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HERMITE-HADAMARD TYPE INEQUALITY FOR STRONGLY CONVEX FUNCTIONS VIA SUGENO INTEGRALS

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ABSTRACT

In this paper, Hermite-Hadamard type inequality for Sugeno integrals based on strongly convex functions is studied. Some examples are given to illustrate the results.
Keywords: Hermite hadamard type inequality, sugeno integrals, strongly convex functions.

1. INTRODUCTION

The theory of fuzzy measures and fuzzy integrals was introduced by Sugeno [1]. The properties and applications of Sugeno-integral have been studied by lots of authors. Between these others, Ralescu and Adams [2] proposed several equivalent definitions of fuzzy integrals; Román-Flores et al. [3, 4] defined the level-continuity of fuzzy integrals and the H-continuity of fuzzy measures; the book by Wang and Klir [5] contains a general overview on fuzzy measurement and fuzzy integration theory.

Many authors generalized the Sugeno integral by using some other operators to replace the special operators $\vee$ and/or $\wedge$.

In recent years, some authors [6]-[10] generalized several classical integral inequalities for fuzzy integral.Caballero and Sadarangani [10] showed off a Hermite-Hadamard type inequality of fuzzy integrals for convex function. Li, Song and Yue [11] served Hermite-Hadamard type inequality for Sugeno Integrals. S. Turhan, N. O. Bekar and H. G. Akdemir [14] have studied Hermite-Hadamard type inequality for log-convex functions via Sugeno Integrals lately. K. Nikodem, J. L. Sάnchez, L. Sάnchez [12] studied Jensen and Hermite-Hadamard inequalities for strongly convex set-valued maps.

The aim of this paper is to prove a Hermite-Hadamard type inequality for Sugeno integrals related to strongly convex functions. .

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Let's see some proporties of fuzzy integral.

2. PRELIMINARY DISCUSSIONS

In this section, we remember some basic definition and properties of fuzzy integral and strongly convex function. For details we refer the readers to Refs [1, 5, 11].

Suppose that Σ is a σ-algebra of subsets of X and that $\mu : \Sigma \to 0, \infty)$ is a non-negative, extended real-valued set function. We say that μ is a fuzzy measure if and only if:

1. $\mu(\varnothing) = 0$;
2. $E, F \in \Sigma$ and $E \subset F$ imply $\mu(E) \leq \mu(F)$ (monotonicity);
3. $\{E_n\} \subset \Sigma, E_1 \subset E_2 \subset ...,$ imply $\lim_{n\to\infty} \mu(E_n) = \mu\left(\bigcup_{n=1}^{\infty} E_n\right)$ (continuity from below);
4. $\{E_n\} \subset \Sigma, E_1 \supset E_2 \supset ..., \mu(E_1) < \infty,$ imply $\lim_{n\to\infty} \mu(E_n) = \mu\left(\bigcap_{n=1}^{\infty} E_n\right)$ (continuity from above).

If f is a non-negative real-valued function defined on $X,$ we denote the set

$\{x \in X : f(x) \geq \alpha\} = \{f \geq \alpha\}$

by F_α for $\alpha \geq 0$. Note that if $\alpha \leq \beta$ then $F_\beta \subset F_\alpha$.

Let (X, Σ, μ) be a fuzzy measure space, we denote M^+ the set of all non-negative measurable functions with respect to Σ.

Definition 2.1 *[1, 5] Let $A \in X$, $f \in M^+$ The fuzzy integral of f on A with respect to μ which is denoted by $(s)\int f d\mu,$ is defined by*

$$(s)\int f d\mu = \bigvee_{\alpha \geq 0} [\alpha \wedge \mu(A \cap \{f \geq \alpha\})].$$

When $A = \Sigma,$ the fuzzy integral may also be denoted by $(s)\int f d\mu.$

Where $\vee$ and $\wedge$ denote the operations inf and sup on $[0, \infty)$, respectively.

The following properties of the Sugeno integral are well known and can be found in.

Proposition 2.1 *Let (X, Σ, μ) be a fuzzy measure space, $A \in \Sigma$ and $f, g \in M^+$*

1. $(s)\int_A f d\mu \leq \mu(A)$;
2. $(s)\int_A k d\mu = k \wedge \mu(A),$ k non-negative constant;

3. If $f \leq g$ on A then $(s)\int_A f d\mu \leq (s)\int_A g d\mu$;
4. $\mu(A \cap \{f \geq \alpha\}) \geq \alpha \Rightarrow (s)\int_A f d\mu \geq \alpha$;
5. $\mu(A \cap \{f \geq \alpha\}) \leq \alpha \Rightarrow (s)\int_A f d\mu \leq \alpha$;
6. $(s)\int_A f d\mu < \alpha \Leftrightarrow$ there exists $\gamma < \alpha$ such that $\mu(A \cap \{f \geq \gamma\}) < \alpha$;
7. $(s)\int_A f d\mu > \alpha \Leftrightarrow$ there exists $\gamma > \alpha$ such that $\mu(A \cap \{f \geq \gamma\}) > \alpha.$

Remark 2.1 *Consider the distribution function F associated to f on A, that is, $F(\alpha) = \mu(A \cap \{f \geq \alpha\})$. Then, due to (4) and (5) of Preposition 2.1, we have that*

$$F(\alpha) = \alpha \Rightarrow (s)\int f d\mu = \alpha.$$

Thus, from a numerical point of view, the fuzzy integral can be calculated solving the equation $F(\alpha) = \alpha.$

In [11], D-O. Li, X-Q. Song,T. Yue proved with the help of certain examples that the classical Hermite-Hadamard inequalities for Sugeno integral.

Definition 2.2 *[13] Let $(X, \|.\|)$ be normed space. D be a convex subset of X and c > 0. A function $f : D \to \mathbb{R}$ is said to be strongly convex with modulus c if*

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y) - c\lambda(1-\lambda)\|x-y\|^2 \tag{1}$$

for all $x, y \in D$ and $t \in [0,1]$.

3. HERMITE-HADAMARD TYPE INEQUALITY FOR PREINVEX FUNCTIONS VIA SUGENO INTEGRALS

The following inequality is well known in the literature as the Hermite-Hadamard inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{2}$$

where $f : I \to \mathbb{R}$ is a convex function on the interval I and $a, b \in I$ with $a < b.$ In [12], the following version of Hermite-Hadamard inequality for strongly convex function was recently proved

$$f\left(\frac{a+b}{2}\right) + \frac{c}{12}(a-b)^2 \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} - \frac{c}{6}(a-b)^2. \tag{3}$$

In this paper, we prove using Sugeno integral another refinement of the Hermite-Hadamard type inequality for strongly convex functions. Some applications for special means are also given.

Example 3.1. Consider $X=[0,1]$ and let μ be the Lebesgue measure on X. If we take the function $f(x)=4x^2$, then $f(x)$ is a strongly convex function. In fact

$$f(1-\lambda)\leq \lambda f(0)+(1-\lambda)f(1)-c\lambda(1-\lambda)(0-1)^2$$
$$4(1-\lambda)^2\leq (1-\lambda)4-c\lambda(1-\lambda)$$
$$4(1-\lambda)\leq 4-c\lambda$$
$$-4\lambda\leq -c\lambda$$
$$0<c\leq 4.$$

Calculating the Sugeno integral related to this function, by Remark 2.1, this is

$$F(\alpha)=\mu(0,1]\cap\{f\geq\alpha\})$$
$$=\mu(0,1]\cap\{4x^2\geq\alpha\})$$
$$=\mu\left(0,1]\cap\left\{x\geq\frac{\sqrt{\alpha}}{2}\right\}\right)=1-\frac{\sqrt{\alpha}}{2}$$

and we solve the equation

$$1-\frac{\sqrt{\alpha}}{2}=\alpha.$$

It is easily proved that the solution of the last equation is achieved and Remark 2.1, we get

$$(s)\int_0^1 4x^2 d\mu=\left(\frac{-1+\sqrt{17}}{4}\right)^2\cong 0.6096. \tag{4}$$

On the other hand,

$$f\left(\frac{1}{2}\right)+\frac{c}{12}=1+\frac{c}{12}. \tag{5}$$

From (6) and (7) inequalities

$$1+\frac{c}{12}\leq 0.6096 \tag{6}$$

$$c\leq -4.6848. \tag{7}$$

From $c>0$, this is contradiction.This proves that the left part of Hermite-Hadamard inequality is not satisfied in Sugeno integral.

Example 3.2. Consider $X=[0,1]$ and let μ be the Lebesgue measure on X. If we take the function $f(x)=\frac{x^2}{4}$, then $f(x)$ is a strongly convex function. To calculate the Sugeno integral related to this function, by Remark 2.1, this is

$$F(\alpha) = \mu(0,1] \cap \{f \geq \alpha\})$$
$$= \mu\left(0,1] \cap \left\{\frac{x^2}{4} \geq \alpha\right\}\right)$$
$$= \mu\left(0,1] \cap \left(x \geq 2\sqrt{\alpha}\right)\right)$$
$$= 1 - 2\sqrt{\alpha}$$

and we solve the equation

$$1 - 2\sqrt{\alpha} = \alpha.$$

It is easily proved that the solutions of the last equation is $\alpha = \left(\frac{-2+\sqrt{8}}{2}\right)^2$, and, Remark 2.1, we get

$$(s)\int_0^1 f d\mu = (s)\int_0^1 \frac{x^2}{4} d\mu = \left(\frac{-2+\sqrt{8}}{2}\right)^2 \cong 0.1716. \tag{8}$$

On the other hand.

$$\frac{f(0)+f(1)}{2} - \frac{c}{6} = \frac{1}{8} - \frac{c}{6}, \quad c > 0. \tag{9}$$

From (8) and (9) equations, it is achieved

$$0.1716 \leq 0.125 - \frac{c}{6} \tag{10}$$

$$c \leq -0.2796. \tag{11}$$

From $c > 0$, it is contradiction.This proves that the left part of Hermite-Hadamard inequality is not satisfied in the fuzzy context.

The aim of the following theorem is to show a Hermite-Hadamard type inequality for the Sugeno integral.

Theorem 3.1 *Let* $g:[0,1] \to [0,\infty)$ *be a preinvex function such that* $g(0) < g(1)$. *Then*

$$(s)\int_0^1 g(x) d\mu \leq \min\{\alpha, 1\}.$$

where α is root of

$$\mu\left(0,1] \cap \left\{x \geq \sqrt{\frac{\alpha - g(0)}{c} + \left(\frac{g(1)-g(0)-c}{2c}\right)^2} - \frac{g(1)-g(0)-c}{2c}\right\}\right) = \alpha.$$

Proof. As g is a strongly convex funtion

$$g(x) = g(x.1 + (1-x).0) \leq x.g(1) + (1-x).g(0) - cx(1-x) = h(x), \quad c > 0, x \in 0,1].$$

By (3) of Proposition 2.1, we have that

$$(s)\int_0^1 g(x)d\mu \leq (s)\int_0^1 g(x.1+(1-x).0)d\mu$$

$$\leq (s)\int_0^1 \left(cx^2+(g(1)-g(0)-c)x+g(0)\right)d\mu = (s)\int_0^1 h(x)d\mu.$$

In order to calculate the integral in the right-hand part of the last inequality, we consider the distribution function F given by

$$\begin{aligned} F(\alpha) &= \mu(0,1]\cap\{h\geq\alpha\}) \\ &= \mu(0,1]\cap\{cx^2+(g(1)-g(0)-c)x+g(0)\geq\alpha\}) \\ &= \mu\left(0,1]\cap\left\{x\geq\sqrt{\frac{\alpha-g(0)}{c}+\left(\frac{g(1)-g(0)-c}{2c}\right)^2}-\frac{g(1)-g(0)-c}{2c}\right\}\right) \end{aligned}$$

and the solution of the equation

$$\mu\left(0,1]\cap\left\{x\geq\sqrt{\frac{\alpha-g(0)}{c}+\left(\frac{g(1)-g(0)-c}{2c}\right)^2}-\frac{g(1)-g(0)-c}{2c}\right\}\right)=\alpha. \quad (12)$$

By (1) of Proposition 2.1, we get that

$$(s)\int_0^1 h(x)d\mu \leq \mu([0,1])=1.$$

By Remark 2.1, we have

$$(s)\int_0^1 g(x)d\mu \leq \min\{\alpha,1\},$$

where α is the root of (12) equation and $g(0)<\alpha<g(1)$. This completes is proof.

Remark 3.1 *In the case* $g(0)=g(1)$ *in Theorem 3.1, the function* $h(x)$ *is*

$$g(x)=g(x.1+(1-x).0)\leq(1-x).g(0)+x.g(1)-cx(1-x)=cx^2-cx+g(0)=h(x)$$

and

$$(s)\int_0^1 g(x)d\mu \leq (s)\int_0^1 h(x)d\mu \leq \min\{\alpha,1\},$$

where α is root of

$$\mu\left([0,1]\cap\left\{x\geq\sqrt{\frac{\alpha-g(0)}{c}+\frac{1}{4}}+\frac{1}{2}\right\}\right)=\alpha.$$

Theorem 3.2 *Let* $g:[0,1]\to[0,\infty)$ *be a strongly convex function such that* $g(0)>g(1)$. *Then*

$$(s)\int_0^1 g(x)d\mu \le \min\{\alpha,1\}$$

where α is root of the equation

$$\mu\left(0,1]\cap\left\{x\ge \frac{g(0)-g(1)+c}{2c}-\sqrt{\frac{\alpha-g(0)}{c}+\left(\frac{g(0)-g(1)+c}{2c}\right)^2}\right\}\right)=\alpha.$$

Proof. Similarly, using the method in Theorem 3.1, we have

$$\begin{aligned}F(\alpha)&=\mu(0,1]\cap\{h\ge\alpha\})\\&=\mu(0,1]\cap\{cx^2+(g(1)-g(0)-c)x+g(0)\ge\alpha\})\\&=\mu\left(0,1]\cap\left\{x\ge \frac{g(0)-g(1)+c}{2c}-\sqrt{\frac{\alpha-g(0)}{c}+\left(\frac{g(0)-g(1)+c}{2c}\right)^2}\right\}\right),\end{aligned}$$

and the solution of the equation

$$\mu\left(0,1]\cap\left\{x\ge \frac{g(0)-g(1)+c}{2c}-\sqrt{\frac{\alpha-g(0)}{c}+\left(\frac{g(0)-g(1)+c}{2c}\right)^2}\right\}\right)=\alpha.$$

where $g(1)<\alpha<g(0)$. The proof of the rest part is similar, so we omit it.

Theorem 3.3 *Let* $g:[a,b]\to[0,\infty)$ *be a strongly convex function. Then*

(i) If $g(a)<g(b)$, then

$$(s)\int_a^b g(x)d\mu \le \min\{\alpha,b-a\}$$

where $g(a)<\alpha<g(b)$ and α is root of the equation

$$\mu\left([a,b]\cap\left\{\left|x+\frac{1}{2}\left(\frac{g(a)-g(b)}{c(a-b)}-(a+b)\right)\right|\right.\right.$$

$$\left.\left.\ge\sqrt{\frac{\alpha}{c}-\frac{ag(b)-bg(a)}{c(a-b)}-ab+\frac{1}{4}\left(\frac{g(a)-g(b)}{c(a-b)}-(a+b)\right)^2}\right\}\right)=\alpha.$$

(ii) If $g(a)=g(b)$, then

$$(s)\int_a^b g(x)d\mu \le \min\{\alpha, b-a\}$$

where α is root of the equation

$$\mu\left([a,b]\cap\left\{\left|x-\frac{a+b}{2}\right| \ge \sqrt{\frac{\alpha-g(a)}{c}-ab+\left(\frac{a+b}{2}\right)^2}\right\}\right)=\alpha.$$

(iii) If $g(a)>g(b)$, then

$$(s)\int_a^b g(x)d\mu \le \min\{\alpha, b-a\},$$

where $g(a)>\alpha>g(b)$ and α is root of the equation

$$\mu\left([a,b]\cap\left\{|x+\frac{1}{2}\left(\frac{g(a)-g(b)}{c(a-b)}-(a+b)\right)|\right.\right.$$

$$\left.\left.\ge\sqrt{\frac{\alpha}{c}-\frac{ag(b)-bg(a)}{c(a-b)}-ab+\frac{1}{4}\left(\frac{g(a)-g(b)}{c(a-b)}-(a+b)\right)^2}\right\}\right)=\alpha.$$

Proof. Note that as g is a strongly function then for $x\in[a,b]$ we have

$$g(x)=g\left(\left(\frac{x-b}{a-b}\right)a+\left(1-\frac{x-b}{a-b}\right)b\right)$$

$$\le\left(\frac{x-b}{a-b}\right)g(a)+\left(1-\frac{x-b}{a-b}\right)g(b)-c\left(\frac{x-b}{a-b}\right)\left(1-\frac{x-b}{a-b}\right)(a-b)^2=h(x).$$

By (3) of Proposition 2.1,

$$(s)\int_a^b g(x)d\mu=(s)\int_a^b g\left(\left(\frac{x-b}{a-b}\right)a+\left(1-\frac{x-b}{a-b}\right)b\right)d\mu$$

$$\le(s)\int_a^b\left(\frac{x-b}{a-b}\right)g(a)+\left(1-\frac{x-b}{a-b}\right)g(b)-c\left(\frac{x-b}{a-b}\right)\left(1-\frac{x-b}{a-b}\right)(a-b)^2d\mu$$

$$=(s)\int_a^b h(x)d\mu.$$

Now, we consider the distribution function F given by

$$F(\alpha)=\mu([a,b]\cap\{h\ge\alpha\})$$

$$= \mu\left([a,b] \cap \left\{ cx^2 + \left(\frac{g(a)-g(b)}{(a-b)} - c(a+b) \right)x + \left(\frac{ag(b)-bg(a)}{(a-b)} \right) + cab \geq \alpha \right\} \right)$$

$$= \mu\left([a,b] \cap \left\{ \left| x + \frac{1}{2}\left(\frac{g(a)-g(b)}{c(a-b)} - (a+b) \right) \right| \right. \right.$$

$$\left. \left. \geq \sqrt{\frac{\alpha}{c} - \frac{ag(b)-bg(a)}{c(a-b)} - ab + \frac{1}{4}\left(\frac{g(a)-g(b)}{c(a-b)} - (a+b) \right)^2} \right\} \right),$$

then for $g(a) < \alpha < g(b)$ and $c > 0$, we obtained

$$\mu\left([a,b] \cap \left\{ \left| x + \frac{1}{2}\left(\frac{g(a)-g(b)}{c(a-b)} - (a+b) \right) \right| \right. \right. \quad (13)$$

$$\left. \left. \geq \sqrt{\frac{\alpha}{c} - \frac{ag(b)-bg(a)}{c(a-b)} - ab + \frac{1}{4}\left(\frac{g(a)-g(b)}{c(a-b)} - (a+b) \right)^2} \right\} \right) = \alpha .$$

Then by (1) of Proposition 2.1 and Remark 2.1, we have

$$(s)\int_a^b g(x)d\mu \leq (s)\int_a^b h(x)d\mu = \min\{\alpha, b-a\}.$$

where α is root of (13) equation. (ii) and (iii) have been proved as proof of (i). We omitted it.

Example 3.3. Consider $X = [0,1/2]$ and $c = 4$ from Example 3.1. Let μ be the Lebesgue measure on X. If we take the function $g(x) = 4x^2$, then $g(x)$ is a strongly convex and increasing function on $[0,1/2]$. From the (i) of theorem 3.3, When we solve

$$\frac{1}{2} - \sqrt{\frac{\alpha}{4} - \frac{0g\left(\frac{1}{2}\right) - \frac{1}{2}g(0)}{4\left(0-\frac{1}{2}\right)} - 0.\frac{1}{2} + \frac{1}{4}\left(\frac{g(0)-g\left(\frac{1}{2}\right)}{4\left(0-\frac{1}{2}\right)} - \left(0+\frac{1}{2}\right) \right)^2}$$

$$+ \frac{1}{2}\left(\frac{g(0)-g\left(\frac{1}{2}\right)}{4\left(0-\frac{1}{2}\right)} - \left(0+\frac{1}{2}\right) \right) = \alpha.$$

From the last equation, $\alpha = \left(\frac{-1+\sqrt{1+8}}{4} \right)^2 = \frac{1}{4}$ is obtained. Then

$$(s)\int_0^{1/2} g(x)\,d\mu = (s)\int_0^{1/2} 4x^2\,d\mu \le \min\left\{\frac{1}{4},\frac{1}{2}\right\} = \frac{1}{4}.$$

4. CONCLUSION

In this paper, we have researched the classical Hermite-Hadamard inequality for Sugeno integral based on strongly convex function. For further investigations we will continue to study Hermite-Hadamard and other integral inequalities for several fuzzy integrals based on strongly convex function.

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