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A NONLINEAR CONSTITUTIVE THEORY FOR DEVIATORIC CAUCHY STRESS TENSOR FOR INCOMPRESSIBLE VISCOUS FLUIDS

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ABSTRACT

Newton's law of visocosity is a commonly used constitutive theory for deviatoric Cauchy stress tensor. In this constitutive theory originally constructed based on experimental observation, the deviatoric Cauchy stress is proportional to the symmetric part of the velocity gradient tensor. The constant of proportionality is the viscosity of the fluid. For all continuous media if the deforming matter is in thermodynamic equilibrium then all constitutive theories including those considered here must satisfy conservation and balance laws. It is well known that only the second law of thermodynamics provides possible conditions or mechanisms for deriving constitutive theories. The constitutive theory for deviatoric stress tensor used here can be shown to be a simplified form of the constitutive theory derived using conditions resulting from the entropy inequality in conjunction with the theory of generators and invariants that contains up to fifth degree terms in the components of the symmetric part of the velocity gradient tensor. In general the constitutive theory for deviatoric stress tensor is basis (covariant, contravariant, or Jaumann) dependent as it uses convected time derivatives of the Green and Almansi strain tensors of orders higher than one. However, the first convected time derivative of the Green and Almansi strain tensors are in fact symmetric part of the velocity gradient tensor which is basis independent. Thus, if the constitutive theory for deviatoric Cauchy stress tensor is only dependent on the symmetric part of the velocity gradient tensor, then it is basis independent. This is the case for the theory presented in this paper.

In this paper we limit the constitutive theory for deviatoric Cauchy stress tensor to contain only up to quadratic terms in the components of the symmetric part of the velocity gradient tensor. The objective is to study the resulting flow physics due to the constitutive theory for deviatoric Cauchy stress tensor that contains up to quadratic terms in the velocity gradient tensor. Model problems consisting of fully developed flow between parallel plates, square lid-driven cavity, and asymmetric sudden expansion are used to present numerical solutions. Numerical solutions of the model problems are calculated using least squares finite element formulation based on residual functional in which the local approximations are considered in higher order scalar product spaces that permit higher order global differentiability solutions. Nonlinear system of algebraic equations are solved using Newton's linear method with line search.

Keywords: *Nonlinear Constitutive Theory, Viscous Fluid, Eulerian Description, Generators and Invariants, Entropy Inequality, Integrity*

INTRODUCTION

For isotropic, homogeneous incompressible viscous fluids the entropy inequality requires that we decompose Cauchy stress tensor into equilibrium and deviatoric parts in order to be able to establish mechanisms of deriving constitutive theories. The entropy inequality in conjunction with incompressibility condition establishes equilibrium stress as mechanical pressure (Lagrange multiplier). In case of deviatoric Cauchy stress the entropy inequality requires that the trace of the product of deviatoric Cauchy stress tensor ${}_{d}\bar{\boldsymbol{\sigma}}$ and the symmetric part of the velocity gradient tensor $\bar{\boldsymbol{D}}$ resulting in rate of work due to deviatoric stress be positive. In this term defining rate of work due to deviatoric Cauchy stress, the deviatoric Cauchy stress and the symmetric part of the velocity gradient tensor are conjugate, implying that symmetric part of the velocity gadient tensor can be an argument tensor of the deviatoric Cauchy stress tensor.

It is well known [1] that the symmetric part of the velocity gradient tensor is the first convected time derivative $\boldsymbol{\gamma}_{(1)}$ of the Green strain tensor (covariant) as well as the first convected time derivative $\boldsymbol{\gamma}^{(1)}$ of the Almansi strain tensor (contravariant basis). Since the symmetric part of the velocity gradient tensor is basis

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independent, it is straightforward to conclude that the first convected time derivatives of the strain tensor in co- and contravariant bases are basis independent. In the rate of work term symmetric part of the velocity gradient tensor can be replaced with the first convected time derivative of the strain tensor in either co- or contravariant bases (as they are both same as the symmetric part of the velocity gradient tensor). Since the first convected time derivatives in both bases are fundamental kinematic tensors, Surana [1] has shown that we could consider higher order convected time derivatives of strain tensors up to order n in co- and contravariant basis (i.e. $\boldsymbol{\gamma}_{(k)}$; $k=1,2,\dots,n$ and $\boldsymbol{\gamma}^{(k)}$; $k=1,2,\dots,n$) to be conjugate with the deviatoric Cauchy stress tensors ${}_d\bar{\boldsymbol{\sigma}}_{(0)}$ and ${}_d\bar{\boldsymbol{\sigma}}^{(0)}$ in co- and contravariant bases. In this case $\boldsymbol{\gamma}_{(k)}$; $k=1,2,\dots,n$ are argument tensors of ${}_d\bar{\boldsymbol{\sigma}}^{(0)}$ and $\boldsymbol{\gamma}^{(k)}$; $k=1,2,\dots,n$ are argument tensors of ${}_d\bar{\boldsymbol{\sigma}}_{(0)}$. The resulting constitutive theories for ${}_d\bar{\boldsymbol{\sigma}}_{(0)}$ and ${}_d\bar{\boldsymbol{\sigma}}^{(0)}$ using the theory of generators and invariants [1–3] are ordered rate constitutive theories of up to order n and are naturally basis dependent. However, if we only consider first convected time derivative $\boldsymbol{\gamma}_{(1)}=\boldsymbol{\gamma}^{(1)}=\bar{\boldsymbol{D}}$ as argument of deviatoric stress tensor then constitutive theory for the Cauchy stress tensor becomes ${}_d\bar{\boldsymbol{\sigma}}$, i.e. basis independent.

In the present work we consider ${}_d\bar{\boldsymbol{\sigma}}={}_d\bar{\boldsymbol{\sigma}}(\bar{\boldsymbol{D}},\bar{\theta})={}_d\bar{\boldsymbol{\sigma}}(\boldsymbol{\gamma}_{(1)},\bar{\theta})={}_d\bar{\boldsymbol{\sigma}}(\boldsymbol{\gamma}^{(1)},\bar{\theta})$ in which $\bar{\theta}$ is the temperature. The constitutive theory for ${}_d\bar{\boldsymbol{\sigma}}$ is derived using theory of generators and invariants. The resulting constitutive theory contains up to fifth degree terms in the components of $\bar{\boldsymbol{D}}$. This theory is based on integrity, hence complete, but unfortunately requires too many material coefficients. In the present work this constitutive theory is simplified to contain only up to quadratic terms in the components of $\bar{\boldsymbol{D}}$. The resulting theory requires only two additional material coefficients. Solutions are presented using this constitutive theory for model problems consisting of fully developed flow between parallel plates, a square lid driven cavity, and an asymmetric 3:2 expansion. A theoretical solution is presented for fully developed flow between parallel plates whereas for the other two model problems numerical solutions are obtained using least squares finite element method based on residual functional. The local approximations are considered in higher order scalar product spaces that permit higher order global differentiability solutions, and permit all integrals over discretization to be Riemann with appropriate choice of the order of approximation space. In these studies the focus is to illustrate the influence of the nonlinear terms in $\bar{\boldsymbol{D}}$ in the constitutive theory for ${}_d\bar{\boldsymbol{\sigma}}$ on the flow physics when compared with Newton's law of viscosity as constitutive theory for ${}_d\bar{\boldsymbol{\sigma}}$ in which ${}_d\bar{\boldsymbol{\sigma}}$ is a linear function of $\bar{\boldsymbol{D}}$.

CONSTITUTIVE THEORY FOR DEVIATORIC STRESS TENSOR

The mathematical models, i.e. conservation and balance laws, for fluids are believed to be Eulerian description [1]. Following the notation in [1], $\boldsymbol{x}$ and $\bar{\boldsymbol{x}}$ are undeformed and deformed coordinates, $Q=Q(\boldsymbol{x},t)$ and $\bar{Q}=\bar{Q}(\bar{\boldsymbol{x}},t)$ are Lagrangian and Eulerian descriptions of a quantity Q in the current configuration. An overbar on a quantity implies Eulerian description except that $\bar{\boldsymbol{x}}$, $\bar{\boldsymbol{A}}$, and $\bar{V}$ are deformed coordinates, areas, and volume. We follow this notation in presenting details of the constitutive theory for ${}_d\bar{\boldsymbol{\sigma}}$ and the complete mathematical models for the model problems. In view of the material presented in the introduction, we can begin with

$$ {}_d\bar{\boldsymbol{\sigma}}={}_d\bar{\boldsymbol{\sigma}}(\bar{\boldsymbol{D}},\bar{\theta}) \tag{1}$$

$$ \text{and} \qquad {}_d\bar{\boldsymbol{\sigma}}(0,\bar{\theta})=0 \tag{2}$$

${}_d\bar{\boldsymbol{\sigma}}$ and $\bar{\boldsymbol{D}}$ are symmetric tensors of rank two and $\bar{\theta}$ is a tensor of rank zero. We derive constitutive theory for ${}_d\bar{\boldsymbol{\sigma}}$ in which ${}_d\bar{\boldsymbol{\sigma}}$ is expressed as a function of $\bar{\boldsymbol{D}}$ and $\bar{\theta}$ using theory of generators and invariants [1,2,4–20]. In this approach we determine combined generators of the argument tensors of ${}_d\bar{\boldsymbol{\sigma}}$ that are symmetric tensors of rank two. These are [1,2,4–20]

$$ {}^{\sigma}\underset{\sim}{\boldsymbol{G}}^0=\boldsymbol{I}\,; \quad {}^{\sigma}\underset{\sim}{\boldsymbol{G}}^1=\bar{\boldsymbol{D}}\,; \quad {}^{\sigma}\underset{\sim}{\boldsymbol{G}}^2=\bar{\boldsymbol{D}}^2 \tag{3}$$

The combined invariants of the same argument tensors are

$$
\begin{aligned}
{}^{\sigma}\underset{\sim}{I}^{1} &= i_{\bar{D}} = \mathrm{tr}(\bar{\boldsymbol{D}}) \\
{}^{\sigma}\underset{\sim}{I}^{2} &= ii_{\bar{D}} = \mathrm{tr}(\bar{\boldsymbol{D}}^2) \\
{}^{\sigma}\underset{\sim}{I}^{3} &= iii_{\bar{D}} = \mathrm{tr}(\bar{\boldsymbol{D}}^3)
\end{aligned}
\tag{4}
$$

We could also have chosen principal invariants I_σ, II_σ, III_σ based on characteristic equation of $\bar{\boldsymbol{D}}$. Since the two sets of invariants are related, the resulting constitutive theory is not affected. The generators in (3) form integrity or basis of the space of ${}_d\bar{\boldsymbol{\sigma}}$, so ${}_d\bar{\boldsymbol{\sigma}}$ can be represented as a linear combination of the generators in (3).

$$
{}_d\bar{\boldsymbol{\sigma}} = \sum_{i=0}^{2} {}^{\sigma}\underset{\sim}{\alpha}^{i}\, {}^{\sigma}\underset{\sim}{\boldsymbol{G}}^{i} = {}^{\sigma}\underset{\sim}{\alpha}^{0}\boldsymbol{I} + {}^{\sigma}\underset{\sim}{\alpha}^{1}\bar{\boldsymbol{D}} + {}^{\sigma}\underset{\sim}{\alpha}^{2}\bar{\boldsymbol{D}}^2 \tag{5}
$$

$$
\text{in which} \qquad {}^{\sigma}\underset{\sim}{\alpha}^{i} = {}^{\sigma}\underset{\sim}{\alpha}^{i}(i_{\bar{D}}, ii_{\bar{D}}, iii_{\bar{D}}, \bar{\theta})\,; \quad i = 0,1,2 \tag{6}
$$

We expand ${}^{\sigma}\underset{\sim}{\alpha}^{i}$; $i = 0,1,2$ in Taylor series in $i_{\bar{D}}$, $ii_{\bar{D}}$, $iii_{\bar{D}}$, and $\bar{\theta}$ about a known configuration $\underline{\Omega}$ and retain only up to linear terms (for simplicity of the resulting constitutive theory) in the invariants $i_{\bar{D}}$, $ii_{\bar{D}}$, $iii_{\bar{D}}$, and temperature $\bar{\theta}$ and we substitute these back in (5) and rearrange the terms such that the quantities in the known configuration become material coefficients. In doing so we obtain the most general constitutive theory based on (1), given in the following [1].

$$
\begin{aligned}
{}_d\bar{\boldsymbol{\sigma}} = {} & \bar{\sigma}_0|_{\underline{\Omega}}\boldsymbol{I} + {}^{\sigma}b_1 i_{\bar{D}}\boldsymbol{I} + {}^{\sigma}b_2 ii_{\bar{D}}\boldsymbol{I} + {}^{\sigma}b_3 iii_{\bar{D}}\boldsymbol{I} \\
& + {}^{\sigma}b_1^1\bar{\boldsymbol{D}} + {}^{\sigma}b_2^1 i_{\bar{D}}\bar{\boldsymbol{D}} + {}^{\sigma}b_3^1 ii_{\bar{D}}\bar{\boldsymbol{D}} + {}^{\sigma}b_4^1 iii_{\bar{D}}\bar{\boldsymbol{D}} \\
& + {}^{\sigma}b_1^2\bar{\boldsymbol{D}}^2 + {}^{\sigma}b_2^2 i_{\bar{D}}\bar{\boldsymbol{D}}^2 + {}^{\sigma}b_3^2 ii_{\bar{D}}\bar{\boldsymbol{D}}^2 + {}^{\sigma}b_4^2 iii_{\bar{D}}\bar{\boldsymbol{D}}^2 \\
& + {}^{\sigma}b_1^3(\bar{\theta} - \bar{\theta}_{\underline{\Omega}})\boldsymbol{I} + {}^{\sigma}b_2^3(\bar{\theta} - \bar{\theta}_{\underline{\Omega}})\bar{\boldsymbol{D}} + {}^{\sigma}b_3^3(\bar{\theta} - \bar{\theta}_{\underline{\Omega}})\bar{\boldsymbol{D}}^2
\end{aligned}
\tag{7}
$$

Using

$$
\frac{\partial\, {}^{\sigma}\underset{\sim}{\alpha}^{i}}{\partial\, {}^{\sigma}\underset{\sim}{I}^{j}} = {}^{\sigma}\underset{\sim}{\alpha}^{i}_{,j}\,; \quad i = 0,1,2 \tag{8}
$$

$$
\begin{aligned}
&\bar{\sigma}_0\big|_{\underline{\Omega}} = {}^{\sigma}\alpha^0\big|_{\underline{\Omega}} - ({}^{\sigma}\alpha^0{}_{,1})_{\underline{\Omega}}(i_{\bar{D}})_{\underline{\Omega}} - ({}^{\sigma}\alpha^0{}_{,2})_{\underline{\Omega}}(ii_{\bar{D}})_{\underline{\Omega}} - ({}^{\sigma}\alpha^0{}_{,3})_{\underline{\Omega}}(iii_{\bar{D}})_{\underline{\Omega}} \\
&{}^{\sigma}b_1 = ({}^{\sigma}\alpha^0{}_{,1})_{\underline{\Omega}} \quad ; \quad {}^{\sigma}b_2 = ({}^{\sigma}\alpha^0{}_{,2})_{\underline{\Omega}} \quad ; \quad {}^{\sigma}b_3 = ({}^{\sigma}\alpha^0{}_{,3})_{\underline{\Omega}} \\
&{}^{\sigma}b_1^1 = {}^{\sigma}\alpha^1\big|_{\underline{\Omega}} - ({}^{\sigma}\alpha^1{}_{,1})_{\underline{\Omega}}(i_{\bar{D}})_{\underline{\Omega}} - ({}^{\sigma}\alpha^1{}_{,2})_{\underline{\Omega}}(ii_{\bar{D}})_{\underline{\Omega}} - ({}^{\sigma}\alpha^1{}_{,3})_{\underline{\Omega}}(iii_{\bar{D}})_{\underline{\Omega}} \\
&{}^{\sigma}b_2^1 = ({}^{\sigma}\alpha^1{}_{,1})_{\underline{\Omega}} \quad ; \quad {}^{\sigma}b_3^1 = ({}^{\sigma}\alpha^1{}_{,2})_{\underline{\Omega}} \quad ; \quad {}^{\sigma}b_4^1 = ({}^{\sigma}\alpha^1{}_{,3})_{\underline{\Omega}} \\
&{}^{\sigma}b_1^2 = {}^{\sigma}\alpha^2\big|_{\underline{\Omega}} - ({}^{\sigma}\alpha^2{}_{,1})_{\underline{\Omega}}(i_{\bar{D}})_{\underline{\Omega}} - ({}^{\sigma}\alpha^2{}_{,2})_{\underline{\Omega}}(ii_{\bar{D}})_{\underline{\Omega}} - ({}^{\sigma}\alpha^2{}_{,3})_{\underline{\Omega}}(iii_{\bar{D}})_{\underline{\Omega}} \\
&{}^{\sigma}b_2^2 = ({}^{\sigma}\alpha^2{}_{,1})_{\underline{\Omega}} \quad ; \quad {}^{\sigma}b_3^2 = ({}^{\sigma}\alpha^2{}_{,2})_{\underline{\Omega}} \quad ; \quad {}^{\sigma}b_4^2 = ({}^{\sigma}\alpha^2{}_{,3})_{\underline{\Omega}} \\
&{}^{\sigma}b_1^3 = \frac{\partial({}^{\sigma}\alpha^0)}{\partial\bar{\theta}}\bigg|_{\underline{\Omega}} \quad ; \quad {}^{\sigma}b_2^3 = \frac{\partial({}^{\sigma}\alpha^1)}{\partial\bar{\theta}}\bigg|_{\underline{\Omega}} \quad ; \quad {}^{\sigma}b_3^3 = \frac{\partial({}^{\sigma}\alpha^2)}{\partial\bar{\theta}}\bigg|_{\underline{\Omega}}
\end{aligned}
\tag{9}
$$

Using (7) we can derive a constitutive theory that is quadratic in $\bar{\boldsymbol{D}}$ (neglecting $(\bar{\theta} - \bar{\theta}_{\underline{\Omega}})$ terms and noting that $\mathrm{tr}(\bar{\boldsymbol{D}}) = 0$ for incompressible fluids).

$$
{}_d\bar{\boldsymbol{\sigma}} = \bar{\sigma}_0|_{\underline{\Omega}}\boldsymbol{I} + {}^{\sigma}b_1^1\bar{\boldsymbol{D}} + {}^{\sigma}b_1^2\bar{\boldsymbol{D}}^2 + {}^{\sigma}b_2\mathrm{tr}(\bar{\boldsymbol{D}}^2)\boldsymbol{I} \tag{10}
$$

If we neglect $\bar{\sigma}_0|_{\underline{\Omega}}\boldsymbol{I}$ and redefine $2\eta = {}^{\sigma}b_1^1$, $\eta_1 = {}^{\sigma}b_1^2$, and $\eta_3 = {}^{\sigma}b_2$, then (10) can be written as

$$
{}_d\bar{\boldsymbol{\sigma}} = 2\eta\bar{\boldsymbol{D}} + \eta_1\bar{\boldsymbol{D}}^2 + \eta_3\mathrm{tr}(\bar{\boldsymbol{D}}^2)\boldsymbol{I} \tag{11}
$$

This constitutive theory given by (11) requires material coefficients η, η_1, and η_3 that must be determined experimentally (calibrating the constitutive theory).

COMPLETE MATHEMATICAL MODEL AND ITS DIMENSIONLESS FORM

In the following we present complete mathematical model for homogeneous, isotropic, incompressible viscous fluid under isothermal conditions. In the absence of sources and sinks: conservation of mass, balance of linear momenta, and the nonlinear constitutive theory for deviatoric stress tensor can be written as

$$\hat{\bar{\rho}}(\hat{\bar{\boldsymbol{\nabla}}}\cdot\hat{\bar{\boldsymbol{v}}}) = 0 \tag{12}$$

$$\hat{\bar{\rho}}\frac{D\hat{\bar{\boldsymbol{v}}}}{D\hat{t}} + \hat{\bar{\boldsymbol{\nabla}}}\hat{\bar{p}} - ({}_d\hat{\bar{\boldsymbol{\sigma}}})^T\cdot\hat{\bar{\boldsymbol{\nabla}}} = 0 \tag{13}$$

$${}_d\hat{\bar{\boldsymbol{\sigma}}} = 2\hat{\eta}\hat{\bar{\boldsymbol{D}}} + \hat{\eta}_1\hat{\bar{\boldsymbol{D}}}^2 + \hat{\eta}_3\mathrm{tr}(\hat{\bar{\boldsymbol{D}}}^2) \tag{14}$$

Compressive pressure $\hat{\bar{p}}$ is assumed positive. In (12) – (14) all quantities have overbar as it is Eulerian description. Hat ($\hat{}$) indicates that the quantitites have their usual dimensions. We choose the following reference quantities (with subscript zero) and dimensionless variables (without hat).

$$\begin{gathered}\bar{x}_i = \frac{\hat{\bar{x}}_i}{L_0}\ ; \quad \bar{v}_i = \frac{\hat{\bar{v}}_i}{v_0}\ ; \quad \bar{\rho} = \frac{\hat{\bar{\rho}}}{\rho_0} \\ {}_d\bar{\boldsymbol{\sigma}} = \frac{{}_d\hat{\bar{\boldsymbol{\sigma}}}}{\tau_0}\ ; \quad \bar{p} = \frac{\hat{\bar{p}}}{p_0}\ ; \quad \eta = \frac{\hat{\eta}}{\eta_0} \\ \underset{\sim}{\eta}_1 = \frac{\hat{\eta}_1}{\eta_0}\ ; \quad \underset{\sim}{\eta}_3 = \frac{\hat{\eta}_3}{\eta_0}\ ; \quad t_0 = \frac{L_0}{v_0}\end{gathered} \tag{15}$$

Using (15) in (12) – (14) we can obtain the following dimensionless form of the mathematical model.

$$\bar{\rho}(\bar{\boldsymbol{\nabla}}\cdot\bar{\boldsymbol{v}}) = 0 \tag{16}$$

$$\bar{\rho}\frac{D\bar{\boldsymbol{v}}}{Dt} + \left(\frac{p_0}{\rho_0 v_0^2}\right)\bar{\boldsymbol{\nabla}}\bar{p} - \left(\frac{\tau_0}{\rho_0 v_0^2}\right)({}_d\bar{\boldsymbol{\sigma}})^T\cdot\bar{\boldsymbol{\nabla}} = 0 \tag{17}$$

$${}_d\bar{\boldsymbol{\sigma}} = \left(\frac{\eta_0 v_0}{\tau_0 L_0}\right)2\eta\bar{\boldsymbol{D}} + \left(\frac{\eta_0 v_0^2}{\tau_0 L_0^2}\right)\underset{\sim}{\eta}_1\bar{\boldsymbol{D}}^2 + \left(\frac{\eta_0 v_0^2}{\tau_0 L_0^2}\right)\underset{\sim}{\eta}_3\mathrm{tr}(\bar{\boldsymbol{D}}^2) \tag{18}$$

Using

$$Re = \frac{\rho_0 v_0 L_0}{\eta_0}\ ; \quad \eta_{10} = \underset{\sim}{\eta}_1\left(\frac{v_0}{L_0}\right)\ ; \quad \eta_{30} = \underset{\sim}{\eta}_3\left(\frac{v_0}{L_0}\right) \tag{19}$$

$$p_0 = \tau_0 = \rho_0 v_0^2 \qquad \text{(characteristic kinetic energy)} \tag{20}$$

We can write (16) – (18) as follows.

$$\bar{\rho}(\bar{\boldsymbol{\nabla}}\cdot\bar{\boldsymbol{v}}) = 0 \tag{21}$$

$$\bar{\rho}\frac{D\bar{\boldsymbol{v}}}{Dt} + \bar{\boldsymbol{\nabla}}\bar{p} - ({}_d\bar{\boldsymbol{\sigma}})^T\cdot\bar{\boldsymbol{\nabla}} = 0 \tag{22}$$

$${}_d\bar{\boldsymbol{\sigma}} = \frac{2\eta}{Re}\bar{\boldsymbol{D}} + \frac{\eta_{10}}{Re}\bar{\boldsymbol{D}}^2 + \frac{\eta_{30}}{Re}\mathrm{tr}(\bar{\boldsymbol{D}}^2) \tag{23}$$

Equations (21) – (23) constitute the dimensionless form of the mathematical model in $\mathbb{R}^3$.

It is perhaps more meaningful to decompose ${}_d\bar{\boldsymbol{\sigma}}$ into linear and nonlinear parts, linear part being standard deviatoric Cauchy stress tensor due to Newton's law of viscosity and the nonlinear part is due to $\bar{\boldsymbol{D}}^2$ and $\mathrm{tr}(\bar{\boldsymbol{D}}^2)\boldsymbol{I}$ in the constitutive theory for ${}_d\bar{\boldsymbol{\sigma}}$.

$$_d\bar{\boldsymbol{\sigma}} = (_d\bar{\boldsymbol{\sigma}})_l + (_d\bar{\boldsymbol{\sigma}})_{nl} \tag{24}$$

$$(_d\bar{\boldsymbol{\sigma}})_l = \frac{2\eta}{Re}\bar{\boldsymbol{D}} \tag{25}$$

$$(_d\bar{\boldsymbol{\sigma}})_{nl} = \frac{\eta_{10}}{Re}\bar{\boldsymbol{D}}^2 + \frac{\eta_{30}}{Re}\mathrm{tr}(\bar{\boldsymbol{D}}^2)\boldsymbol{I} \tag{26}$$

It is obvious that $(_d\bar{\sigma}_{xx})_{nl} = (_d\bar{\sigma}_{yy})_{nl} = (_d\bar{\sigma}_{zz})_{nl}$ due to the second term on the right side of (26), however this is not so obvious due to first term on right side of (26) as in order to show this (if possible) we need to make use of continuity equation (21). Instead of showing this in $\mathbb{R}^3$, in section 5 we show that $(_d\bar{\sigma}_{xx})_{nl} = (_d\bar{\sigma}_{yy})_{nl}$ holds in $\mathbb{R}^2$, i.e. in 2D flows.

DIMENSIONLESS MATHEMATICAL MODEL IN $\mathbb{R}^2$

Using (21) – (23) and defining $\eta_{10} = \eta_1$ and $\eta_{30} = \eta_3$ we can obtain the explicit form of the mathematical model in $\mathbb{R}^2$.

$$\bar{\rho}\left(\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}}\right) = 0 \tag{27}$$

$$\bar{\rho}\left(\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}}\right) + \frac{\partial \bar{p}}{\partial \bar{x}} - \left(\frac{\partial _d\bar{\sigma}_{xx}}{\partial \bar{x}} + \frac{\partial _d\bar{\sigma}_{xy}}{\partial \bar{y}}\right) = 0 \tag{28}$$

$$\bar{\rho}\left(\bar{u}\frac{\partial \bar{v}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{v}}{\partial \bar{y}}\right) + \frac{\partial \bar{p}}{\partial \bar{y}} - \left(\frac{\partial _d\bar{\sigma}_{xy}}{\partial \bar{x}} + \frac{\partial _d\bar{\sigma}_{yy}}{\partial \bar{y}}\right) = 0 \tag{29}$$

$$_d\bar{\sigma}_{xx} = \frac{1}{Re}\left[2\eta\frac{\partial \bar{u}}{\partial \bar{x}} + \eta_1\left(\left(\frac{\partial \bar{u}}{\partial \bar{x}}\right)^2 + \frac{1}{4}\left(\frac{\partial \bar{u}}{\partial \bar{y}} + \frac{\partial \bar{v}}{\partial \bar{x}}\right)^2\right) + \eta_3\left(\left(\frac{\partial \bar{u}}{\partial \bar{x}}\right)^2 + \left(\frac{\partial \bar{v}}{\partial \bar{y}}\right)^2 + \frac{1}{2}\left(\frac{\partial \bar{u}}{\partial \bar{y}} + \frac{\partial \bar{v}}{\partial \bar{x}}\right)^2\right)\right] \tag{30}$$

$$_d\bar{\sigma}_{xx} = \frac{1}{Re}\left[2\eta\frac{\partial \bar{v}}{\partial \bar{y}} + \eta_1\left(\left(\frac{\partial \bar{v}}{\partial \bar{y}}\right)^2 + \frac{1}{4}\left(\frac{\partial \bar{u}}{\partial \bar{y}} + \frac{\partial \bar{v}}{\partial \bar{x}}\right)^2\right) + \eta_3\left(\left(\frac{\partial \bar{u}}{\partial \bar{x}}\right)^2 + \left(\frac{\partial \bar{v}}{\partial \bar{y}}\right)^2 + \frac{1}{2}\left(\frac{\partial \bar{u}}{\partial \bar{y}} + \frac{\partial \bar{v}}{\partial \bar{x}}\right)^2\right)\right] \tag{31}$$

$$_d\bar{\sigma}_{xy} = \frac{\eta}{Re}\left(\frac{\partial \bar{v}}{\partial \bar{x}} + \frac{\partial \bar{u}}{\partial \bar{y}}\right) \tag{32}$$

Equations (27) – (32) are a system of six nonlinear partial differential equations in six dependent variables: $\bar{u}$, $\bar{v}$, $\bar{p}$, $_d\bar{\sigma}_{xx}$, $_d\bar{\sigma}_{yy}$, and $_d\bar{\sigma}_{xy}$. Using (30) and (31) and the stress decomposition shown in section 4, we can write

$$\begin{aligned}
(_d\bar{\sigma}_{xx})_l &= \frac{2\eta}{Re}\frac{\partial \bar{u}}{\partial \bar{x}} \\
(_d\bar{\sigma}_{yy})_l &= \frac{2\eta}{Re}\frac{\partial \bar{v}}{\partial \bar{y}} \\
(_d\bar{\sigma}_{xy})_l &= \frac{\eta}{Re}\left(\frac{\partial \bar{v}}{\partial \bar{x}} + \frac{\partial \bar{u}}{\partial \bar{y}}\right)
\end{aligned} \tag{33}$$

and

$$(_d\bar{\sigma}_{xx})_{nl} = \frac{\eta_1}{Re}\left(\left(\frac{\partial \bar{u}}{\partial \bar{x}}\right)^2 + \frac{1}{4}\left(\frac{\partial \bar{u}}{\partial \bar{y}} + \frac{\partial \bar{v}}{\partial \bar{x}}\right)^2\right) + \frac{\eta_3}{Re}\left(\left(\frac{\partial \bar{u}}{\partial \bar{x}}\right)^2 + \left(\frac{\partial \bar{v}}{\partial \bar{y}}\right)^2 + \frac{1}{2}\left(\frac{\partial \bar{u}}{\partial \bar{y}} + \frac{\partial \bar{v}}{\partial \bar{x}}\right)^2\right) \tag{34}$$

$$({}_d\bar{\sigma}_{yy})_{nl} = \frac{\eta_1}{Re}\left(\left(\frac{\partial \bar{v}}{\partial \bar{y}}\right)^2 + \frac{1}{4}\left(\frac{\partial \bar{u}}{\partial \bar{y}} + \frac{\partial \bar{v}}{\partial \bar{x}}\right)^2\right) + \frac{\eta_3}{Re}\left(\left(\frac{\partial \bar{u}}{\partial \bar{x}}\right)^2 + \left(\frac{\partial \bar{v}}{\partial \bar{y}}\right)^2 + \frac{1}{2}\left(\frac{\partial \bar{u}}{\partial \bar{y}} + \frac{\partial \bar{v}}{\partial \bar{x}}\right)^2\right) \tag{35}$$

$$({}_d\bar{\sigma}_{xy})_{nl} = 0 \tag{36}$$

If we use $\frac{\partial \bar{u}}{\partial \bar{x}} = -\frac{\partial \bar{v}}{\partial \bar{y}}$ (continuity) in (34) or (35) we find that the following holds.

$$({}_d\bar{\sigma}_{xx})_{nl} = ({}_d\bar{\sigma}_{yy})_{nl} \tag{37}$$

Remarks.

(1) Since $({}_d\bar{\sigma}_{xx})_{nl} = ({}_d\bar{\sigma}_{yy})_{nl}$, the nonlinear normal stresses are like a pressure field, their presence will undoubtedly influence the pressure values.

(2) From the momentum equations we note that the velocity field is influenced by the pressure gradients and not values of the pressure. The equal magnitude additional normal stresses result when the pressure values are shifted but the pressure gradients remain unaffected. As a consequence, the velocity field remains unaltered.

(3) In model problem studies, remark (2) is an extremely important aspect that the calculated solutions must exhibit. That is, $\frac{\partial \bar{p}}{\partial \bar{x}}$ and $\frac{\partial \bar{p}}{\partial \bar{y}}$ must remain same due to nonlinear constitutive theory as in case of linear theory and as a consequence the velocity field must remain invariant of the coefficients η_1 and η_3.

DIMENSIONLESS MODEL FOR FULLY DEVELOPED FLOW BETWEEN PARALLEL PLATES

If $\bar{x}$ is the direction of the flow, then in the case of fully developed flow between parallel plates, the flow is independent of the $\bar{x}$ coordinate, hence gradients of all dependent variables in the $\bar{x}$-direction and the velocity $\bar{v}$ in the $\bar{y}$-direction are zero. Using these conditions and the mathematical model in $\mathbb{R}^2$ (equations (27) – (32)) we can derive the following mathematical model for fully developed flow between parallel plates. We redifine $\frac{\eta_1}{4} = \eta_1$ and $\frac{\eta_3}{2} = \eta_3$.

$$\frac{\partial \bar{p}}{\partial \bar{x}} - \frac{\partial\, {}_d\bar{\sigma}_{xy}}{\partial \bar{y}} = 0 \tag{38}$$

$$\frac{\partial \bar{p}}{\partial \bar{y}} - \frac{\partial\, {}_d\bar{\sigma}_{yy}}{\partial \bar{y}} = 0 \tag{39}$$

$${}_d\bar{\sigma}_{xx} = \frac{\eta_1+\eta_3}{Re}\left(\frac{\partial \bar{u}}{\partial \bar{y}}\right)^2 = ({}_d\bar{\sigma}_{xx})_{nl} \tag{40}$$

$${}_d\bar{\sigma}_{yy} = \frac{\eta_1+\eta_3}{Re}\left(\frac{\partial \bar{u}}{\partial \bar{y}}\right)^2 = ({}_d\bar{\sigma}_{yy})_{nl} \tag{41}$$

$${}_d\bar{\sigma}_{xy} = \frac{\eta}{Re}\frac{\partial \bar{u}}{\partial \bar{y}} \tag{42}$$

In this mathematical model ${}_d\bar{\sigma}_{xx} = {}_d\bar{\sigma}_{yy}$, and these are purely due to nonlinear constitutive theory. A theoretical solution of this mathematical model can be obtained for a fixed pressure gradient $\frac{\partial \bar{p}}{\partial \bar{x}}$. Let the distance between the plates be $2\bar{H}$ and the origin of the coordinate system $\bar{x}\bar{y}$ be at the center between the plates with positive $\bar{x}$ pointing to the right. From (38) we obtain the following (after integrating with respect to $\bar{y}$ and using boundary condition ${}_d\bar{\sigma}_{xy} = 0$ at $\bar{y} = 0$).

$${}_d\bar{\sigma}_{xy} = \frac{\partial \bar{p}}{\partial \bar{x}}\bar{y} \tag{43}$$

Using (43) in (42) and integrating with respect to $\bar{y}$ (since $\frac{\partial \bar{p}}{\partial \bar{x}}$ is constant) and using $\bar{u} = 0$ at $\bar{y} = \bar{H}$, we obtain

$$\bar{u} = \frac{Re}{2\eta}\frac{\partial \bar{p}}{\partial \bar{x}}(\bar{y}^2 - \bar{H}^2) \tag{44}$$

From (44) we can determine $\frac{\partial \bar{u}}{\partial \bar{y}} = \frac{Re}{\eta}\frac{\partial \bar{p}}{\partial \bar{x}}\bar{y}$, and then using (40), (41), and (42), ${}_d\bar{\sigma}_{xx}$, ${}_d\bar{\sigma}_{yy}$, and ${}_d\bar{\sigma}_{xy}$ can be determined.

NONLINEAR MODEL BOUNDARY VALUE PROBLEMS AND THEIR SOLUTIONS

We consider fully developed flow between parallel plates, a square lid driven cavity, and asymmetric sudden expansion as model boundary value problems. In each model problem the objective is to study and compare the solutions obtained using nonlinear constitutive theory for ${}_d\bar{\boldsymbol{\sigma}}$ with those from linear constitutive theory (Newton's law of viscosity).

The finite element formulations of nonlinear partial differential equations in the mathematical models are constructed using least squares finite element method based on residual functional. We illustrate the details using a single ordinary differential equation (ODE), the boundary value problem

$$A\phi - f = 0 \quad \forall x \in \Omega \subset \mathbb{R}^1 \tag{45}$$

Let $\bar{\Omega}^T = \underset{e}{\cup}\bar{\Omega}^e$ be the discretization of $\bar{\Omega} = \Omega \cup \Gamma$ in which $\bar{\Omega}^e$ is an element e and Γ is boundary of Ω (two endpoints of $\bar{\Omega}$ in this case). Let ϕ_h be approximation of ϕ over $\bar{\Omega}^T$ and ϕ_h^e be the approximation of ϕ over $\bar{\Omega}^e$. Then

$$E = A\phi_h - f \quad \forall x \in \bar{\Omega} \tag{46}$$

is the residual function over $\bar{\Omega}$ and the residual functional $I(\phi_h)$ can be defined using

$$I(\phi_h) = \int_{\bar{\Omega}^T} (E)^2 d\Omega = (E,E)_{\bar{\Omega}^T} \tag{47}$$

If

$$I^e(\phi_h^e) = (E^e, E^e)_{\bar{\Omega}^e} \tag{48}$$

in which

$$E^e = A\phi_h^e - f \tag{49}$$

then

$$I(\phi_h) = \sum_e I^e(\phi_h^e) = \sum_e (E^e, E^e)_{\bar{\Omega}^e} \tag{50}$$

$I(\phi_h)$ is the desired residual functional over $\bar{\Omega}^T$. If $I(\phi_h)$ is differentiable in its arguments, then the first variation of $I(\phi_h)$, i.e. $\delta I(\phi_h)$, set to zero is necessary condition for an extremum of the functional $I(\phi_h)$. Using (50)

$$\delta I(\phi_h) = \sum_e \delta I^e(\phi_h^e) = 2\sum_e (E^e, \delta E^e)_{\bar{\Omega}^e} = 2\sum_e g^e = 2g = 0 \tag{51}$$

When A is a nonlinear differential operator in the boundary value problem (45), then g is a nonlinear function of ϕ_h, hence we must find a ϕ_h iteratively that would satisfy the condition (51). This is done using Newton's linear method with line search [21]. The final result is that if ϕ_h^0 is an assumed or starting solution then

$$\phi_h = \phi_h^0 + \alpha\Delta\phi_h \tag{52}$$

and

$$\Delta\phi_h = -\frac{1}{2}\left(\delta^2 I(\phi_h)\right)^{-1}_{\phi_h^0}(g)_{\phi_h^0} \tag{53}$$

and

$$\delta^2 I(\phi_h) \simeq 2(\delta E, \delta E) = 2\sum_e (\delta E^3, \delta E^e) > 0 \tag{54}$$

$$0 < \alpha \leq 2 \text{ is such that } I(\phi_h) \leq I(\phi_h^0) \tag{55}$$

$$\max_i |g_i(\phi_h)| \leq \Delta \tag{56}$$

where Δ is a preset tolerance of computed zero (generally $O(10^{-6})$ or lower). When (56) then ϕ_h is the converged solution in the iterative process. If not, then ϕ_h^0 is set to ϕ_h and (52) – (54) are repeated until converged. Since Newton's linear method has quadratic convergence, accuracy of $\Delta = O(10^{-6})$ is generally achievable within 3-5 iterations.

When there are more than one partial differential equations, say 'm' in 'm' dependent variables, in the mathematical model, then we have residual functions $E_i \quad i = 1,2,\dots,m$ corresponding to each equation and

$$I = \sum_{i=1}^{m} (E_i, E_i) = \sum_e I^e = \sum_e \big(\sum_{i=1}^{m} I_i^e \big) \tag{57}$$

in which

$$I_i^e = (E_i^e, E_i^e)_{\bar{\Omega}^e} \ ; \quad i = 1,2,\dots,m \tag{58}$$

Rest of the details follow the procedure presented for one nonlinear differential equation using I and I^e.

Model Problem 1: Fully Developed Flow Between Parallel Plates

Consider parallel plates separated by a distance of $2\hat{\bar{H}}$ meters. Let the origin of $\bar{x}\bar{y}$ coordinate system be at the center of the plates with positive $\bar{x}$ pointing towards the right and positive $\bar{y}$ pointing upward. Let the flow be pressure drive, i.e. $\frac{\partial \bar{p}}{\partial \bar{x}}$ is known. Dimensions, reference quantities, the dimensionless quantities, and input data are shown in the following.

$$\begin{aligned} \hat{\bar{\rho}} = \rho_0 &= 998.2 \text{ kg/m}^3 \ ; \qquad \bar{\rho} = 1 \\ \hat{\bar{\eta}} = \eta_0 &= 0.001002 \text{ Pa-s} \ ; \qquad \eta = 1 = \bar{\eta} \\ \hat{\bar{H}} = L_0 &= 0.015 \text{ m} \ ; \qquad \bar{H} = 1 \\ v_0 &= 0.015325 \end{aligned} \tag{59}$$

Thus, we have $Re = \frac{\rho_0 v_0 L_0}{\eta_0} = 229$.

In section 6, a mathematical model and its theoretical solution have been given. Using (59) in (44), and then using (44) in (42), (40), and (41) we can obtain velocity $\bar{u}$, shear stress ${}_d\bar{\sigma}_{xy}$, and normal stresses ${}_d\bar{\sigma}_{xx}$ and ${}_d\bar{\sigma}_{yy}$ as functions of $\bar{y}$. We consider $(\eta_1 + \eta_3) = 0$, 0.2, 0.24, 0.3, and 0.4 in the calculations of ${}_d\bar{\sigma}_{xx}$ and ${}_d\bar{\sigma}_{yy}$. Figure 1 shows a plot of $\bar{u}$ versus $\bar{y}$ (independent of $(\eta_1 + \eta_3)$). Figure 2 shows a plot of deviatoric shear stress ${}_d\bar{\sigma}_{xy}$ versus $\bar{y}$ that is also independent of $(\eta_1 + \eta_3)$. Plots of deviatoric normal stresses ${}_d\bar{\sigma}_{xx} = {}_d\bar{\sigma}_{yy}$ (same as $({}_d\bar{\sigma}_{xx})_{nl}$ and $({}_d\bar{\sigma}_{yy})_{nl}$) for different values of $(\eta_1 + \eta_3)$ are shown in figure 3. Since $\frac{\partial \bar{u}}{\partial \bar{y}}$ is linear, the stresses ${}_d\bar{\sigma}_{xx}$ and ${}_d\bar{\sigma}_{yy}$ are quadratic in $\bar{y}$. When $(\eta_1 + \eta_3)$ is as low as only 0.2, the maximum value of ${}_d\bar{\sigma}_{xx} = {}_d\bar{\sigma}_{yy} = 0.004$, more than one fourth the maximum value of ${}_d\bar{\sigma}_{xy}$. At $(\eta_1 + \eta_3) = 0.4$, maximum value of ${}_d\bar{\sigma}_{xx} = {}_d\bar{\sigma}_{yy}$ is more than half of ${}_d\bar{\sigma}_{xy}$.

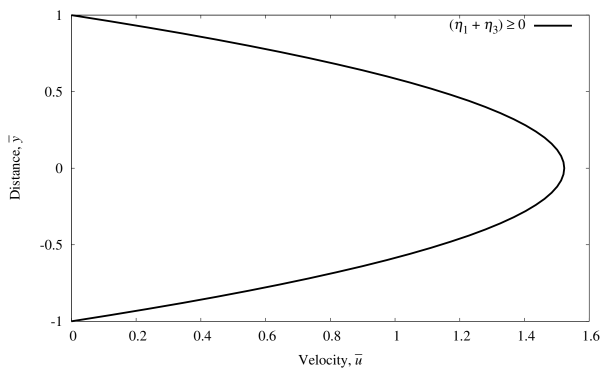


Figure 1: Velocity $\bar{u}$ versus $\bar{y}$: invariant of $(\eta_1 + \eta_3)$

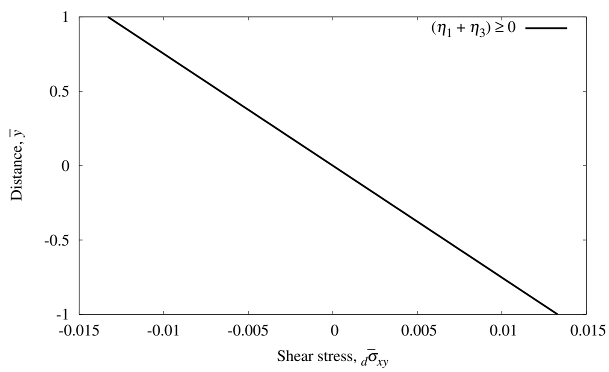


Figure 2: Deviatoric shear stress ${}_d\bar{\sigma}_{xy}$ versus $\bar{y}$: invariant of $(\eta_1 + \eta_3)$

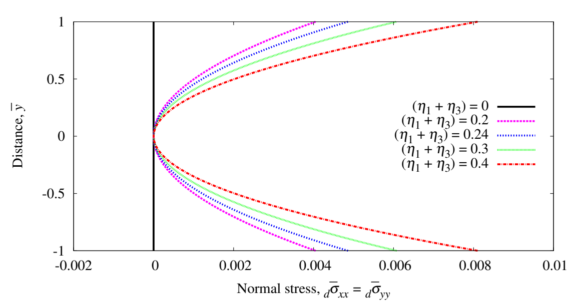


Figure 3: Deviatoric normal stress ${}_d\bar{\sigma}_{xx}$ or ${}_d\bar{\sigma}_{yy}$ versus $\bar{y}$ for different values of $(\eta_1 + \eta_3)$

We note with this nonlinear constitutive theory for ${}_d\bar{\boldsymbol{\sigma}}$, there is no pure shear flow. Normal deviatoric Cauchy stresses are always present. Their magnitude of course depends on $\frac{\partial \bar{u}}{\partial \bar{y}}$ and the new material coefficient $(\eta_1 + \eta_3)$ that must be determined experimentally. When $\eta_1 + \eta_3 = 0$, the constitutive theory, hence the mathematical model, reduces to that for incompressible Newtonian fluids in which case ${}_d\bar{\sigma}_{xx} = {}_d\bar{\sigma}_{yy} = 0$.

Model Problem 2: Square Lid-Driven Cavity

Figure 4(a) shows a schematic of $(0.05$ m $\times 0.05$ m$)$ square cavity with lid velocity of $\hat{u}$. Figure 4(b) shows (1×1) dimensionless square cavity with boundary conditions. Table 1 shows details of a 256 element graded discretization. The length $\bar{h}_d$ is chosen to be 0.0025, sufficiently small so that the constant lid velocity assumption remains valid. We consider the following fluid properties and reference quantities.

$$
\begin{aligned}
&\hat{\hat{\rho}} = 868 \text{ kg/m}^3\,; && \hat{\hat{\eta}} = 3 \text{ Pa-s} \\
&\hat{\hat{\rho}} = \bar{\rho}_0\,; && \hat{\hat{\eta}} = \eta_0 \\
&\hat{\rho} = 1\,; && \hat{\eta} = \eta = 1 \\
&\hat{\hat{H}} = L_0 = 0.05 \text{ m}\,; && u_0 = 69.124 \text{ m/s}
\end{aligned}
\tag{60}
$$

These yield a one unit square cavity with $Re = 1000$.

Numerical solutions are calculated using p-version least squares finite element formulation based on residual functional with local approximations in higher order scalar product spaces $H^{k,p}(\bar{\Omega}^e)$ using p-level of 7 with $k = 2$, i.e. solutions of class $C^{11}(\bar{\Omega}^e)$. For these choices of p and k the residual functional is of the order of $O(10^{-4})$ or lower, confirming good accuracy of the computed solutions.

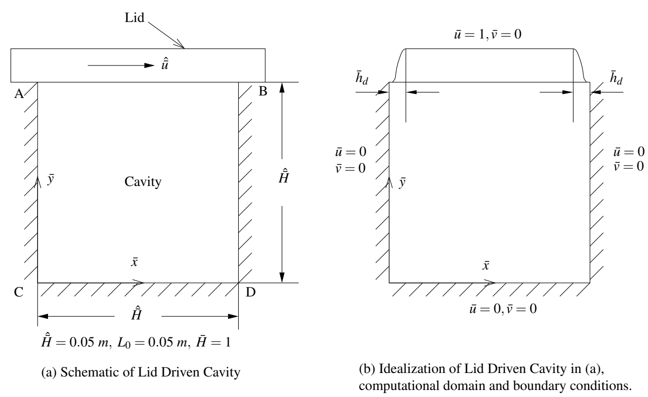


Figure 4: Schematic of lid-driven square cavity and boundary conditions

Table 1: A 256 element graded discretization, $\bar{h}_d = 0.0025$

Side	Element Lengths						
$\overrightarrow{CA}$	0.0025	0.005	0.01	0.02	0.04	0.08	0.16
	0.1825	0.1825	0.16	0.08	0.04	0.02	0.01
	0.005	0.0025					
$\overrightarrow{CD}$	0.0025	0.005	0.01	0.02	0.04	0.08	0.16
	0.1825	0.1825	0.16	0.08	0.04	0.02	0.01
	0.005	0.0025					

Figure 5 shows contours of streamlines for $\eta_1 = \eta_3 = 0$ and for $\eta_1 \neq 0$ and $\eta_3 \neq 0$. That is, contours of streamlines for $\eta_1 \neq 0$ and $\eta_3 \neq 0$ remain the same as for Newtonian fluid. As discussed earlier for 2D case ${}_d\bar{\sigma}_{xx} = {}_d\bar{\sigma}_{yy}$, i.e. normal stresses are similar to pressure field, hence the pressure field $\bar{p}(\bar{x},\bar{y})$ is shifted but $\frac{\partial \bar{p}}{\partial \bar{x}}$, $\frac{\partial \bar{p}}{\partial \bar{y}}$ remain the same in nonlinear constitutive theory as in case of Newton's law of viscosity.

Figures 6(a) – (b) show plots of $\frac{\partial \bar{p}}{\partial \bar{x}}$ and $\frac{\partial \bar{p}}{\partial \bar{y}}$ versus $\bar{y}$ at $\bar{x} = 0.5$ and figures 6(c) – (d) show plots of $\frac{\partial \bar{p}}{\partial \bar{x}}$ and $\frac{\partial \bar{p}}{\partial \bar{y}}$ versus $\bar{x}$ at $\bar{y} = 0.5$ for $\eta_1 = \eta_3 = 0$ and for $\eta_1 = 0.15$ and $\eta_3 = 0.15$. We clearly observe that gradients of $\bar{p}$ remain unaffected in majority of the domain due to nonzero values of η_1 and η_3, except near the upper and lower boundaries due to lack of mesh refinement, hence lack of adequate resolution of the gradients.

Figures 7(a) – (b) show plots of pressure $\bar{p}$ versus $\bar{y}$ at $\bar{x} = 0.5$ and $\bar{p}$ versus $\bar{x}$ at $\bar{y} = 0.5$ for $\eta_1 = \eta_3 = 0$ as well as for different values of η_1 and η_3 ($\eta_1 = \eta_3$). We clearly observe that different values of η_1,η_3 only cause a shift in the pressure value without affecting its gradients with respect to $\bar{x}$ and $\bar{y}$. Remarks in section 5 and the details in figures 5 – 7 confirm that the presence of nonlinear terms in the constitutive theory does

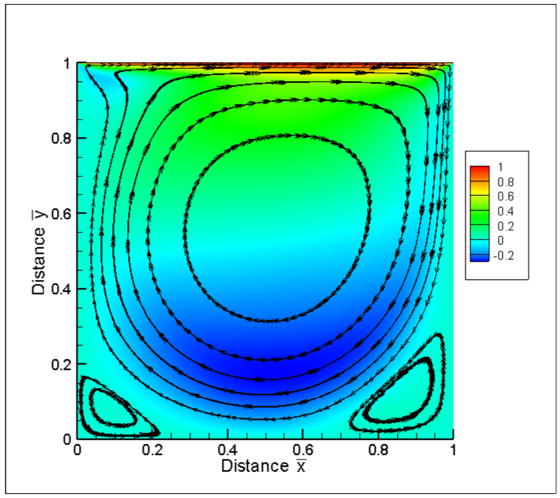
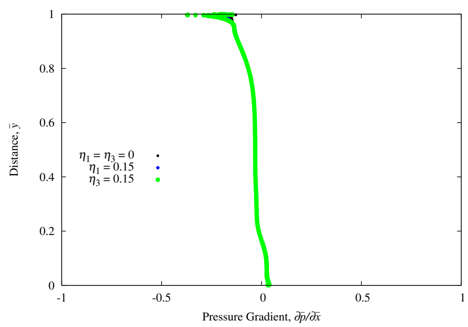
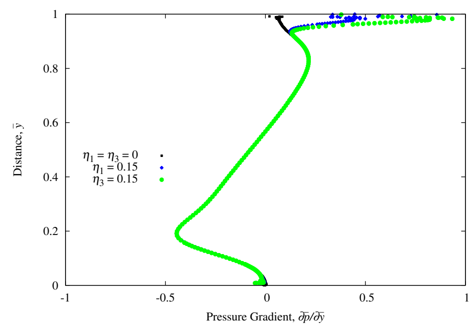


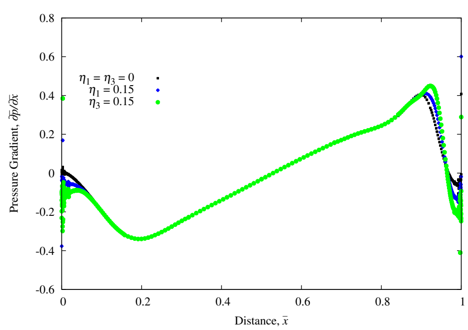
Figure 5: Streamlines in the lid-driven cavity



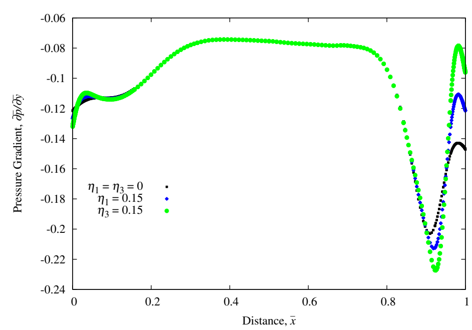
(a) $\frac{\partial \bar{p}}{\partial \bar{x}}$ versus $\bar{y}$ at $\bar{x} = 0.5$



(b) $\frac{\partial \bar{p}}{\partial \bar{y}}$ versus $\bar{y}$ at $\bar{x} = 0.5$



(c) $\frac{\partial \bar{p}}{\partial \bar{x}}$ versus $\bar{x}$ at $\bar{y} = 0.5$

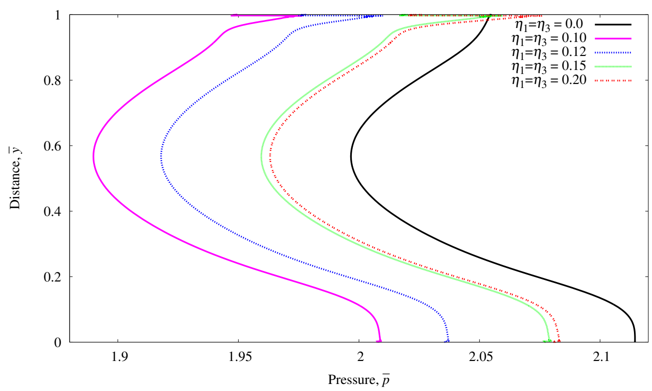


(d) $\frac{\partial \bar{p}}{\partial \bar{y}}$ versus $\bar{x}$ at $\bar{y} = 0.5$

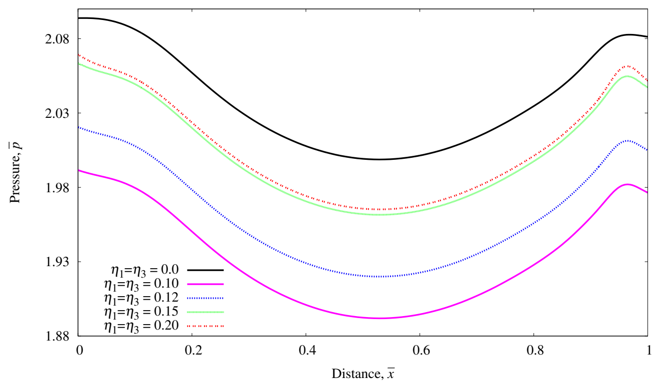
Figure 6: Pressure gradients for various values of η_1 and η_3 at cavity centerlines

not disturb the velocity field, hence the linear stresses (due to Newton's law of viscosity) remain unaffected as well. When comparing results for $\eta_1 = \eta_3 = 0$ and $\eta_1, \eta_3 \neq 0$ the only differences between the two are in the value of the pressure field $\bar{p}$ such that $\frac{\partial \bar{p}}{\partial \bar{x}}$ and $\frac{\partial \bar{p}}{\partial \bar{y}}$ remain unaffected and the nonzero normal stresses ${}_d\bar{\sigma}_{xx} = {}_d\bar{\sigma}_{yy}$ for $\eta_1 \neq 0$ and $\eta_3 \neq 0$.

Figures 8(a) – (d) show plots of velocities $\bar{u}$ and $\bar{v}$ that remain unaffected by the choices of η_1 and η_3. These results are in excellent agreement with those reported in reference [22] that are considered to be of benchmark quality. Plots of deviatoric shear stress ${}_d\bar{\sigma}_{xy}$ at vertical and horizontal centerlines of the cavity



(a) $\bar{p}$ versus $\bar{y}$ at $\bar{x} = 0.5$

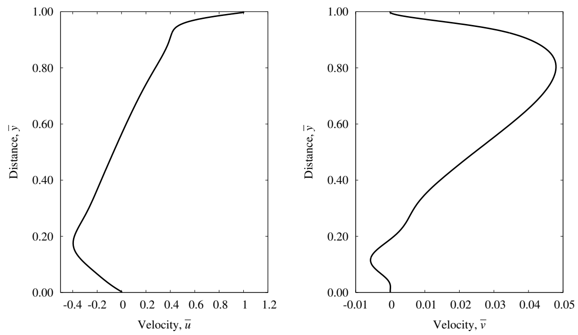


(b) $\bar{p}$ versus $\bar{x}$ at $\bar{y} = 0.5$

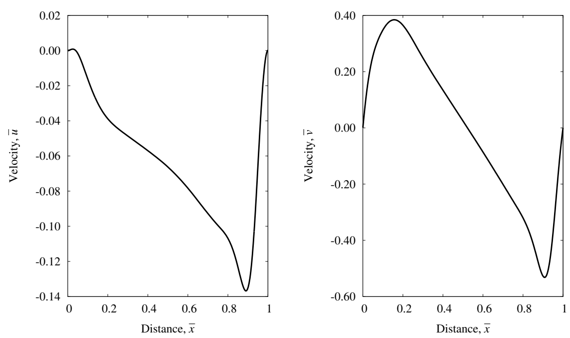
Figure 7: Pressure for various values of η_1 and η_3 at cavity centerlines

shown in figure 9(a) – (b) are also invariant of the choices of η_1 and η_3. Figures 10(a) – (d) show plots of linear deviatoric normal stresses $({}_d\bar{\sigma}_{xx})_l$ and $({}_d\bar{\sigma}_{yy})_l$ at the vertical and the horizontal centerlines of the cavity. Since velocities $\bar{u}$ and $\bar{v}$ are not affected by the choices of η_1 and η_3, results presented in figures 8 – 10 hold for all values of η_1,η_3, including $\eta_1 = \eta_3 = 0$ (Newtonian Fluid). We observe from figure 10 that $({}_d\bar{\sigma}_{xx})_l = -({}_d\bar{\sigma}_{yy})_l$ holds as expected.

Figure 11 shows plots of nonlinear deviatoric normal stresses $({}_d\bar{\sigma}_{xx})_{nl} = ({}_d\bar{\sigma}_{yy})_{nl}$ for $\eta_1 = \eta_3 \neq 0$. These stresses are obviously zero in case of Newtonian fluids. As expected, increasing values of η_1, η_3 produce progressively higher stresses. Their magnitudes are quite significant compared to $({}_d\bar{\sigma}_{xx})_l$ and $({}_d\bar{\sigma}_{yy})_l$ (figure 10). Even for the smallest value of $\eta_1 = \eta_3 = 0.1$, the magnitudes of nonlinear deviatoric normal stresses over a large portion are quite significant compared to linear deviatoric normal stresses.

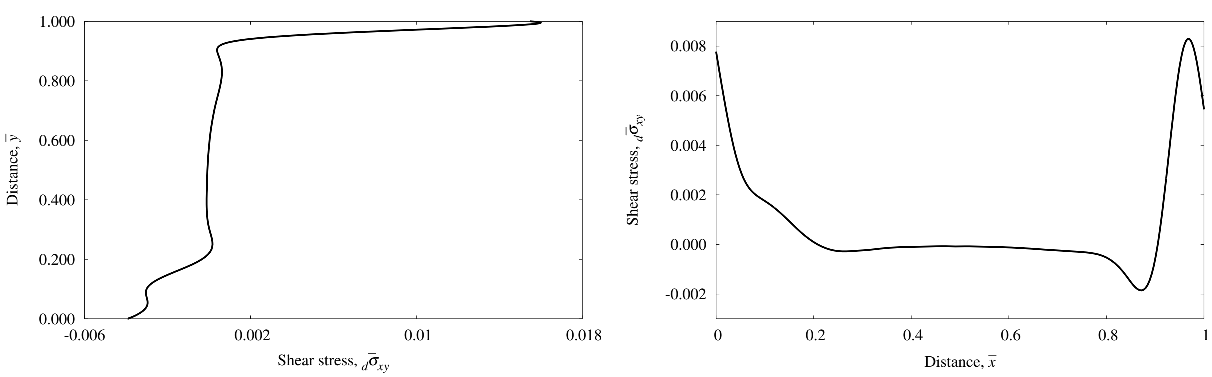


(a) $\bar{u}$ versus $\bar{y}$ at $\bar{x} = 0.5$ **(b)** $\bar{v}$ versus $\bar{y}$ at $\bar{x} = 0.5$



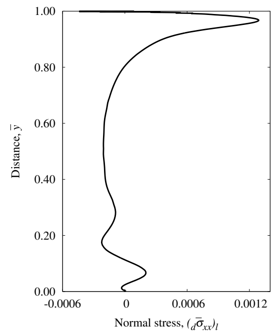
(c) $\bar{u}$ versus $\bar{x}$ at $\bar{y} = 0.5$ **(d)** $\bar{v}$ versus $\bar{x}$ at $\bar{y} = 0.5$

Figure 8: Velocities $\bar{u}$ and $\bar{v}$ for all values of η_1 and η_3 at cavity centerlines

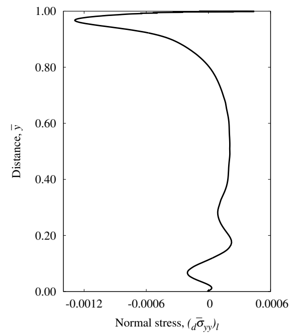


(a) $\bar{\tau}_{xy}$ versus $\bar{y}$ at $\bar{x} = 0.5$ **(b)** $\bar{\tau}_{xy}$ versus $\bar{x}$ at $\bar{y} = 0.5$

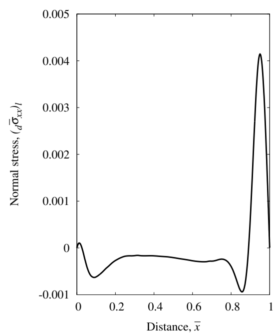
Figure 9: Shear stress $\bar{\tau}_{xy}$ for all values of η_1 and η_3 at cavity centerlines



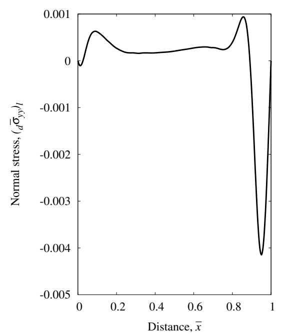
(a) $({}_d\bar{\sigma}_{xx})_l$ versus $\bar{y}$ at $\bar{x} = 0.5$



(b) $({}_d\bar{\sigma}_{yy})_l$ versus $\bar{y}$ at $\bar{x} = 0.5$

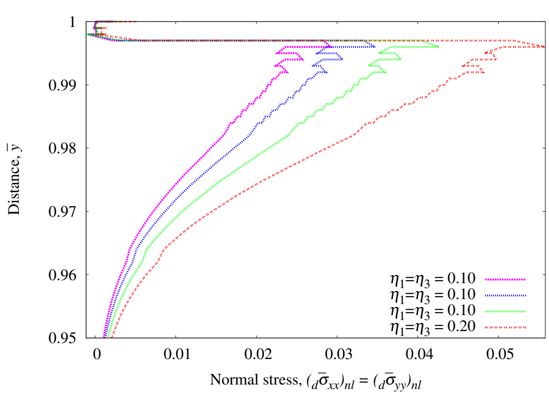


(c) $({}_d\bar{\sigma}_{xx})_l$ versus $\bar{x}$ at $\bar{y} = 0.5$

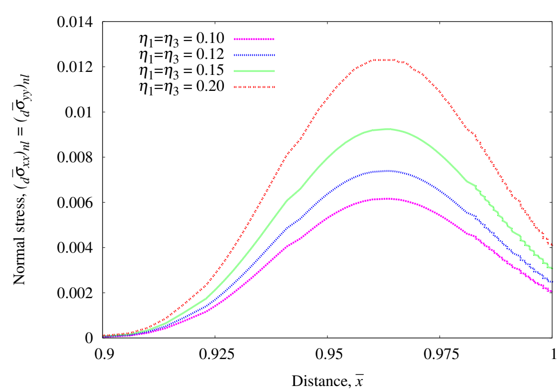


(d) $({}_d\bar{\sigma}_{yy})_l$ versus $\bar{x}$ at $\bar{y} = 0.5$

Figure 10: Linear normal deviatoric stresses $({}_d\bar{\sigma}_{xx})_l$ and $({}_d\bar{\sigma}_{yy})_l$ for all values of η_1 and η_3 at cavity centerlines



(a) $({}_d\bar{\sigma}_{xx})_{nl}$ and $({}_d\bar{\sigma}_{yy})_{nl}$ versus $\bar{y}$ at $\bar{x} = 0.5$



(b) $({}_d\bar{\sigma}_{xx})_{nl}$ and $({}_d\bar{\sigma}_{yy})_{nl}$ versus $\bar{x}$ at $\bar{y} = 0.5$

Figure 11: Nonlinear normal deviatoric stresses $({}_d\bar{\sigma}_{xx})_{nl}$ and $({}_d\bar{\sigma}_{yy})_{nl}$ for all values of η_1 and η_3 at cavity centerlines

Model Problem 3: Asymmetric Sudden Expansion

In this model problem we consider a 3:2 asymmetric sudden expansion. Dimensionless schematic with boundary conditions is shown in figure 12(a). A graded finite element discretization using 174 p-version hierarchical nine-node 2D finite elements is shown in figure 12(b). The following material constants and reference values are used in the numerical studies.

$$\begin{aligned} &\hat{\rho} = 998.2 \text{ kg/m}^3\,; \quad \hat{\eta} = 0.001002 \text{ Pa-s} \\ &\hat{\rho} = \rho_0\,; \qquad\qquad\qquad \hat{\eta} = \eta_0 \\ &\hat{H} = L_0 = 0.015 \text{ m}\,; \quad u_0 = 0.015325 \text{ m/s} \end{aligned} \tag{61}$$

These yield $Re = 229$, $\bar{\rho} = 1$, and $\eta = 1 = \bar{\eta}$.

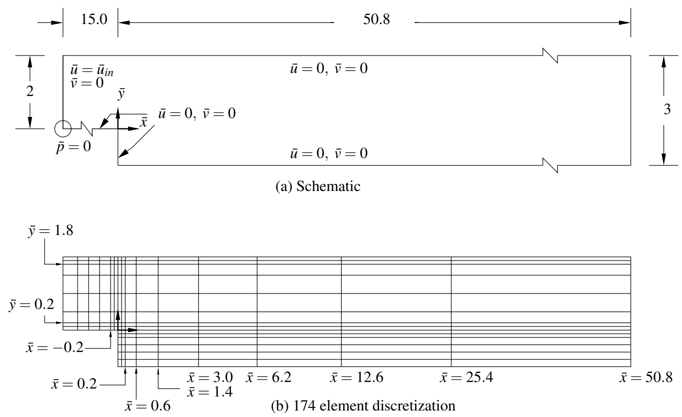


Figure 12: Asymmetric 3:2 sudden expansion: schematic and discretization

A fully developed velocity profile is applied at the inlet that results in flow rate of 2.02. The outflow boundary is open, as the length of the channel beyond the expansion point may not be sufficient for the flow to be fully developed at the outflow. Computations are performed at $p = 9$, $k = 2$ for which the residual functional I for the whole discretization is of the order $O(10^{-7})$.

First, we show that velocity field and the linear deviatoric Cauchy stress field is not affected by the choices of η_1 and η_3, hence remains the same as that for the Newtonian case for all values of η_1 and η_3. Figure 13 shows streamlines for all values of η_1, η_3, confirming that $\bar{u}, \bar{v}$ are independent of η_1, η_3, hence the velocities $\bar{u}, \bar{v}$ in case of nonlinear constitutive theory are same as those for Newtonian case. Graphs of velocity $\bar{u}$ versus $\bar{y}$ at $\bar{x} = 0.0$, 0.2, 5.5, and 50.8 for all values of η_1, η_3 (figure 14) confirm these to be invariant of the values of η_1 and η_3.

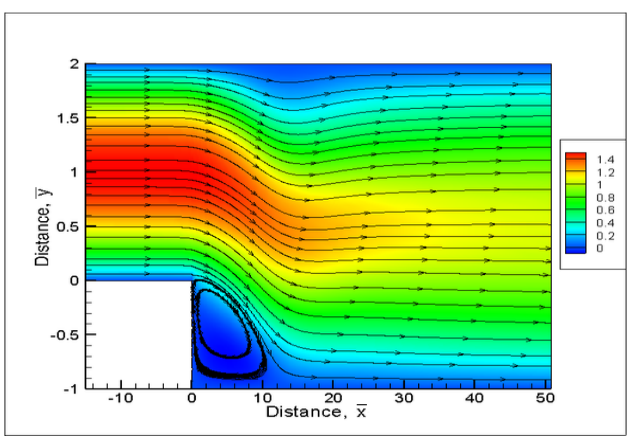
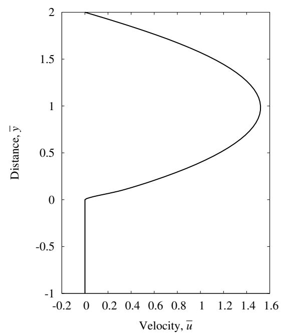
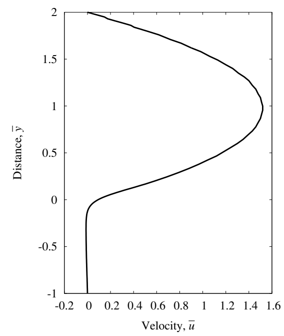


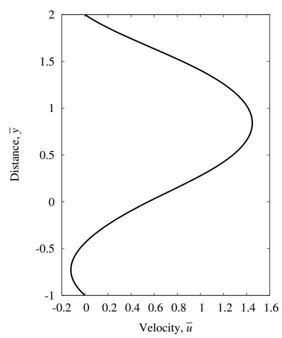
Figure 13: Streamlines in the asymmetric expansion



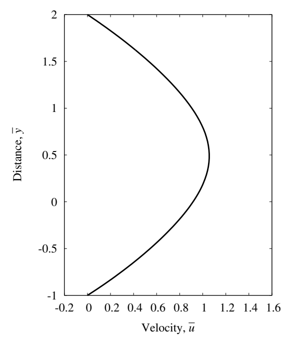
(a) $\bar{x} = 0.0$ (expansion point)



(b) $\bar{x} = 0.2$



(c) $\bar{x} = 5.5$



(d) $\bar{x} = 50.8$ (outlet)

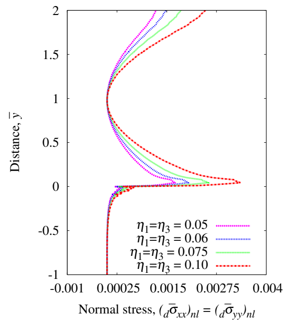
Figure 14: Velocity $\bar{u}$ versus $\bar{y}$ for all values of η_1 and η_3 for different values of $\bar{x}$

Since linear deviatoric Cauchy stresses are functions of the derivatives of $\bar{u}$,$\bar{v}$ with respect to $\bar{x}, \bar{y}$, these also are invariant of η_1 and η_3 and $({}_d\bar{\sigma}_{xx})_l = -({}_d\bar{\sigma}_{yy})_l$ (graphs not shown for the sake of brevity). These are quite small in magnitude compared to ${}_d\bar{\sigma}_{xy}$.

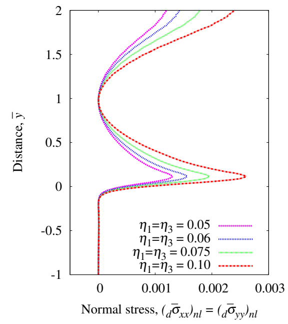
Plots of nonlinear deviatoric normal Cauchy stresses $(({}_d\bar{\sigma}_{xx})_{nl} = ({}_d\bar{\sigma}_{yy})_{nl})$ at $\bar{x} = 0.0$, 0.2, 5.5, and 50.8 for different values of $\eta_1 = \eta_3$ are shown in figures 15(a) – (d). For Newton's law of viscosity these are obviously zero. We observe that $({}_d\bar{\sigma}_{xx})_{nl} = ({}_d\bar{\sigma}_{yy})_{nl}$ increase with progressively increasing values of η_1, η_3 as expected.

Plots of ${}_d\bar{\sigma}_{xy}$ are shown in figure 16 for $\bar{x} = 0.0$, 0.2, 5.5, and 50.8. For all values of η_1 and η_3, including $\eta_1 = \eta_3 = 0$ (Newtonian fluid), the shear stress is clearly independent of η_1, η_3.

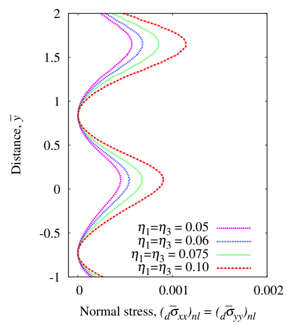
Figures 17(a) – (d) show plots of $\frac{\partial \bar{p}}{\partial \bar{x}}$ versus $\bar{y}$ at $\bar{x} = 0.0$, 5.5, 25.3, and 50.8 for $\eta_1 = \eta_3 = 0.0$ (Newtonian fluid) and $\eta_1 = \eta_3 = 0.05$ and $\eta_1 = \eta_3 = 0.06$. We note that $\frac{\partial \bar{p}}{\partial \bar{x}}$ is independent of η_1 and η_3. Minor deviations near the boundaries and some scatter near the expansion point $\bar{x} = 0.0$ are obviously due to inadequate discretization.



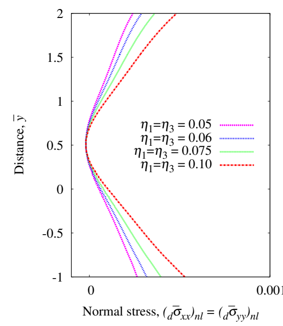
(a) $\bar{x} = 0.0$ (expansion point)



(b) $\bar{x} = 0.2$

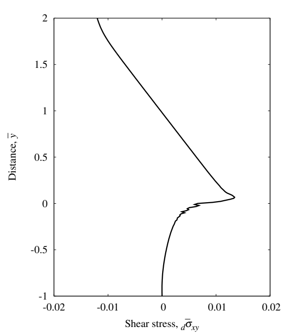


(c) $\bar{x} = 5.5$

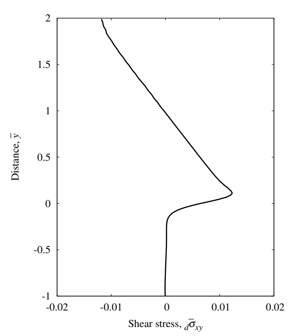


(d) $\bar{x} = 50.8$ (outlet)

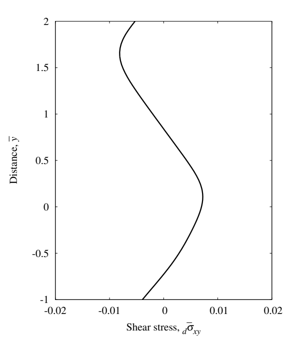
Figure 15: Nonlinear normal deviatoric stress $({}_d\bar{\sigma}_{xx})_{nl}$ and $({}_d\bar{\sigma}_{yy})_{nl}$ for various values of η_1 and η_3 for different values of $\bar{x}$



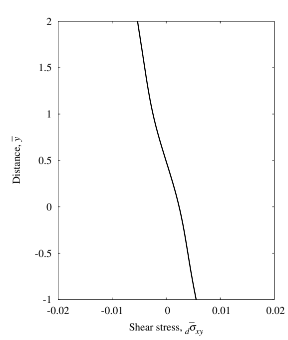
(a) $\bar{x} = 0.0$ (expansion point)



(b) $\bar{x} = 0.2$

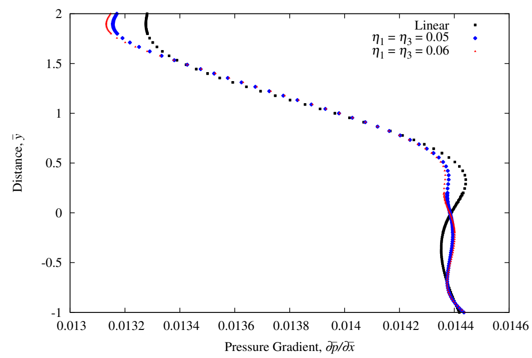


(c) $\bar{x} = 5.5$

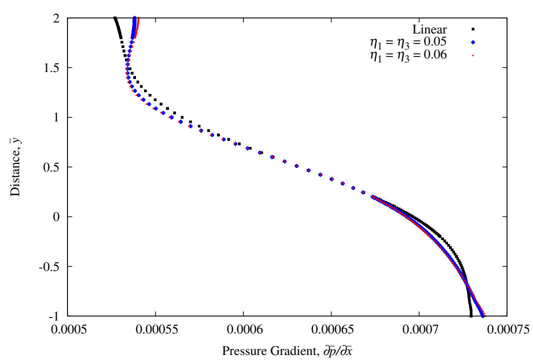


(d) $\bar{x} = 50.8$ (outlet)

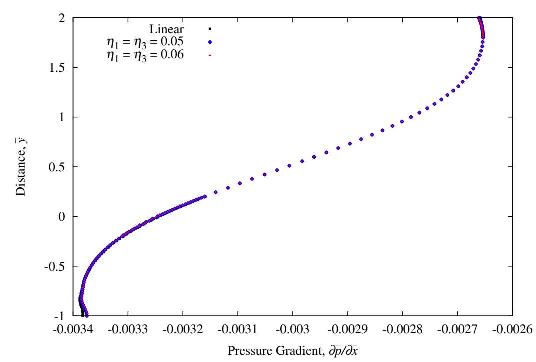
Figure 16: Deviatoric shear stress ${}_d\bar{\sigma}_{xy}$ for all values of η_1 and η_3 for different values of $\bar{x}$



(a) $\bar{x} = 5.5$



(b) $\bar{x} = 25.3$



(c) $\bar{x} = 50.8$ (outlet)

Figure 17: Pressure gradient $\frac{\partial \bar{p}}{\partial \bar{x}}$ versus $\bar{y}$ for various values of η_1 and η_3 for different values of $\bar{x}$

SUMMARY AND CONCLUSIONS

A nonlinear constitutive theory is derived for the deviatoric Cauchy stress tensor for homogeneous, isotropic, incompressible thermoviscous fluids using condition resulting from the entropy inequality in conjunction with the theory of generators and invariants. Based on the conjugate pair in the entropy inequality, the symmetric part of the velocity gradient tensor $\bar{\boldsymbol{D}}$ and temperature $\bar{\theta}$ are considered as argument tensors of deviatoric Cauchy stress tensor. The linear combination of the combined generators of $\bar{\boldsymbol{D}}$ and $\bar{\theta}$ that are symmetric tensors of rank two ($\boldsymbol{I}$, $\bar{\boldsymbol{D}}$, and $\bar{\boldsymbol{D}}^2$) are used to form a constitutive theory for ${}_d\bar{\boldsymbol{\sigma}}$. The material coefficients in this constitutive theory for ${}_d\bar{\boldsymbol{\sigma}}$ are determined by expanding the coefficients in the linear combination in terms of the combined invariants of $\bar{\boldsymbol{D}}$,$\bar{\theta}$ about a reference configuration. The resulting constitutive theory is shown to be of fifth degree in the components of $\bar{\boldsymbol{D}}$. This constitutive theory is complete as it is based on integrity, but requires too many material coefficients that must be determined experimentally. A simplified form of this constitutive theory is considered in which ${}_d\bar{\boldsymbol{\sigma}}$ is a quadratic function of the components of $\bar{\boldsymbol{D}}$ that requires only three material coefficients: viscosity η (standard, known for a given fluid) and η_1 and η_3, two new coefficients.

It is shown that for $\mathbb{R}^1$ and $\mathbb{R}^2$, the nonlinear constitutive theory produces additional normal stresses that are equal in magnitude, hence act like pressure field, thus cause a uniform shift in the pressure values while leaving $\frac{\partial \bar{p}}{\partial \bar{x}}$ and $\frac{\partial \bar{p}}{\partial \bar{y}}$ unaffected. A consequence of this is (1) that due to nonlinear constitutive theory for ${}_d\bar{\boldsymbol{\sigma}}$ presented here, the velocity field and the deviatoric shear stress ${}_d\bar{\sigma}_{xy}$ remain unaffected and (2) additional normal stresses of equal magnitude are created that are dependent on the squares of velocity gradients and the material coefficients η_1 and η_3.

First we remark that this constitutive theory is only quadratic in $\bar{\boldsymbol{D}}$, but is certainly closer to that based on integrity when compared with linear constitutive theory, hence a better representation of physics than

Newton's law of viscosity. Based on this constitutive theory there are no pure shear flows. We have seen existence of normal deviatoric stresses in flow between parallel plates when nonlinear constitutive theory is used, i.e. in present theory the fully developed flow between parallel plates is not a pure shear flow. In fact, within the framework of the nonlinear constitutive theory for ${}_d\bar{\boldsymbol{\sigma}}$ pure shear flows do not exist.

Three model problems (fully developed flow between parallel plates, square lid-driven cavity, and 3:2 asymmetric expansion) are presented to demonstrate various features of the physics discussed above in this section due to nonlinear constitutive theory for ${}_d\bar{\boldsymbol{\sigma}}$.

We remark that additional material coefficients η_1 and η_3 need to be determined experimentally, i.e. the nonlinear constitutive theory for ${}_d\bar{\boldsymbol{\sigma}}$ needs to be calibrated for a fluid of interest. Values of the coefficients η_1 and η_3 and severity of the velocity gradients obviously determine the magnitudes of nonlinear normal stresses, hence determine their relevance or lack thereof in a given application. However, based on the constitutive theory presented here these nonlinear normal stresses exist in all flows, but their magnitude may or may not be significant as it is dependent on the fluid and actual application.

Acknowledgements

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