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A ROBUST STABILITY TEST FOR BIMODAL SYSTEMS

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ABSTRACT

This paper studies robust stability problem for bimodal systems with continuous vector field. Firstly, a corollary is presented to construct common quadratic Lyapunov function to check the robust stability of the related system with norm-bounded uncertainties. Then, a mechanical system in bimodal configuration is provided and the switching mechanism of the system is simulated with nominal and norm-bounded uncertain parameters. The effectiveness of the study is demonstrated with simulation studies on spring-mass-damper system.
Keywords: Bimodal systems, norm-bounded uncertainty, robust stability.

BİMODAL SİSTEMLER İÇİN GÜRBÜZ KARARLILIK TESTİ

ÖZ

Bu çalışmada sürekli vektör alanına sahip çift durumlu sistemler için gürbüz kararlılık testi incelenmiştir. Çift durumlu sistemlerin normu sınırlı belirsizlikler içermeleri durumunda kararlı olup olmamalarının test edilmesi için doğrusal matris eşitsizliklerinin tanımlılığı üzerinden bir kriter geliştirilmiştir. Daha sonra geliştirilen kararlılık kriteri ile literatürde kullanılan, çift durumlu yapıda modellenmiş, mekanik bir sistem için kararlılık durumu test edilmiştir. Ayrıca, aynı model üzerinde değişken belirsizlik durumu da incelenip, bulunan kararlılık şartının etkin bir araç olduğu grafik sonuçları üzerinden yorumlanmıştır.
Anahtar Sözcükler: Çift durumlu sistemler, norm sınırlı belirsizlik, gürbüz kararlılık.

1. INTRODUCTION

Piecewise affine linear(PWL) systems, which are one of the fundamental classes of hybrid systems, consist of some pairs of linear time invariant dynamics and a switching surface which divides the state space into subspaces according to a criterion depending on the system dynamics. In this study, we deal with bimodal system with a continuous vector field that is a particular class of PWL systems with state depended switching. For those systems, it is assumed that the state space is covered by two conical regions and linear dynamics are separate on each regions in the continuous vector field. In this regard, as bimodal systems are quite simple compared to the

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multimodal subclasses of PWL systems, those systems are important for the notions of stability and stabilization to establish hybrid control theory.

Basic concepts of linear control theory such as well-posedness in [1,2], controllability in [3,4,5] and observability in [6,7] reported for the class of bimodal systems. Also, stability and stabilization notions of the PWL systems discussed deeply in the literature. Various system classes and different techniques considered in the literature; a stability test based on poles and zeros of subsystems for piecewise linear systems offered in [8] and stability analysis with a unified dissipativity approach of piecewise smooth systems presented in [9]. Stability tests for piecewise linear systems and quadratic stabilization for bimodal systems studied in [10] and [11]. Stability condition of hybrid system formed by a family of simultanously triangularizable matrices proposed in [12] by utilising the technique of common Lyapunov function. For stability of the linear complementarity system, which is a intrinsically piecewise linear, investigated in [13] and the authors adopted Lyapunov stability approach to present sufficient conditions for exponential stability. The authors also generalized LaSalle invariance theorem for linear complementarity system. Besides, stability and stabilization concepts of finite-dimensional hybrid systems classes presented in [14]. Last but not least, it should be noted that the remarkable study of J. L. Willems on stability conditions for nonlinear time-dependent feedback systems that is considered as a threshold matter to extend the stability results for the bimodal systems [15]. J. L. Willems obtained the stability condition by the technique of optimal quadratic Lyapunov function and showed that the results obtained for a second order system by means of the circle criterion which is the same with quadratic stability technique. He also proved that the results are valid in general for nonlinear time-dependent feedback systems which have feedback gains in open interval.

As seen from the brief summary of the literature, Lyapunov stability approach is the most adopted techniques to check stability for the related systems. But, it is very complicated issue to construct Lyapunov function especially for PWL systems. In this context, Hassibi and Boyd studied analysis and controller synthesis of PWL systems and their method was based on constructing quadratic and piecewise-quadratic Lyapunov functions [16]. They derived sufficient conditions for stability and performance analysis by the help of Lyapunov approach in the linear matrix inequality(LMI) context that can be also turned into convex optimization problem. They emphasized the importance of Lyapunov approach for the analysis and controller synthesis for related systems to get less conservative results and also proposed ellipsoidal outer approximation to the operating regions to reduce the conservatism for piecewise systems. Johansson and Rantzer developed a uniform and computationally tractable approach for stability analysis of a class of nonlinear systems with PWL dynamics and the solution tool for piecewise quadratic Lyapunov functions for nonlinear and hybrid systems which stated as a convex optimization problem in terms of the LMIs [17]. They also specified that, this approach is promising to generalize in a large number of directions such as performance analysis, global linearization, controller optimization and model reduction. Moreover, Samadi and Rodriguez presented a unified dissipative approach for stability analysis of piecewise smooth systems with continuous and discontinuous vector fields [9].

In spite of the the fact that, there has been established scientific background for bimodal systems, there still remain fundamental problems to check stability and robust stability characterization of such a simple class of piecewise affine system. However, it is known that the existence of a quadratic Lyapunov function is necessary and sufficient for asymptotic stability of linear time invariant(LTI) systems [10]. To this respect, we dealt with the stability problem such that under which conditions a number of convex combination of LTI systems share a common quadratic Lyapunov function(CQLF). We made an inference that the result of J. L. Willems' conceptual study on the quadratic stability of nonlinear time-dependent feedback systems may be benefitted as a starting point to derive better conditions for stability analysis of bimodal systems. From this point of view, we extended the stability result in [15] into the bimodal systems with norm-bounded uncertainty as considering the case that the feedback gains are in a closed interval. By this way, we derived novel LMI conditions on the dynamics for the bimodal systems with continuous vector field such that the robust stability of the related system is guaranteed.

The paper is organized as follows. This section ends with introducing the notational conventions used in this paper. After providing the definition of quadratic stability of bimodal systems, equivalent statements for the stability is presented in Section 2. This is followed by giving robust stability test condition for the related systems in Section 3. Next, it contains numerical examples and simulation studies to illustrate the proposed stability methods in Section 4. Finally, conclusion and directions for future research is presented in Section 5.

The following notational conventions will be in force through the paper. The transpose of a vector x(or matrix M) is denoted by x^T (M^T) and conjugate transpose by *. All inequalities involving a vector are understood componentwise.

2. BIMODAL SYSTEMS

Consider the bimodal piecewise linear system given by

$$\dot{x} = \begin{cases} A_1x(t) + bu(t) \;\; if \;\; c^Tx(t) \leq 0 \\ A_2x(t) + bu(t) \;\; if \;\; c^Tx(t) \geq 0 \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}$ input, $b \in \mathbb{R}^n$ input vector and $c \in \mathbb{R}^m$ is the slope of the hyperplane separating modes and all the matrices involved are of appropriate dimensions. In this case, the right-hand side of (1) is a Lipschitz continuous function. Hence, for each initial state x_0 and locally integrable input u there exists a unique absolutely continuous function $x^{(x_0,u)}$ such that (1) holds for almost all $t \in \mathbb{R}$ and $x^{(x_0,u)} = x_0$.

Before investigation of the robust stability conditions of the bimodal systems, we need to give nominal stability test of the related systems. So that, let us reconsider the autonomous bimodal system (1) without external input.

$$\dot{x} = \begin{cases} A_1x(t) \;\; if \;\; c^Tx(t) \leq 0 \\ A_2x(t) \;\; if \;\; c^Tx(t) \geq 0 \end{cases} \tag{2}$$

Through this paper, the authors assume that the right-hand side is a continuous function in x, or equivalently, there exists a vector $e \in \mathbb{R}^n$ such that

$$A_1 - A_2 = ec^T. \tag{3}$$

One can say that the bimodal system is quadratically stable if there exists a quadratic function $V: \mathbb{R}^n \longrightarrow \mathbb{R}$ such that $V(x) > 0$ for all $x \neq 0 \in \mathbb{R}^n$ and $dV(x(t))/dt < 0$ for all state trajectories x of (2) with $x \neq 0$. The following theorem gives us necessary and sufficient condition to check the stability of bimodal PWL systems.

Theorem 1: The following statements are equivalent.

1. The bimodal system (2) is quadratically stable.
2. There exists a symmetric positive definite matrix P such that

$$(A_1 - \mu ec^T)^T P + P(A_1 - \mu ec^T) < 0 \tag{4}$$

for all $\mu \in [0,1]$.

3. There exists a symmetric positive definite matrix K such that

$$\begin{bmatrix} A_1^T K + KA_1 & Ke - c \\ e^T K - c^T & -2 \end{bmatrix} < 0. \tag{5}$$

Note that, this theorem was proved in our earlier work [11].

3. ROBUST STABILITY TEST FOR BIMODAL SYSTEMS

In this chapter, we deal with robust stability problem of bimodal systems with norm-bounded uncertainties. In a similar fashion with stability analysis of bimodal systems, we also utilize the method of CQLF which gives necessary and sufficient conditions for stability of convex combination of LTI systems with norm-bounded uncertainties [18].Those uncertainties could occur due to disturbances, external inputs and unmodelled dynamics and should be clarified precisely in the stability formulation. For this purpose, let us to describe bimodal system with norm-bounded uncertainties as follows

$$\dot{x} = \begin{cases} (A_1 + \Delta A_1(t))x \ \ if \ \ c^T x(t) < 0 \\ (A_2 + \Delta A_2(t))x \ \ if \ \ c^T x(t) \geq 0 \end{cases} \tag{6}$$

We assume

$$A_1 - A_2 + \Delta A_1(t) - \Delta A_2(t) = ec^T \tag{7}$$

$$\Delta A_1(t) = E_1^T \Delta_1(t) D_1 \tag{8}$$

$$\Delta A_2(t) = E_2^T \Delta_2(t) D_2 \tag{9}$$

where E_1, E_2, D_1 and D_2 are known uncertainty matrices of appropriate dimensions. Note that, Δ_1 and Δ_2 are unknown Lebesque measurable functions of time which satisfy $\Delta_1^T \Delta_1 \leq I$ and $\Delta_2^T \Delta_2 \leq I$. With this preparition, we are ready to give the following corollary which presents a robust stability condition for system (6) in the view of (7-9).

Corollary 2: System (6) with (7-9) is robustly stable if there exist a positive symmetric matrix M and a positive constant ϵ such that the following condition holds.

$$\begin{bmatrix} A_1^T M + M A_1 + \epsilon E_1^T E_1 & Me - c & D_1^T \\ * & -2 & 0 \\ * & * & -\epsilon I \end{bmatrix} < 0 \tag{10}$$

Proof: Consider the inequality given in (5), exchanging A_1 with $A_1 + \Delta A_1(t)$ and K with $M = M^T > 0$ the condition turns into

$$\begin{bmatrix} (A_1 + \Delta A_1)^T + M(A_1 + \Delta A_1) & Me - c \\ (Me - c)^T & -2 \end{bmatrix} < 0 \tag{11}$$

and substituting (8) and (9) into (11) yields,

$$\begin{bmatrix} A_1{}^T K + D_1^T \Delta_1^T(t) E_1 \mathrm{M} \ + M A_1 + M E_1^T \Delta_1(t) D_1 & Me - c \\ (Me - c)^T & -2 \end{bmatrix} < 0. \tag{12}$$

Note that (12) can be decomposed as,

$$\begin{bmatrix} A_1^T M + M A_1 & Me - c \\ e^T M - c^T & -2 \end{bmatrix} + \begin{bmatrix} D_1^T \\ 0 \end{bmatrix} \Delta_1^T(t) [E_1 \quad 0] + \begin{bmatrix} E_1^T \\ 0 \end{bmatrix} \Delta_1^T(t) [D_1 \quad 0] < 0. \tag{13}$$

Let us to define

$$\tilde{A} = \begin{bmatrix} A_1^T M + M A_1 & Me - c \\ e^T M - c^T & -2 \end{bmatrix}, \tilde{E} = \begin{bmatrix} E_1^T \\ 0 \end{bmatrix} \text{ and } \widetilde{D} = [D_1 \quad 0].$$

Then, some straightforward calculations yield

$$\tilde{A} + \widetilde{D} \Delta_1(t) \breve{E} + \tilde{E}^T \Delta_1(t) \widetilde{D}^T < 0. \tag{14}$$

However, one can find a positive constant ϵ such that

$$E_1^T \Delta_1(t)^T D_1 + D_1^T \Delta_1(t) E_1 \leq \frac{1}{\epsilon} E_1^T E_1 + \epsilon D_1^T D_1 \tag{15}$$

holds [19]. Therefore in the view of (15), if

$$\tilde{A} + \frac{1}{\epsilon}\tilde{D}\tilde{D}^T + \epsilon\tilde{E}^T\tilde{E} < 0 \tag{16}$$

holds then (14) holds too. Then applying Schur complement to (16) allows to write

$$\begin{bmatrix} \tilde{A} + \epsilon\tilde{E}\tilde{E}^T & \tilde{D} \\ \tilde{D}^T & -\epsilon I \end{bmatrix} < 0 \tag{17}$$

which is nothing but (10).

Remark 3: Corollary 2 clearly states that one needs only check the above LMI based condition to determine the stability of bimodal systems with norm-bounded uncertainty.

4. SIMULATION STUDIES

Considering the mechanical system shown in Figure 1, let the mass of the cart denoted by m_1, damping constant by d and the spring constants by k for the left most and k_1 for the other. Defining the state variables as x_1 and x_2 which represents position from the reference point and velocity of the mass respectively, allow to obtain governing equations

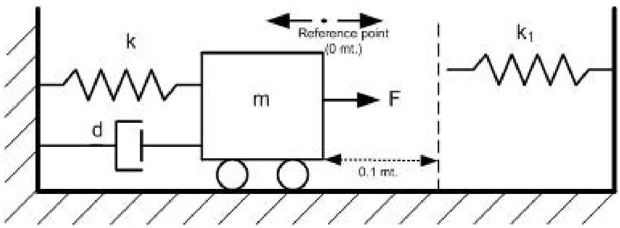

Figure 1. Bimodal configurated spring-mass-damper mechanical system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{cases} \begin{bmatrix} 0 & 1 \\ -k/m & -d/m \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} & if\ c^T X < 0.1 \\ \begin{bmatrix} 0 & 1 \\ -(k+k_1)/m & -d/m \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} & if\ c^T X \geq 0.1 \end{cases} \tag{18}$$

where c and state vector X are $[0.1 \quad 0]^T$ and $[x_1 \quad x_2]$, respectively.

According to Theorem 1, let us to check the nominal stability of bimodal system given in (18) as having the system parameters be as $m = 250$ kg, $k = 160000$ N/m, $k_1 = 16000$ N/m and $d = 1000$ Ns/m. In this context, firstly, we need to solve the LMI in (5) to find a common symmetric definite positive matrix K. Hence, we conduct a computer simulation using LMILab toolbox in MATLAB© that is giving the following result

$$K = \begin{bmatrix} 1.2185 & 0.0016 \\ 0.0016 & 0.0019 \end{bmatrix} < 0. \tag{19}$$

Now, reformulate the system in Figure 1 with parametric uncertainty under the assumption of $||\Delta|| \leq I$. Immediately, the governing equations can be obtained as follows.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{cases} \begin{bmatrix} 0 & 1 \\ -k/m & -d/m \end{bmatrix} + [\Delta A_1(t)]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} & if\ c^T X < 0.1 \\ \begin{bmatrix} 0 & 1 \\ -(k+k_1)/m & -d/m \end{bmatrix} + [\Delta A_2(t)]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} & if\ c^T X \geq 1 \end{cases} \tag{20}$$

It is assumed that the spring parameter k and damper parameter d have 60% admissible uncertainties in their exact values. So that, one can assign the vectors E_1 and D_1 as $[0 \quad 1]^T$ and $[0.6 * k/m \quad 0.6 * d/m]$, respectively. Now, Corollary 2 can be applied to check the stability of the bimodal system with the norm-bounded uncertainities. Let the system parameters are same as the previous example and solving the LMI in (10) to find a positive real matrix M via Matlab-LMIlab solver, one get the following results with $\epsilon = 1.3989$

$$K = \begin{bmatrix} 42.2501 & 2.1272 \\ 2.1272 & 0.7813 \end{bmatrix}. \tag{21}$$

Finally, we present elementary simulation to visualize the switching mechanism of bimodal configured spring-mass-damper mechanical system. For this purpose, we deal with three cases to check the behaviour of the system. As a first step, as regarding exact knowledge case which the bimodal system operates with nominal parameters, we present Figure 2 to illustrate the changing of state variable x_1. Then, we deal with the worst constant uncertain case which is allowed minus 60% change in k and d parameters and render the changing of x_1 in Figure 3. Finally, we consider time varying uncertainty case for the parameter of k that is taken as $0.6 * k * sin(2 * pi * t)$ and we provide Figure 4 to show the the change of location(x_1) of the system. Consequently, it should be noted that decreasing on damper coefficient affects to the system at most. Hence, one can state that constant type uncertainty affects the stability of system more than the sinusoidal type uncertainty with respect to time. Besides, it could be underlined that sinusoidal type uncertainty affects to attenuation path more than the other type of uncertainties. It should be also noted that, the approach in the paper incorparates LMI condition to check the stability which means that one can transform the condition into optimization problem easily. Hence, this issue could be seen as the main advantageous of the approach. On the other hand, one needs to limit parametric uncertainty as a specific value to use the norm-bounded approach. In other words, the approach only provides sufficient condition. So, an assumption to bound uncertainty is needed to get the stability condition. In the example, we determine the parametric uncertainty bound as 60% of the nominal value to find a CQLF from the optimization problem formulation, which can be considered as weakness of the approach. But, it must be underlined that this approach provides both ease of handling the problem formulation and numerical solution. Consequently, numerical examples and simulation studies demonstrate the applicability of the results provided.

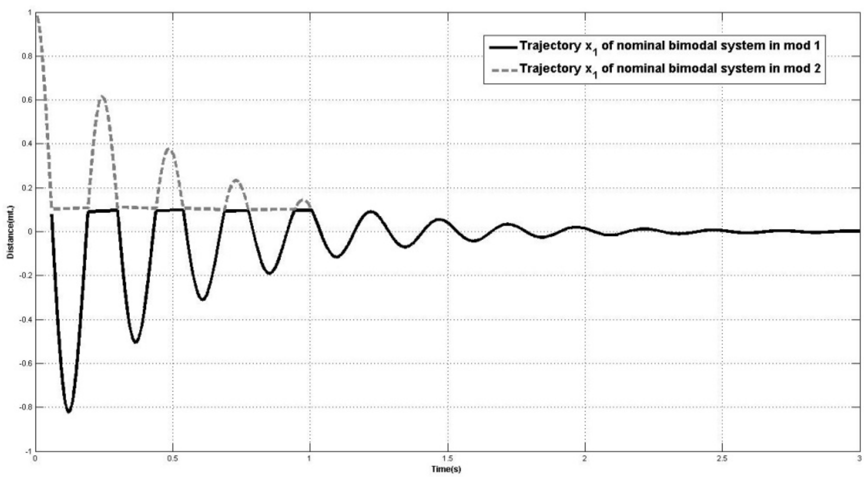

Figure 2. Simulation of bimodal configurated nominal spring-mass-damper mechanical system

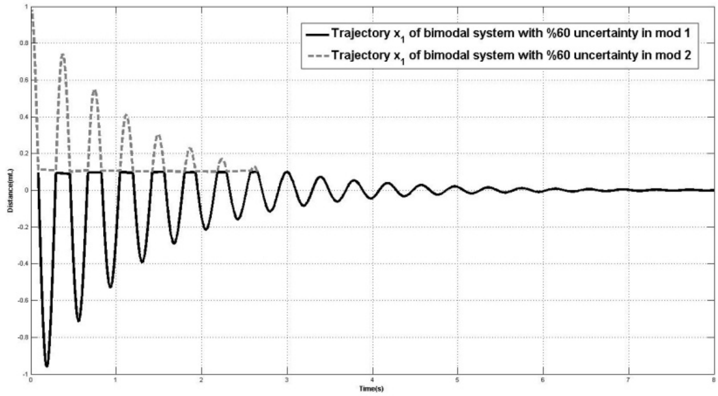

Figure 3. Simulation of bimodal configurated spring-mass-damper mechanical system with constant uncertainty

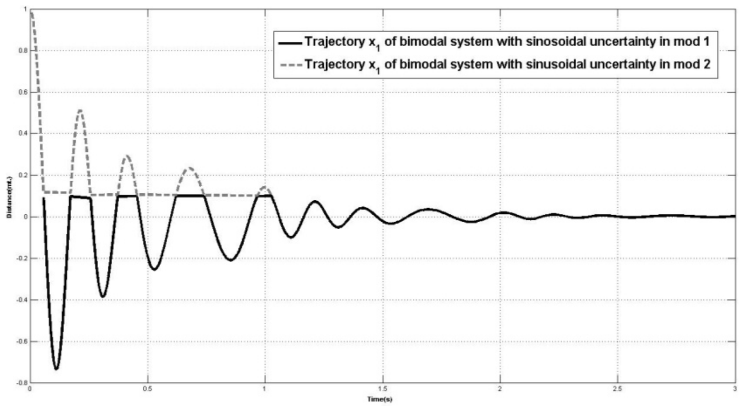

Figure 4. Simulation of bimodal configurated spring-mass-damper mechanical system with sinusoidal uncertainty

5. CONCLUSION

Stability checking problem for bimodal systems with norm-bounded uncertainties is addressed and sufficient condition is derived for robust stability notion of bimodal systems based on nominal stability test of bimodal systems. We present a corollary to test the stability of the related systems in the presence of some unmodeled uncertainty which may cause a possible loss of exponential stability. The numerical examples illustrate the computational effectiveness of the LMI based approach. As a furture study, the stability results obtained in this paper can be extended into the method that is covering the larger classes of uncertainty families by the help of geometric control approach and optimization techniques.

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