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AN EXTENDED COUPLED COINCIDENCE POINT THEOREM AND RELATED RESULTS

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ABSTRACT

In this paper, we give an extended coupled coincidence point theorem for a mixed g −monotone mapping $F: X \to X$ satisfying a weaker contractive condition. As a result of this theorem, we introduce an extended coupled fixed point theorem. We also explain that there exist a relationship between Theorem 2.1 which is our main theorem and Theorem 1.3 introduced by Choudhury et. al. [Choudhury, BS, Kundu, A: Appl. Math. Lett. 25,6-10(2012)].
Keywords: Fixed point, coupled coincidence point, coupled fixed point, Partially ordered set.

BİR GENELLEŞTİRİLMİŞ ÇİFTE ÇAKIŞMA NOKTASI TEOREMİ VE İLGİLİ SONUÇLARI

ÖZ

Bu makalede, zayıf şartlar içeren karma g −monoton bir dönüşüm için genelleştirilmiş çifte çakışma noktası teoremi vereceğiz. Bu teoremin bir sonucu olarak bir çifte sabit nokta teoremi ifade edilmiştir. Ayrıca, ana teoremimiz olan Teorem 2.1 ile Choudhury et. al. [Choudhury, BS, Kundu, A: Appl. Math. Lett. 25,6-10(2012)] tarafından ifade edilen Teorem 1.3 arasında bir ilişki olduğunu göstereceğiz.
Anahtar Sözcükler: Sabit nokta, çifte çakışma noktası, çifte sabit nokta, kısmi sıralı küme.

1. INTRODUCTION

Fixed point theory is a very useful tool in many branch of applied sciences. Recently, the existence of fixed points and coincidence points in partially ordered metric space are considered by many mathematicans [2]-[9]. They established very important results to study the solutions of the matrix equations, the ordinary differential equationsons and integral equations.

Recall that the pair $(X, \preccurlyeq)$ is called a partially ordered set (in short Poset), if the relation $\preccurlyeq$ on X is transitive, reflexive and antisymmetric [1]. Also, a mapping $f: X \to X$ is called monotone nondecreasing if

$$x, y \in X, x \preccurlyeq y \text{ implies } fx \preccurlyeq fy.$$

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Ran and Reurings [2] established an analogue of Banach's fixed point theorem in partially ordered set and they discussed several applications to linear and nonlinear matrix equations.

Theorem 1.1 Let $(X,\preccurlyeq)$ be a partially ordered set endowed with a metric d and (X,d) be a complete metric space. Furthermore, every $x,y\in X$ has a lower bound and an upper bound. If $f:X\to X$ is a continuous, monotone (i.e., either order-preserving or order-reversing) map from X into X such that

$$\exists 0<c<1: d(fx,fy)\le cd(x,y), x\succcurlyeq y$$

$$\exists x_0\in X: x_0\preccurlyeq fy_0 \; or \; x_0\succcurlyeq fy_0$$

then f has a unique fixed point $\bar{x}$. Moreover, for every $\in X$ $\lim_{n\to\infty} f^n x=\bar{x}$.

Nieto and Lopez [3] extended some results to study problems of solving ordinary differential equations. Also, They gave an alternative condition to replace the continuity of f as following:

" $if\ any\ nondecreasing\ sequence\ \{x_n\}\ in\ X\ converges\ to\ z, then\ x_n\preccurlyeq z\ for\ all\ n\ge 0$".

Forthermore, to guarantee the uniqueness of the fixed point, Nieto and Lopez [3] gave a condition of the pairs $(x,y)\in X\times X$ as follows

" for every $x,y\in X$; there exists $z\in X$ which is comparable to x and y "

In many works, there is a trend to weaken the requirement conditions of the existence theorems. In [4], Ćirić et. al. introduced the concept of g-monotone mapping and prove some fixed and common fixed point theorems for g −nondecreasing generalized nonlinear contractions in partially ordered metric spaces. Choudhury and Kundu [5], using this concept, introduced the concept of (ψ,α,β) −weak contraction which is generalized weak contraction principle to coincidence point and common fixed point results in Posets.

Definition 1.2. Suppose $(X,\preccurlyeq)$ is a partially ordered set and $f,g: X\to X$ are mappings. f is said to be g −non-decreasing if for $x,y\in X$, $gx\preccurlyeq gy$ implies $fx\preccurlyeq fy$.

Theorem 1.3. [Choudhury and Kundu, [5] **]** Let $(X,\preccurlyeq,d)$ be a partially ordered complete metric space. Let $f,g:X\to X$ be such that $f(X)\subseteq g(X)$, f is g-nondecreasing, $g(X)$ is closed and

$$\psi(d(fx,fy))\le\alpha(d(gx,gy))-\beta(d(gx,gy)), for\ all\ x,y\in X\ with\ gx\preccurlyeq gy,$$

where $\psi,\alpha,\beta:[0,\infty)\to[0,\infty)$ are such that, ψ is continuous and monotone nondecreasing, α is continuous, β is lower semi-continuous,

$$\psi(t)=0\ if\ and\ only\ if\ t=0, \alpha(0)=\beta(0)=0$$

and for all $t>0$, $\psi(t)-\alpha(t)+\beta(t)>0$. Also, if any nondecreasing sequence $\{x_n\}$ in X converges to z, then we assume $x_n\preccurlyeq z$ for all $n\ge 0$.

If there exists $x_0\in X$ such that $gx_0\preccurlyeq fx_0$, then f and g have coincidence point.

On the other hand, the notion of coupled fixed point and related theorems are considered by many researcher. Bhaskar and Lakshmikantham [8] introduced the notion of the mixed monotone property of a given mapping and established related fixed point theorems.

Definition 1.4. Let $(X,\preccurlyeq)$ be a partially ordered set and $F:X\times X\to X$. The mapping F is said to has the mixed monotone property if F is monotone non-decreasing in its first argument and is monotone non-increasing in its second argument, that is, for any $x,y\in X$, we have

$$x_1,x_2\in X,\quad x_1\preccurlyeq x_2\ implies\ F(x_1,y)\preccurlyeq F(x_2,y)$$

and

$$y_1,y_2\in X,\quad y_1\preccurlyeq y_2\ implies\ F(x,y_1)\succcurlyeq F(x,y_2).$$

Definition 1.5. An element $(x,y)\in X^2$ is called a coupled fixed point of the mapping $F:X\times X\to X$ if $F(x,y)=x$ and $F(y,x)=y$.

Theorem 1.6. [8] Let $(X, <)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is complete metric space. Let $F: X \times X \to X$ be a continuous mapping having the mixed monotone property on X. Assume that there exists a constant $k \in [0,1)$ with

$$d(F(x,y), F(u,v)) \leq (k/2)[d(x,u) + d(y,v)], for\ each\ x \succcurlyeq u, y \preccurlyeq v.$$

If there exists $x_0, y_0 \in X$ such that $x_0 \preccurlyeq F(x_0, y_0)$ and $y_0 \succcurlyeq F(y_0, x_0)$, then there exists $x, y \in X$ such that $x = F(x,y)$ and $y = F(y,x)$.

Many authours have studied on the existence of coupled fixed point theorems in the partially ordered metric space (see [8]-[14]). Lakshmikantham and Ciric [9] introduced the concept of mixed g −monotone mapping and proved related coupled coincidence point theorems. Also, Berinde [13] gave some more general results than the results given by Luong and Thuan [12].

Definition 1.7. [9] Let $(X, \preccurlyeq)$ be a partially ordered set and $F: X \times X \to X, g: X \times X \to X$ are mappings. We say F has the mixed g-monotone property if F is monotone g-non-decreasing in its first argument and monotone g-non-increasing in its second argument, that is for any $x, y \in X$,

$$x_1, x_2 \in X, \ \ g(x_1) \preccurlyeq g(x_2)\ implies\ F(x_1, y) \preccurlyeq F(x_2, y)$$

and

$$y_1, y_2 \in X, g(\ y_1) \preccurlyeq g(y_2)\ implies\ F(x, y_1) \succcurlyeq F(x, y_2).$$

It is clear that if g is considered an identity mapping, then we get the definition of mixed monotone property.

Definition 1.8. [9] An element $(x, y) \in X^2$ is called a coupled coincidence point of a mapping $F: X \times X \to X$, if $F(x,y) = g(x)$ and $F(y,x) = g(y)$.

Also, it is clear that when we consider g is identity mapping, then we obtain the definition of coupled fixed point.

Definition 1.9. [9] Let X be a non-empty set and $F: X \times X \to X$ and $g: X \to X$ be mappings. We say F and g are commutative if for all $x, y \in X g(F(x,y)) = F(gx, gy)$.

In this paper, we give an extended coupled coincidence point theorem for a mixed g-monotone operator $F: X \times X \to X$ satisfying a contractive condition weaker than the conditions given in some previous paper. As a result of our coupled coincidence point theorem, we obtain an extended coupled fixed point theorem. Also, we give a relationship between our main Theorem 2.1 and the Theorem 1.3 given by Choudhury and Kundu [5].

2. MAIN RESULTS

We denote by Ψ the set of functions $\psi: [0, \infty) \to [0, \infty)$ satisfying the properties;

1. ψ is continuous and monotone nondecreasing,
2. $\psi(t) = 0$ if and only if $t = 0$.

We denote by Φ
the set of functions $\alpha: [0, \infty) \to [0, \infty)$ satisfying the properties;

1. α is continuous,
2. $\alpha(t) = 0$ if and only if $t = 0$.

We denote by Γ the set of functions $\beta: [0, \infty) \to [0, \infty)$ satisfying the properties;

1. β is lower semi-continuous,
2. $\beta(t) = 0$ if and only if $t = 0$.

Teorem 2.1. Let $(X, \preccurlyeq)$ be a partially ordered set endowed with a metric d and (X, d) be a complete metric space. Let $F: X \to X$ and $g: X \to X$ be mappings such that F is mixed

g −monotone map and $\psi \in \Psi, \alpha \in \Phi, \beta \in \Gamma$. If for all $x, y, u, v \in X$ for which $g(x) \preccurlyeq g(u), g(y) \succcurlyeq g(v)$ such that

$$\psi\left(\frac{1}{2}\left[d\left(F(x,y), F(x,y) + d\big(F(y,x) + F(v,u)\big)\right)\right]\right)$$
$$\leq \alpha\left(\frac{1}{2}[d(gx,gu) + d(gy,gv)]\right) - \beta\left(\frac{1}{2}[d(gx,gu) + d(gy,gv)]\right) \ldots (E1)$$

where,

a) $F(X \times X) \subset g(X)$, g is continuous and commute with F,
b) for all $t > 0$, we have

$$\psi(t) - \alpha(t) + \beta(t) > 0 \ldots\ldots (E2)$$

Also, suppose that either

C1) F is continuous or
C2) X has the following property
i) if any nondecreasing sequence $\{x_n\}$ in X converges to x, then $x_n \preccurlyeq x$ for all $n \geq 0$,
ii) if any nonincreasing sequence $\{y_n\}$ in X converges to y, then $y \preccurlyeq y_n$ for all $n \geq 0$.

If there exists $x_0, y_0 \in X$ such that $g(x_0) \preccurlyeq F(x_0, y_0)$ and $g(y_0) \succcurlyeq F(y_0, x_0)$ then there exist $x, y \in X$ such that

$$g(x) = F(x,y) \text{ and } g(y) = F(y,x)$$

that is, F and g have a coupled coincidence point.

Moreover, if for every $(x,y), (y', x') \in X \times X$ there exists a $(u,v) \in X \times X$ such that

$$\big(F(u,v), F(v,u)\big) is\ comparable\ to\ \big(F(x,y), F(y,x)\big) and\ \big(F(x',y'), F(y',x')\big) \ldots (E3)$$

then F and g have a unique coupled coincidence point.

Proof. Let $x_0, y_0 \in X$ be an arbitrary point such that $g(x_0) \preccurlyeq F(x_0, y_0)$ and $g(y_0) \succcurlyeq F(y_0, x_0)$. Since $F(X \times X) \subset g(X)$, we can choose $x_1, y_1 \in X$ such that

$$g(x_1) = F(x_0, y_0)\ and\ g(y_1) = F(y_0, x_0).$$

Continueing this way, we have the sequences $\{g(x_n)\}$ and $\{g(y_n)\}$ in X such that

$$g(x_{n+1}) = F(x_n, y_n)\ and\ g(y_{n+1}) = F(y_n, x_n), \forall\, n \geq 0.$$

From mixed g-monotone property of F, we have

$$gx_0 \preccurlyeq F(x_0, y_0) = gx_1 \preccurlyeq F(x_1, y_1) = gx_2 \preccurlyeq \cdots$$

thus, we get

$$gx_n \preccurlyeq gx_{n+1}.$$

Again, from mixed g −monotone property of F, we have

$$gy_0 \succcurlyeq F(y_0, x_0) = gy_1 \succcurlyeq F(y_1, x_1) = gy_2 \succcurlyeq \cdots$$

thus, we get

$$gy_n \succcurlyeq gy_{n+1}.$$

If any two consecutive terms of above sequence are equal, then the proof is completed. Assume that all terms of the sequences are different.

For the convenience, we use the following notation;

$$G_n(x_0, y_0) := \frac{d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1})}{2}$$

If possible, let $G_{n-1}(x_0, y_0) < G_n(x_0, y_0)$ then we have

$\psi\big(G_{n-1}(x_0,y_0)\big) \le \psi\big(G_n(x_0,y_0)\big) = \psi\left(\frac{1}{2}[d(gx_n,gx_{n+1})+d(gy_n,gy_{n+1})]\right)$
$= \psi\left(\frac{1}{2}\begin{bmatrix} d\big(F(x_{n-1},y_{n-1}),F(x_n,y_n)\big) \\ +d\big(F(y_{n-1},x_{n-1}),F(y_n,x_n)\big)\end{bmatrix}\right)$
$\le \alpha\big(G_{n-1}(x_0,y_0)\big) - \beta\big(G_{n-1}(x_0,y_0)\big) \ldots.$ (1)

From the (E2), the last inequality implies that $G_n(x_0,y_0)=0$, but this ia a contradiction. Thus, we get that $\{G_n(x_0,y_0)\}$ is monotone decreasing sequence of non-negative real numbers. Therefore, there is $r \ge 0$ such that

$$\lim_{n\to\infty} G_n(x_0,y_0) = r.$$

Letting $n \to \infty$ in the (1), we have

$\psi(r) \le \alpha(r) - \beta(r) \ldots\ldots\ldots$ (2)

From, the condition (E2), the inequality (2) implies that

$\lim_{n\to\infty} G_n(x_0,y_0) = 0 \ldots\ldots\ldots$ (3)

Now, we prove that the sequences $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences in (X,d). Suppose at least one of $\{gx_n\}$ or $\{gy_n\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$ for which we can find two subsequences of integers $\{n(k)\}$ and $\{m(k)\}$ with $n(k) > m(k) \ge k$ such that

$\frac{1}{2}\big[d\big(gx_{n(k)},gx_{m(k)}\big)+d\big(gy_{n(k)},gy_{m(k)}\big)\big] \ge \varepsilon \ldots\ldots\ldots$ (4)

Note that we can choose $n(k)$ for the smallest integer satisfying the condition $n(k) > m(k) \ge k$ and

$$\frac{1}{2}\big[d\big(gx_{n(k)-1},gx_{m(k)}\big)+d\big(gy_{n(k)-1},gy_{m(k)}\big)\big] < \varepsilon$$

From the inequality (4) and by triangle inequality, we have

$$\begin{aligned}\varepsilon &\le \frac{1}{2}\big[d\big(gx_{n(k)},gx_{m(k)}\big)+d\big(gy_{n(k)},gy_{m(k)}\big)\big] \\ &\le \frac{1}{2}\begin{bmatrix} d\big(gx_{n(k)},gx_{n(k)-1}\big)+d\big(gy_{n(k)},gy_{n(k)-1}\big) \\ d\big(gx_{n(k)-1},gx_{m(k)}\big)+d\big(gy_{n(k)-1},gy_{m(k)}\big)\end{bmatrix}\end{aligned}$$

taking limit as $k \to \infty$, we get

$$\varepsilon \le \lim_{k\to\infty}\frac{1}{2}\big[d\big(gx_{n(k)},gx_{m(k)}\big)+d\big(gy_{n(k)},gy_{m(k)}\big)\big] \le \varepsilon$$

this implies that

$\lim_{k\to\infty}\frac{1}{2}\big[d\big(gx_{n(k)},gx_{m(k)}\big)+d\big(gy_{n(k)},gy_{m(k)}\big)\big] = \varepsilon \ldots\ldots\ldots$ (5)

Again, from (4), we have

$\psi(\varepsilon) \le \psi\left(\frac{1}{2}\big[d\big(gx_{n(k)},gx_{m(k)}\big)+d\big(gy_{n(k)},gy_{m(k)}\big)\big] \le \alpha\left(\frac{1}{2}\big[d\big(gx_{n(k)-1},gx_{m(k)-1}\big)+\right.\right.$
$\left.\left.d\big(gy_{n(k)-1},gy_{m(k)-1}\big)\big]\right)\right) - \beta\left(\frac{1}{2}\big[d\big(gx_{n(k)-1},gx_{m(k)-1}\big)+d\big(gy_{n(k)-1},gy_{m(k)-1}\big)\big]\right) \ldots\ldots\ldots$
(6)

Letting $k \to \infty$ in inequality (6) and use (5), we have

$\psi(\varepsilon) \le \alpha(\varepsilon) - \beta(\varepsilon) \ldots\ldots\ldots$ (7)

From the condition (E2), the inequality (7) implies that $\varepsilon = 0$. But, this case is in contradiction with $\varepsilon > 0$. Therefore our acceptance is incorrect and we conclude that the sequences $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences in (X,d). Since (X,d) is complete there exists $\bar{x},\bar{y} \in X$ such that

$$\lim_{n\to\infty} gx_n = \bar{x} \ and \ \lim_{n\to\infty} gy_n = \bar{y}$$

As g is continuous, we have

$$\lim_{n\to\infty} g(gx_n) = g(x) \ and \ \lim_{n\to\infty} g(gy_n) = g(y) \ldots\ldots \tag{8}$$

Also, note that F and g commutative, then we have

$$g(g(x_{n+1})) = g(F(x_n, y_n)) = F(gx_n, gy_n)$$

and

$$g(gy_{n+1}) = g(F(y_n, x_n)) = F(gy_n, gx_n)$$

Now, we show that the point $(\bar{x}, \bar{y}) \in X \times X$ is a coupled coincidence point of F and g. In here we have two cases.

Case 1: Let C1 holds, then we have

$$\begin{aligned} g(\bar{x}) = \lim_{n\to\infty} g(gx_{n+1}) = \lim_{n\to\infty} F(gx_n, gy_n) \\ = F\left(\lim_{n\to\infty} gx_n, \lim_{n\to\infty} gy_n,\right) \\ = F(\bar{x}, \bar{y}) \ldots\ldots.. \end{aligned} \tag{9}$$

Similarly, we have

$$\begin{aligned} g(y) = \lim_{n\to\infty} g(gy_{n+1}) = \lim_{n\to\infty} F(gy_n, gx_n) \\ = F(\lim_{n\to\infty} gy_n, \lim_{n\to\infty} gx_n) = F(\bar{y}, \bar{x}) \ldots\ldots \end{aligned} \tag{10}$$

From (9) and (10), we get

$$g(x) = F(x, y) \ and \ g(y) = F(y, x)$$

that is, $(\bar{x}, \bar{y}) \in X \times X$ is a coupled coincidence point of F and g.

Case 2: Let C2 holds. Note that $\{gx_n\}$ is nondecreasing and converges to $\bar{x}$, then $g(x_n) \preccurlyeq \bar{x}$ and similarly, $\{gy_n\}$ is nonincreasing and converges to $\bar{y}$, then $\bar{y} \preccurlyeq gy_n$. From triangle inequality of the metric function, we have

$$\begin{aligned} \psi\left(d\big(g(\bar{x}), F(\bar{x}, \bar{y})\big)\right) &\leq \psi\left(d\big(g(\bar{x}), g(gx_{n+1})\big) + d\big(g(gx_{n+1}), F(\bar{x}, \bar{y})\big)\right) \\ &\leq \psi\left(d\big(g(\bar{x}), g(gx_{n+1})\big)\right) + \psi\left(d\big(g(gx_{n+1}), F(\bar{x}, \bar{y})\big)\right) \\ &\leq \psi\left(d\big(g(\bar{x}), g(gx_{n+1})\big)\right) + \psi\left(d\big(F(gx_n, gy_n), F(\bar{x}, \bar{y})\big)\right) \\ &\leq \psi\left(d\big(g(\bar{x}), g(gx_{n+1})\big)\right) \\ &\quad + \alpha\left(\frac{d\big(g(gx_n), g(\bar{x})\big) + d\big(g(gy_n), g(\bar{y})\big)}{2}\right) \\ &\quad - \beta\left(\frac{d\big(g(gx_n), g(\bar{x})\big) + d\big(g(gy_n), g(\bar{y})\big)}{2}\right) \ldots\ldots\ldots \end{aligned} \tag{11}$$

Letting $n \to \infty$ in the last inequality (11) and use (8), we have

$$\psi\left(d\big(g(\bar{x}), F(\bar{x}, \bar{y})\big)\right) \leq 0$$

which implies that $g(\bar{x}) = F(\bar{x}, \bar{y})$. Similarly, we have $g(\bar{y}) = F(\bar{y}, \bar{x})$Thus, we get that $(\bar{x}, \bar{y}) \in X \times X$ is a coupled coincidence point of F and g.

Now, we prove that the coupled coincidence point is unique under additional condition (E3). Condition. Note that the set of coupled coincidence point of F and g is nonempty. Assume that (x', y') is another coincidence point of F and g. By assumption (E3), there is $(u, v) \in X^2$ such that $(F(u,v),F(v,u))$ is comparable to $(F(\bar{x},\bar{y}),F(\bar{y},\bar{x}))$ and $(F(x',y'),F(y',x'))$. From the (E1), we have

$$\psi\left(\frac{d\big(g(\bar{x}),g(x')\big)+d\big(g(\bar{y}),g(y')\big)}{2}\right)=\psi\left(\frac{d\big(F(\bar{x},\bar{y}),F(x',y')\big)+d\big(F(\bar{y},\bar{x}),F(y',x')\big)}{2}\right)$$
$$\le\alpha\left(\frac{d\big(g(\bar{x}),g(x')\big)+d\big(g(\bar{y}),g(y')\big)}{2}\right)$$
$$-\beta\left(\frac{d\big(g(\bar{x}),g(x')\big)+d\big(g(\bar{y}),g(y')\big)}{2}\right)$$

From (E2), the last inequality implies

$$\frac{d\big(g(\bar{x}),g(x')\big)+d\big(g(\bar{y}),g(y')\big)}{2}=0$$

and so we have

$$g(\bar{x})=g(x')\ and\ g(\bar{y})=g(y').$$

That is, the coupled coincidence point is unique.

Remark 2.2. In our Theorem 2.1, if we consider g as $g(x)=x$, then we obtain the following coupled fixed point theorem.

Theorem 2.3. Let $(X,\preccurlyeq)$ be a partially ordered set endowed with a metric d and (X,d) be a complete metric space. Let $F: X\to X$ be a mixed monotone mapping and there exist $\psi\in\Psi,\alpha\in\Phi$ and $\beta\in\Gamma$. If for all $x,y,u,v\in X$ for which $x\preccurlyeq u, y\succcurlyeq v$ such that

$$\psi\left(\frac{1}{2}\left[d\left(F(x,y),F(x,y)+d\big(F(y,x)+F(v,u)\big)\right)\right]\right)$$
$$\le\alpha\left(\frac{1}{2}[d(x,u)+d(y,v)]\right)-\beta\left(\frac{1}{2}[d(x,u)+d(y,v)]\right)$$

where for $\psi\in\Psi,\alpha\in\Phi,\beta\in\Gamma$, $\psi(t_1)\le\alpha(t_2)$, implies $t_1\le t_2$.

Also, suppose that either

C1) F is continuous or

C2) X has the following property

i) if any nondecreasing sequence $\{x_n\}$ in X converges to x, then $x_n\preccurlyeq x$ for all $n\ge 0$,

ii) if any nonincreasing sequence $\{y_n\}$ in X converges to y, then $y\preccurlyeq y_n$ for all $n\ge 0$.

If there exists $x_0,y_0\in X$ such that $x_0\preccurlyeq F(x_0,y_0)$ and $y_0\succcurlyeq F(y_0,x_0)$ then there exist $x,y\in X$ such that

$$x=F(x,y)\text{ and }y=F(y,x)$$

that is, F has a coupled fixed point.

Moreover, if for every $(x,y),(y',x')\in X\times X$ there exists a $(u,v)\in X\times X$ such that

$$\big(F(u,v),F(v,u)\big) is\ comparable\ to\ \big(F(x,y),F(y,x)\big) and\ \big(F(x',y'),F(y',x')\big)$$

then F has a unique coupled fixed point.

Remark 2.4 In Theorem 2.3, under the restriction $\psi(t)=\alpha(t)=\phi(t)$ and $\beta(t)=\psi(t)$, we obtain the Berinde's coupled fixed point theorem [11].

Corollary 2.5 [Berinde,11] Let (X,d) be a partially ordered set and suppose there is a metric d on X such that (X,d) is complete metric space. Let $F: X\times X\to X$ be mixed monotone mapping for which there exists $\phi\in\Phi$ and $\psi\in\Psi$ such that for all $x,y,u,v\in X$ with

$\phi\left(\frac{1}{2}\left[d\big(F(x,y),F(u,v)\big)+d\big(F(y,x),F(v,u)\big)\right]\right)\leq \phi\left(\frac{1}{2}[d(x,u)+d(y,v)]\right)-\psi\left(\frac{1}{2}[d(x,u)+d(y,v)]\right)$

Suppose either

(a) F is continuous or
(b) X satisfies
i) if any non-decreasing sequence $\{x_n\}$ in X converges to x, then $x_n \preccurlyeq x$ for all $n \geq 0$,
ii) if any nonincreasing sequence $\{y_n\}$ in X converges to y, then $y \preccurlyeq y_n$ for all $n \geq 0$.

If there exists $x_0, y_0 \in X$ such that $x_0 \preccurlyeq F(x_0, y_0)$ and $y_0 \succcurlyeq F(y_0, x_0)$ then there exist $x, y \in X$ such that

$$x = F(x,y) \text{ and } y = F(y,x)$$

that is, F has a coupled fixed point.

Example 2.6 [Choudhury and Kundu, 5**]** Let $X=\mathbb{R}$, $d(x,y)=|x-y|$ and $F\colon X\times X\to X$ defined by

$$F(x,y)=\begin{cases}\dfrac{x-3y}{5}, & x>3y\\ 0, & other\ cases\end{cases}$$

For all $x \geq u, y \leq v$, we have

$$\frac{d\big(F(x,y),F(u,v)\big)+d\big(F(y,x),F(v,u)\big)}{2}=\frac{1}{2}\left[\left|\frac{x-3y}{5}-\frac{u-3v}{5}\right|+\left|\frac{y-3x}{5}-\frac{v-3u}{5}\right|\right]=\frac{2}{5}[|x-u|+|y-v|]$$

And for $gx = 2x$, we have

$$\frac{d(gx,gu)+d(gy,gv)}{2}=|x-u|+|y-v|$$

If taken $\psi(t)=\alpha(t)=t$ and $\beta(t)={}^{3t}/_{7}$, then the mapping F fulfill all conditions of the Theorem 2.1 with the mapping g. Thus, F and g have a coupled coincidence point. Indeed, $(0,0)\in\mathbb{R}\times\mathbb{R}$ is a coupled coincidence point of F and g, that is,

$$g(x=0)=F(0,0)\ and\ g(y=0)=F(0,0).$$

At the present time, we prove a theorem that given a relation between a coupled coincidence point theorem and coincidence point theorem in the context of partially ordered set. As a result of this theorem we also obtain a relation between coupled fixed point theorem and fixed point theorem.

Theorem 2.7 Theorem 2.1 is equivalent to Theorem 1.3.

Proof. Let take a metric $d_2\colon X^2\times X^2\to[0,\infty)$ defined by

$$d_2(Y,V)=\frac{d(x,u)+d(y,v)}{2}$$

for all $(x,y)=Y\in X^2$ and $(u,v)=V\in X^2$.

It is clear that (X^2,d_2) is complete metric space whenever (X,d) complete metric space. Now, consider a map $T\colon X^2\to X^2$ defined for all $(x,y)=Y\in X^2$ by

$$T(Y)=\big(F(x,y),F(y,x)\big)$$

Therefore, we have

$d_2(TY,TV) = d_2\left(\left(F(x,y),F(y,x)\right),\left(F(u,v),F(v,u)\right)\right)$
$= \frac{1}{2}\left[d\left(F(x,y),F(u,v)\right) + d\left(F(y,x),F(v,u)\right)\right] \ldots\ldots\ldots \quad (12)$

And

$d_2\left(g(Y),g(V)\right) = d_2\left(g(x,y),g(u,v)\right)$
$= d_2\left((gx,gy),(gu,gv)\right)$
$= \frac{1}{2}\left[d(gx,gv) + d(gy,gv)\right] \ldots\ldots\ldots \quad (13)$

Thus, using the (12) and the (13) in the inequality (E1), we get

$\psi\left(d_2(TY,TV)\right) \le \alpha\left(d_2\left(g(Y),g(V)\right)\right) - \beta\left(d_2\left(g(Y),g(V)\right)\right) \ldots \quad (14)$

for all $Y,V \in X^2$ with order $Y \preccurlyeq V$.

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