G. Yaylacı, M. and Öner, E. (2015), "Analysis of Continuous and Discontinuous Cases of a Contact Problem Using Analytical Method and FEM", Latin American Journal of Solids And Structures, 12(9), 1771-1789.

[6] Bostan, V. and Han, W. (2006), "A posteriori error analysis for finite element solutions of a frictional contact problem", Comput. Methods Appl. Mech. Eng. 195(9-12), 1252–1274.

[7] Chan, S. K. and Tuba, I. S. (1971), "A finite element method for contact problems of solid bodies-I. Theory and Validation", Int. J. mech. Sci., 13, 615-625.

[8] Chidlow, S.J., Teodorescu, M. (2013), "Two-dimensional contact mechanics problems involving inhomogeneously elastic solids split into three distinct layers", International Journal of Engineering Science, 70, 102–123.

[9] Comez, İ. Birinci, A. and Erdol, R. (2004), "Double receding contact problem for a rigid stamp and two elastic layers", European Journal of Mechanics A/Solids, 23, 301–309.

[10] Francavilla, A. and Zienkiewicz, O.C. (1975), "A note on numerical computation of elastic contact problems", Int. J. Numer. Meth. Eng., 17, 913-924.

[11] Fredriksson, B. (1976), "Finite element solution of surface nonlinearities in structural mechanics with special emphasis to contact and fracture mechanics problems", Comput. Sfruct., 6, 281-290.

[12] Gladwell, G. M. L. (1980), "Contact Problems in the Classical Theory of Elasticity", Sijthoff and Nordhoff.

[13] Guyot, N. Kosior, F. and Maurice, G. (2000), “Coupling of finite elements and boundary elements methods for study of the frictional contact problem”, Comput. Methods Appl. Mech. Eng. 181(1–3), 147–159.
[14] Gun, H. and Gao, X.W. (2014), “Analysis of frictional contact problems for functionally graded materials using BEM”, Eng. Anal. Bound. Elem., 38, 1–7.
[15] Hertz, H. (1881), "Über die Berührung fester elastischer Körper“, Journal für die reine und angewandte Mathematik,92, 156-171.
[16] Hertz, H. (1896), “Miscellaneous Papers on the Contact o1” Elastic Solids. Translation by D. E. Jones. McMillan, London.
[17] Jing, H.S. and Liao, M.L. (1990), “An improved finite element scheme for elastic contact problems with friction”, Comput. Struct., 35(5), 571-578.
[18] Johnson, K. L. (1985), “Contact Mechanics”, Cambridge University Press, Cambridge, U.K.
[19] Klarbring, A. and Orkman, G. (1992), “Solution of large displacement contact problems with friction usingNewton's method for generalized equations”, International Journal.lbr Numerical Methods in Engineering, 34, 249-269.
[20] Laursen, T.A. (2003), “Computational contact and impact mechanics”, Springer, Berlin.
[21] Li, X.Y., Zheng, R.-F. and Chen, W.-Q. (2014), “Fundamental solutions to contact problems of a magneto-electro-elastic half-space indented by a semi-infinite punch”, Int. J. Solids Struct., 51, 164–178.
[22] Long, J.M., Wang, G.F. (2013), “Effects of surface tension on axisymmetric Hertzian contact problem”, Mechanics of Materials, 56: 65–70.
[23] Oden, J. T. and Pires, E. B. (1984), “Algorithms and numerical results for finite element approximations of contact problems with non-classical friction laws”, Compuf. Struct., 19, 137-147.
[24] Okamoto, N. and Nakazawa, M. (1979), “Finite element incremental contact analysis with various frictional conditions”, Int. J. numer. Meth. Engng, 14, 337-357.
[25] Oner, E. and Birinci, A. (2014), “Continuous Contact Problem For Two Elastic Layers Resting On An Elastic Half-Infinite Plane”, Journal Of Mechanics Of Materials And Structures, 9 (1), 105-119.
[26] Oysu, C. (2007), “Finite element and boundary element contact stress analysis with remeshing technique”, Appl. Math. Model., 31, 2744–2753.
[27] Ozsahin, T.S. Kahya, V. Birinci, A. and Cakiroglu, A.O. (2007), “Contact Problem for an Elastic Layered Composite Resting on Rigid Flat Supports”, International Journal of Computational and Mathematical Sciences, 1, Spring, 154-159.
[28] Papadopoulos, P. and Solberg, J. M. (1998), “A Lagrange multiplier method for the finite element solution of frictionless contact problems”, Math. Comput. Model. 28(4–8), 373–384.
[29] Renaud, C. and Feng, Z.Q. (2003), “BEM and FEM analysis of Signorini contact problems with friction”, Computational Mechanics, 31, 390-399.
[30] Schwarzer, N. Djabella, H. Richter, F. and Arnell, R.D. (1995), “Comparison between analytical and FEM calculations for the contact problem of spherical indenters on layered materials”, Thin Solid Films, 270, 279–282.
[31] Solberg, J. M. Jones, R. E. and Papadopoulos, P. (2007), “A family of simple two-pass dual formulations for the finite element solution of contact problems”, Comput. Methods Appl. Mech. Eng. 196 (4–6), 782–802.
[32] Soldatenkov, I.A. (2013), “The periodic contact problem of the plane theory of elasticity. Taking friction, wear and adhesion into account” Pmm-J. Appl. Math. Mec., 77, 245–255.
[33] Wriggers, P. (1995), “Finite Element Algorithms for Contact Problems”, Archives of Computational Methods in Engineering, 2(2), 1-49.

[34] Wriggers, P. and Nackenhorst, U. (2006), “Analysis and simulation of contact problems”, Lecture notes in applied and computational mechanics, Springer, Vol. 27, Berlin.

[35] Yang, Y.Y. (2013), “Solutions of dissimilar material contact problems”, Engineering Fracture Mechanics, 100, 92–107.

[36] Zhang, H.W., Xie, Z.Q., Chen, B.S., Xing, H.L. (2012). “A finite element model for 2D elastic-plastic contact analysis of multiple cosserat materials”, European Journal of Mechanics A/Solids, 31, 139-151.

[37] Zienkiewicz, O.C. (2005), “The Finite Element Method for Solid and Structural Mechanics”, Elsevier.

[38] Zhu, C. (1995) “A finite element-mathematical programming method for elastoplastic contact problems with friction”, Finite Elem. Anal. Des. 20 (4), 273–282.