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A MORE COMPLETE THERMODYNAMIC FRAMEWORK FOR FLUENT CONTINUA

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Dedicated to Professor J.N. Reddy's 70th Birthday

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ABSTRACT

Polar decomposition of the changing velocity gradient tensor in a deforming fluent continua into pure stretch rates and rates of rotations shows that a location and its neighboring locations can experience different rates of rotations during evolution. Alternatively, we can also consider decomposition of the velocity gradient tensor into symmetric and skew symmetric tensors. The skew symmetric tensor is also a measure of pure rates of rotations whereas the symmetric tensor is a measure of strain rates. The measures of the internal rates of rotations due to deformation in the two approaches describe the same physics but in different forms. Polar decomposition gives the rate of rotation matrix and not the rates of rotation angles whereas the skew symmetric part of the velocity gradient tensor yields rates of rotation angles that are explicitly defined in terms of velocity gradients. These varying rates of rotations at neighboring locations arise due to varying deformation of the continua, hence are internal to the volume of matter and are explicitly defined by deformation. If the internal varying rates of rotations are resisted by the continua, then there must exist internal moments corresponding to these. The internal rates of rotations and the corresponding moments can result in additional rate of energy storage or rate of dissipation. This physics is all internal to the deforming continua and exists in all deforming isotropic, homogeneous fluent continua but is completely neglected in the presently used thermodynamic framework for fluent continua. In this paper we present derivation of a more complete thermodynamic framework in which the derivation of the conservation and balance laws consider additional physics due to varying rates of rotations. The currently used thermodynamic framework for fluent continua is a subset of the thermodynamic framework presented in this paper. The continuum theory presented here considers internal varying rates of rotations and the associated conjugate moments in the derivation of conservation and balance laws, thus the theory presented in this paper can be called "a polar continuum theory" but is different than micropolar continuum theories published currently in which material points have six external degrees of freedom i.e. the rotation rates are additional external degrees of freedom. In the remainder of the paper we refer to this new thermodynamic framework as 'a polar continuum theory'.

The continuum theory presented here only accounts for internal rotation rates and associated moments that exist as a consequence of deformation but are neglected in the present theories hence this theory results in a more complete thermodynamic framework. The polar continuum theory and the resulting thermodynamic framework presented in this paper is suitable for compressible as well as incompressible thermoviscous fluent continua such as Newtonian, Power law, Carreau-Yasuda fluids etc. and thermoviscoelastic fluent continua such as Maxwell, Oldroyd-B, Giesekus etc. The thermodynamic framework presented here is applicable to all isotropic, homogeneous fluent continua. Obviously the constitutive theories will vary depending upon the choice of physics. These are considered in subsequent papers.

1 INTRODUCTION

In deforming fluent continua, the velocities and the velocity gradients are fundamental quantities of the measure of deformation of the matter. In general, velocity gradients may vary between different locations i.e. they may vary between a location and its neighboring locations. Polar decomposition of the velocity gradient tensor at a location into rates of stretches (left or right) and rates of rotations shows that if the velocity gradient tensor varies between a location and the neighboring locations so does the rate of rotation tensor. We could also consider the decomposition of the velocity gradient tensor into symmetric and skew symmetric tensors. The skew symmetric tensor is a measure of pure rates of rotations while the symmetric tensor is a measure of strain rates. The measures of the internal rates of rotations due to deformation in these two approaches describe the same physics but in different forms. Polar decomposition gives the rates of rotation matrix and not the rates of rotation angles, whereas the skew symmetric part of the velocity gradient tensor yields rates of rotation angles that are explicitly defined in terms of velocity gradients. Strain rate measures are purely a function of stretch rates or alternatively symmetric part of the velocity gradient tensor. In these measures, the rate of rotation tensor plays no role. If these varying internal rates of rotations between neighboring locations in the deforming fluent continua are resisted by the continua then there must exist internal moments corresponding to these. The internal rates of rotations and the corresponding moments can result in rate of energy storage or rate of dissipation. This physics exists in all deforming fluent continua, but its degree may vary depending upon the constitution of the matter and the type of the deformation field. This physics is not considered in the derivation of conservation and balance laws that constitute the thermodynamic framework we are currently using for fluent continua. The answer to the question of what we should call the resulting continuum theory that incorporates the physics associated with internal rates of rotations and the corresponding moments is inherent in the description of the physics that the derivation of the theory incorporates. Since the theory accounts for internal rotation rates and associated moments, it is undoubtedly 'a polar continuum theory': (i) that only accounts for internal physics of rates of rotations resulting from the velocity gradient tensor and the conjugate moments (ii) that does not require rotations as additional external degrees of freedom as this theory is only intended to accommodate physics associated with internally varying rates of rotations that arise due to the varying velocity gradient tensor between points. Thus, henceforth we shall refer to the continuum theory presented here as 'a polar continuum theory' implying that there may be others that account for different physics of rates of rotation and moments than considered here. In non-polar continuum theories (current thermodynamic framework for fluent continua) used mostly for fluent continua, stress and strain rates alone contribute to the dissipation i.e. entropy production due to mechanical work. In such theories the influence of varying internal rates of rotations is completely neglected in the theory, hence on the dissipation mechanism as well.

In the present work we consider fluent continua in which the rates of rotations that exist between neighboring locations are resisted by the constitution of the matter, hence can result in additional dissipation. Thus, the polar continuum theory presented here considers strain rate tensor as well as rate of rotation tensor arising from the velocity gradient tensor in the derivation of the conservation and balance laws. The theory presented here should not be confused with micropolar continuum theories [1–21] that are designed to accommodate effects at scales smaller than the continuum scale. Micropolar theories require definitions of additional strain measures [14] related to the micromechanics. The polar continuum theory presented here uses standard measures of strain as used currently in non-polar continuum theories. In the polar continuum theory presented here, the motivation is to account for the influence of different rotation rates at neighboring locations that arise due to different velocity gradient tensors as this can result in mechanical energy dissipation in some fluent media. Polar decomposition of the velocity gradient tensor at neighboring locations clearly substantiates the validity of this concept, hence the motivation. Another significant point to note is that the theory considered here can only account for internal local rates of rotations due to deformation, hence this is an intrinsically local polar continuum theory and thus cannot account for nonlocal effects.

In the following we present a brief literature review of the published works that are pertinent in context with the work considered in this paper. The literature review on micropolar theories, stress couple theories and non-local theories is considered as these consider the effects of rotations generally as external degrees of freedom. Even though some of these works may appear to have no direct connection with the work presented in this paper, many of the concepts and derivation details in the cited references are quite helpful in following the details presented in this paper. A thorough exposition of micropolar theories has been given in references [1–21]. These theories consider measures of microdeformation due to microconstituents in the continuum. In references [22–35] various aspects of micropolar theories, stress couple theories for bending, buckling, vibration of beams, microstructure dependent beam theories, rotation gradient theory and strain gradient theory are considered for solid continua. In most cases use of strain energy density function and principle of virtual work is made in the derivations. These obviously hold for thermoelastic solids only in which the deformation process is irreversible. These concepts and derivations cannot be used for thermoviscoelastic solids with or without memory as in such cases the deformation process is not reversible. We also remark that rotation gradient theory and others cited here for solid matter are not applicable for fluent continua considered in this paper as the displacements of the material points are not available and the fluent continua require consideration of varying internal rotation rates due to varying velocity gradient tensor between neighboring locations without regards to displacements. Nonetheless we have cited these works due to some similarity of concepts between the solid media and the fluent media.

Much of the published work on polar continuum theories is for solid matter, based on consideration of displacements as well as rotations as independent degrees of freedom at the material points in Lagrangian description. This is obviously different than what is considered in the work presented in this paper. First, we consider fluent continua, hence displacements and rotations, as used for solid continua are not available as measures of deformation. Based on the physics of deformation of fluent continua, velocities and velocity gradients must be considered. Secondly, the present work focuses and incorporates the physics due to internal varying rates of rotations due to varying velocity gradient tensor between neighboring locations, hence does not require additional external rotations as degrees of freedom. These two differences clearly distinguish the present work from the published works cited above and those mentioned in the following. In reference [36] Altenbach and Eremeyev present a linear theory for micropolar plates. Each material point is regarded as a small rigid body with six degrees of freedom. Kinematics of plates is described using the vector of translations and the vector of rotations as dependent variables. Equations of equilibrium are established in $\mathbb{R}^3$ and $\mathbb{R}^2$. Strain energy density function is used to present linear constitutive theory. The mathematical models of reference [37] are extended by the same authors to present strain rate tensors and the constitutive equations for inelastic micropolar materials. In reference [38], authors consider the conditions for the existence of the acceleration waves in thermoelastic micropolar media. The work concludes that the presence of the energy equation with Fourier heat conduction law does not influence the wave physics in thermoelastic micropolar media. Thus, from the point of view of acceleration waves in thermoelastic polar media, thermal effects i.e. temperature can be treated as a parameter. In reference [39], authors present a collection of papers related to the mechanics of continua dealing with micro-macro aspects of the physics (largely related to solid matter). In reference [40] a micro-polar theory is presented for binary media with applications to phase-transitional flow of fiber suspensions. Such flows take place during the filling state of injection molding of short fiber reinforced thermoplastics. A similarity solution for boundary layer flow of a polar fluid is given in reference [41]. In specific the paper borrows constitutive equations that are claimed to be valid for flow behavior of a suspension of very fine particles in a viscous fluid. Kinematics of micropolar continuum is presented in reference [42]. References [43, 44] consider material symmetry groups for linear Cosserat continuum and non-linear polar elastic continuum. Grekova et. al. [45–47] consider various aspects of wave processes in ferromagnetic medium and elastic medium with microrotations as well as some aspects of linear reduced Cosserat medium. In references [48–66] various aspects of the kinematics of micropolar theories, stress couple theories, etc. are discussed and presented including some applications to plates and shells.

Based on the literature review we make some remarks. First, most literature is related to micropolar theories that require consideration of additional measures of strains related to micromechanics. Such theories necessitate rotation (or rates of rotations) as additional degrees of freedom. Conjugate to the rotations or rates of rotations are of course moments. In case of so called stress couple theories the physics considered is not clear at the onset. It is only after the derivation of balance laws that one gets some idea regarding what these theories can possibly do. The fundamental question of what these theories correct or complete or supplement when compared with current non-polar continuum theories is not clear.

The work presented in this paper is formulated based on observed physics, that in any deforming fluent media the polar decomposition of the velocity gradient tensor shows that the rates of rotations vary between neighboring locations. If the varying rotation rates and their gradients result in energy storage or dissipation, then its energy conjugate moment tensor must exist in the deforming matter. This necessitates the existence of moment (per unit area) on the oblique plane of the deformed tetrahedron. Thus, at the onset, we consider average force per unit area and velocities, and average moment per unit area and the rates of rotations on the oblique plane of the deformed tetrahedron. The work presented here follows strictly thermodynamic approach i.e. for fluent continua we present derivations of: (i) conservation of mass and present reasons for not deriving conservation of inertia (ii) balance of linear momenta (iii) balance of angular momenta (iv) balance of moments of moments (or couples) (v) first law of thermodynamics and (vi) second law of thermodynamics based on stress and strain rates, moment and rotation rates as energy conjugate pairs. The mathematical description for fluent continua derived here is applicable to compressible and incompressible thermoviscous fluids as well as thermoviscoelastic fluids when augmented with the appropriate constitutive theories. We reiterate and point out that the theory for fluent continua presented here incorporates additional physics due to rates of rotations which is neglected in the currently used thermodynamic framework. Thus this theory presents a more complete form of thermodynamic framework for isotropic, homogeneous fluent continua as it incorporates additional physics due to varying rates of rotations which is neglected in the current thermodynamic framework. The currently used thermodynamic framework is retained as a subset of the thermodynamic framework presented in this paper.

2 MATHEMATICAL DESCRIPTION FOR FLUENT CONTINUA

For a deforming volume of matter, whether solid or fluid, material particles and their motion i.e. displacements are the most fundamental quantities. If x_i is the position of a material particle in the reference configuration then its coordinates $\bar{x}_i$ in the current configuration can be determined using $\bar{x}_i = x_i + u_i$ in which u_i are the displacements. This physics exists in all deforming continua. Based on this we can derive conservation and balance laws using deformed tetrahedron in the current configuration (Fig. 1 (b)) and its corresponding undeformed counterpart in the reference configuration (Fig. 1 (a)). If the resulting equations are expressed as functions of x_i and t, then we have a Lagrangian description. On the other hand, if the resulting equations are a function of $\bar{x}_i$ and t then we have Eulerian description. Due to the fact that $\bar{x}_i = x_i + u_i$, the Lagrangian and Eulerian descriptions are identical mathematical representations of the same physics. Using $\bar{x}_i = x_i + u_i$ we can easily convert one type of description to another type without any loss of information. At this stage the Lagrangian and the Eulerian descriptions are equally suited for solid as well as fluent continua and have total transparency in deriving one from the other. If some special consideration of the physics in a continua requires some modification in either one of the two descriptions, then the transparency between the two will obviously be lost. We consider specific cases in the following. Refer to reference [67] (Chapters 6 and 7) for details.

In case of solids the material points are identified (x_i) and their displacements are monitored (u_i) hence $\bar{x}_i = x_i + u_i$ holds at each material point, thus the Lagrangian and Eulerian descriptions are equivalent, therefore either one can be used for the mathematical description of the physics.

Due to complex motion of fluid particles, monitoring of their motion i.e. displacements is not feasible. Thus, in the case of fluent continua, the first adjustment required by physics of complex motion is not to monitor material point displacements (u_i). This of course suggests that we do not know the whereabouts of the material points during evolution. Deformed positions $\bar{x}_i$

of the material points in the current configuration is only due to displacements u_i which we do not have anymore. Since we cannot monitor displacements of the material particles in fluent continua, it is perhaps fitting in case of fluent continua not to label the material points. Thus, in case of fluent continua we ignore material point displacements i.e. the motion of the material points during the evolution. The only other alternative left at this stage is that we consider fixed locations in the flow at which we monitor the state of the continua (temperature, velocity, etc.) during evolution. These fixed locations are occupied by different fluid particles during evolution. Thus, we could view these locations as current positions of different fluid particles for different values of time. As time elapses the fluid particles currently occupying these positions leave their positions which in turn are occupied by other fluid particles. Here, there are two important things to note: (i) each fixed location is the current position of some fluid particle, hence perhaps appropriate to label these as $\bar{x}_i$, keeping in mind that there are no x_i as u_i are not monitored (ii) we do not know which fluid particles are at which locations. Monitoring the state of fluent continua (velocities, temperature, etc.) at each location provides the evolution of the deforming continua.

We need to determine what mathematical model would be able to describe the physics that we have just discussed. Since the locations at which the evolution is monitored, though fixed, are current locations of different material particles at different values of time. This perhaps suggests that we can begin by choosing Eulerian description in which $\bar{x}_i$ are the fixed locations. In order for this mathematical model to be applicable for fluids, $\bar{u}_i$, $\bar{u}_{i,j}$ must be eliminated. The resulting mathematical model does not contain u_i and x_i nor does it require their use. We must decide what to call this mathematical model, certainly not Eulerian as a true Eulerian description requires x_i and u_i so that its counterpart Lagrangian description can be obtained transparently. In this model, u_i do not exist, hence neither do the strains. This is perfectly fine for fluids as in the case of fluid motion description displacements and strain measures play no role; instead velocities at $\bar{x}_i$ (fixed) and their gradients (strain rates) are fundamental measures of deformation. In summary we have: (i) Eulerian description in which $\bar{x}_i$ are fixed locations (ii) u_i (or $\bar{u}_i$) are assumed zero hence all strain measures are zero as well (iii) velocities $\bar{v}_i$ and its gradients $\frac{\partial \bar{v}_i}{\partial \bar{x}_j}$ are fundamental quantities in the kinematic description of motion using conservation and balance laws. This description is what is used currently in fluid mechanics. In the absence of $\bar{u}_i$ and x_i this description can not be a true Eulerian description. The origin of the derivation of this mathematical model is true Eulerian description with the restriction that we do not have $\bar{u}_i$ and x_i available to us. The derivation of the conservation and balance laws for polar fluent continua in this paper are presented utilizing this approach, i.e. configurations in figure 1 (a) and (b) are assumed to exist at the onset and during the derivation of conservation and balance laws, but at the end only the Eulerian description is retained with the restriction that $\bar{u}_i = 0$ and $\bar{x}_i$ in the model are the fixed locations at which the evolution is monitored. In simple terms we follow Eulerian description but ensure that x_i and $\bar{u}_i$ are not part of the formal mathematical model. Thus, in all subsequent material in this paper use of 'Eulerian description' refers to what has been defined here as Eulerian description for fluent continua.

We use an over bar on quantities to express quantities in the current configuration in Eulerian description, that is, all quantities with over bars are functions of current coordinates $\bar{x}_i$ and time t. We denote $\bar{\rho}$ to be the density of the fluid in the current configuration and $\bar{\phi}$, $\bar{\theta}$, and $\bar{\eta}$ denote the Helmholtz free-energy density, temperature, and entropy density, respectively. $\bar{\boldsymbol{\sigma}}^{(0)}$ is the Cauchy stress tensor (in Eulerian description in contravariant basis). The superscript '0' is used to signify that it is rate of order zero and the lowercase parenthesis distinguish it from the second Piola-Kirchhoff stress tensor $\boldsymbol{\sigma}^{[0]}$ used in Lagrangian description. Dot on any quantity refers to the material derivative.

If the existence of different rates of rotation at neighboring locations, as evident from the polar decomposition of the velocity gradient tensor, can result in additional mechanical energy dissipation, then there must also coexist energy conjugate moments in the deforming matter. Just like forces and velocities result in rate of work, moments and rates of rotation can also result in rate of work. Thus in the development of the polar continuum theory in Eulerian description for fluent media we consider existence of moments and rotation rates independent of forces and velocities. Consider a volume of matter $\underset{\sim}{V}$ in the reference configuration (figure 1 (a)) with closed boundary $\partial \underset{\sim}{V}$. The volume V is isolated from $\underset{\sim}{V}$ by a hypothetical surface ∂V as in the cut principle of Cauchy. Consider a tetrahedron T_1 shown in figure 1 (a) such that its oblique plane is part of ∂V and its other three planes are orthogonal to each other and parallel to the planes of the x-frame. Upon deformation, $\underset{\sim}{V}$ and $\partial \underset{\sim}{V}$ occupy $\bar{\underset{\sim}{V}}$ and $\partial \bar{\underset{\sim}{V}}$ and likewise V and ∂V deform into $\bar{V}$ and $\partial \bar{V}$. The tetrahedron T_1 deforms into $\bar{T}_1$ whose edges (under finite deformation are non-orthogonal covariant base vectors $\tilde{g}_i$. The planes of the tetrahedron formed by the covariant base vectors are flat but obviously non-orthogonal to each other. We assume the tetrahedron to be the small neighborhood of material point $\bar{o}$ so that the assumption of the oblique plane $\bar{A}\bar{B}\bar{C}$ being flat but still part of $\partial \bar{V}$ is valid. When the deformed tetrahedron is isolated from volume $\bar{V}$ it must be in equilibrium under the action of disturbance on surface $\bar{A}\bar{B}\bar{C}$ from the volume surrounding $\bar{V}$ and the internal fields that act on the flat faces which equilibrate with the mating faces in volume $\bar{V}$ when the tetrahedron T_2 is placed back in the volume $\bar{V}$. Consider the deformed tetrahedron $\bar{T}_1$. Let $\bar{\boldsymbol{P}}$ be the average stress per unit area on plane $\bar{A}\bar{B}\bar{C}$, $\bar{\boldsymbol{M}}$ be the average moment per unit area on plane $\bar{A}\bar{B}\bar{C}$ henceforth referred to as moment for short, and $\bar{\boldsymbol{n}}$ be the normal to the face $\bar{A}\bar{B}\bar{C}$. $\bar{\boldsymbol{P}}$, $\bar{\boldsymbol{M}}$, and $\bar{\boldsymbol{n}}$ all have different directions.

2.1 Polar decomposition of velocity gradient tensor and consideration of local rotation rates

Polar decomposition of the velocity gradient tensor is helpful in decomposing deformation into stretch rate tensor and rotation rate tensor. Whether we use left stretch rate tensor or right stretch rate tensor, the rotation rate tensor is unique. Thus, at each location with infinitesimal volume surrounding it, the velocity gradient tensor $[\bar{L}]$ can be decomposed into pure rates of rotation $[{}^t\bar{R}]$ and right or left stretch rate tensors $[{}^t\bar{S}_r]$ and $[{}^t\bar{S}_l]$. $[{}^t\bar{R}]$ is orthogonal and $[{}^t\bar{S}_r]$ and $[{}^t\bar{S}_l]$ are symmetric and positive definite. The rotation rate tensor can equivalently be obtained due to rotation rates ${}^t\bar{\boldsymbol{\Theta}}$ at each location in the flow domain.

Thus, at each location int the flow domain the rotation rate $[{}^t\bar{R}]$ matrix can be viewed as being due to ${}^t\bar{\boldsymbol{\Theta}}$. If varying rotation rates at varying locations in the flow domain are resisted by the constitution of the fluent continua then this must result in additional dissipation that requires existence of energy conjugate moments $\bar{\boldsymbol{M}}$ in the deforming matter. Thus, at the onset ${}^t\bar{\boldsymbol{\Theta}}$ and its conjugate $\bar{\boldsymbol{M}}$ are considered in the derivation of the polar continuum theory for the fluent continua. Details of polar decomposition of $[\bar{L}]$ and rotation rates ${}^t\bar{\boldsymbol{\Theta}}$ are given in the following. Let

$$[\bar{L}] = [{}^t\bar{R}][{}^t\bar{S}_r] = [{}^t\bar{S}_l][{}^t\bar{R}] \tag{2.1}$$

Let $({}^t\lambda_i, \{\phi\}_i)$; $i = 1,2,3$ be the eigenvalues of $[\bar{L}]^T[\bar{L}]$ in which $\{\phi\}_i^T\{\phi\}_j = \delta_{ij}$, then

$$[\bar{L}]^T\,[\bar{L}] = \left[\bar{\Phi}\right]\,[{}^t\bar{\lambda}]\,\left[\bar{\Phi}\right]^T = [{}^t\bar{S}_r]^2 \tag{2.2}$$

The columns of $[\bar{\Phi}]$ are eigenvectors $\{\phi\}_i$ and $[{}^t\bar{\lambda}]$ is a diagonal matrix of ${}^t\lambda_i,\ i = 1,2,3$. If we choose

$$[{}^t\bar{S}_r] = \left[\bar{\Phi}\right]\left[\sqrt{{}^t\bar{\lambda}}\right]\left[\bar{\Phi}\right]^T \tag{2.3}$$

Then (2.2) holds, hence $[{}^t\bar{S}_r]$ can be defined using (2.3). $[{}^t\bar{R}]$ can now be determined using (2.1)

$$[{}^t\bar{R}] = [\bar{L}][{}^t\bar{S}_r]^{-1} \tag{2.4}$$

Thus, we have established $[{}^t\bar{R}]$ and $[{}^t\bar{S}_r]$ in polar decomposition (2.1). Using

$$[\bar{L}]\,[\bar{L}]^T = [{}^t\bar{S}_l]^2 \tag{2.5}$$

and following a similar procedure we can establish the following

$$[{}^t\bar{S}_l] = \left[\bar{\Phi}\right]\left[\sqrt{{}^t\bar{\lambda}}\right]\left[\bar{\Phi}\right]^T \tag{2.6}$$

$$[{}^t\bar{R}] = [{}^t\bar{S}_l]^{-1}[\bar{L}] \tag{2.7}$$

in which $({}^t\lambda_i, \{\phi\}_i)$; $i = 1,2,3$ are eigenpairs of $[\bar{L}][\bar{L}]^T$. $[{}^t\bar{R}]$ defined by (2.4) or (2.7) is unique. The rate of rotation matrix $[{}^t\bar{R}]$ can equivalently be obtained due to rotation rates ${}^t\bar{\boldsymbol{\Theta}}$ at each location. Thus, at each location $[{}^t\bar{R}]$ can be viewed as being due to rates of rotations ${}^t\bar{\boldsymbol{\Theta}}$. Rate of energy dissipation due to ${}^t\bar{\boldsymbol{\Theta}}$ requires coexistence of moments $\bar{\boldsymbol{M}}$ (per unit area) on the oblique surface of the tetrahedron in the deforming matter. Thus we have

$$[\bar{L}] = \frac{\partial\{\bar{v}\}}{\partial\{\bar{x}\}} = [{}^t\bar{R}][{}^t\bar{S}_r] = [{}^t\bar{S}_l][{}^t\bar{R}] \tag{2.8}$$

where

$$[{}^t\bar{R}] = \left[{}^t\bar{R}({}^t\bar{\boldsymbol{\Theta}})\right] \tag{2.9}$$

Explicit forms of ${}^t\bar{\boldsymbol{\Theta}}$ i.e. ${}^t\bar{\Theta}_{x_1}$, ${}^t\bar{\Theta}_{x_2}$, and ${}^t\bar{\Theta}_{x_3}$ or ${}^t\bar{\Theta}_1$, ${}^t\bar{\Theta}_2$, and ${}^t\bar{\Theta}_3$ in terms of velocity gradients are given in section 2.6.

2.2 Rotation rate gradients and strain rate gradients

Even though the presence of varying rates of rotations between neighboring locations in the flow domain may influence the dissipation in some fluent continua, the precise manner in which this occurs is not yet established. All we know at this stage is that in fluent continua forces and velocities, the rotation rates and moments can also be work conjugate if the deforming fluent continua resists varying rotation rates between between the neighboring locations in the flow domain. Through the derivations of the balance laws presented in section 3 we establish that the symmetric part of the rotation rate gradient tensor is energy conjugate to the moment tensor. Thus, it is fair to say that the polar part of the theory presented here is due to rates of rotation gradients. The purpose of the material presented in this section is to demonstrate that the polar continuum theory presented here is not the same as the strain rate gradient theory published or referenced in the literature.

In case of solid matter, author in reference [68] shows a relationship between the gradients of local rotations in terms of gradients of strain tensor and rotation tensor. Based on similar works, it is argued and mostly accepted that the continuum theories that incorporate rotation gradients are same as those derived using strain gradients. Surana et. al. [69]: (i) first derived a relationship between the gradients of rotations and the gradients of strain tensor (similar to reference [68]) and (ii) then demonstrated using these relations that the continuum theories based on rotation gradients and those based on strain gradients are in fact not the same. The resulting theories from the two approaches describe different physics. In the following we present a derivation similar to reference [69] but for fluent continua to demonstrate that the theories based on rotation rate gradients are not the same as those that are derived using strain rate gradients. This is necessary to differentiate the work presented in this paper from the published works on strain rate gradient theories. For simplicity consider a two dimensional state of deformation in x_1x_2-plane. Velocity gradients tensor $[\bar{L}]$ is given by

$$[\bar{L}] = \left[\frac{\partial\{\bar{v}\}}{\partial\{\bar{x}\}}\right] = [\bar{D}] + [\bar{W}] \tag{2.10}$$

$[\bar{D}]$ and $[\bar{W}]$ are symmetric and skew symmetric tensors.

$$[\bar{W}] = \begin{bmatrix} 0 & \frac{1}{2}\left(\frac{\partial\bar{v}_1}{\partial\bar{x}_2} - \frac{\partial\bar{v}_2}{\partial\bar{x}_1}\right) \\ \frac{1}{2}\left(\frac{\partial\bar{v}_2}{\partial\bar{x}_1} - \frac{\partial\bar{v}_1}{\partial\bar{x}_2}\right) & 0 \end{bmatrix} = \begin{bmatrix} 0 & {}^t\bar{\Theta}_{x_3} \\ -{}^t\bar{\Theta}_{x_3} & 0 \end{bmatrix} \tag{2.11}$$

in which

$${}^t\bar{\Theta}_{x_3} = \frac{1}{2}\left(\frac{\partial\bar{v}_1}{\partial\bar{x}_2} - \frac{\partial\bar{v}_2}{\partial\bar{x}_1}\right) = {}^t\bar{\Theta}_3 \tag{2.12}$$

is the rate of rotation tensor about the x_3 axis. Gradients of ${}^t\bar{\Theta}_3$ with respect to $\bar{x}_1$ and $\bar{x}_2$ are

$$\begin{aligned} {}^t\bar{\Theta}_{3,1} &= \frac{1}{2}\left(\frac{\partial^2\bar{v}_1}{\partial\bar{x}_1\partial\bar{x}_2} - \frac{\partial^2\bar{v}_2}{\partial\bar{x}_1^2}\right) \\ {}^t\bar{\Theta}_{3,2} &= \frac{1}{2}\left(\frac{\partial^2\bar{v}_1}{\partial\bar{x}_2^2} - \frac{\partial^2\bar{v}_2}{\partial\bar{x}_1\partial\bar{x}_2}\right) \end{aligned} \tag{2.13}$$

The strain rates are defined by $[\bar{D}]$ (same in co- and contravariant bases and Jaumann rates)

$$[\bar{D}] = \begin{bmatrix} \frac{\partial \bar{v}_1}{\partial \bar{x}_1} & \frac{1}{2}\left(\frac{\partial \bar{v}_2}{\partial \bar{x}_1} + \frac{\partial \bar{v}_1}{\partial \bar{x}_2}\right) \\ \frac{1}{2}\left(\frac{\partial \bar{v}_2}{\partial \bar{x}_1} + \frac{\partial \bar{v}_1}{\partial \bar{x}_2}\right) & \frac{\partial \bar{v}_2}{\partial \bar{x}_2} \end{bmatrix} = \begin{bmatrix} \dot{\bar{\varepsilon}}_{11} & \dot{\bar{\varepsilon}}_{12} \\ \dot{\bar{\varepsilon}}_{21} & \dot{\bar{\varepsilon}}_{22} \end{bmatrix} = \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix} \tag{2.14}$$

in which $\gamma_{21} = \gamma_{12}$.
Substituting from (2.14) into (2.13) we can obtain

$$\begin{aligned} {}^t\bar{\Theta}_{3,1} &= \frac{\partial \gamma_{11}}{\partial \bar{x}_2} - \frac{\partial \gamma_{12}}{\partial \bar{x}_1} \\ {}^t\bar{\Theta}_{3,2} &= \frac{\partial \gamma_{12}}{\partial \bar{x}_2} - \frac{\partial \gamma_{22}}{\partial \bar{x}_1} \end{aligned} \tag{2.15}$$

In (2.15), the gradients ${}^t\bar{\Theta}_{3,1}$ and ${}^t\bar{\Theta}_{3,2}$ of rotation rate ${}^t\bar{\Theta}_3$ are completely expressed in terms of the gradients of γ_{11} and γ_{22} with respect to $\bar{x}_2$ and $\bar{x}_1$ and γ_{12} with respect to $\bar{x}_1$ as well as $\bar{x}_2$

Remarks

1. From (2.15) we note that gradients of ${}^t\bar{\Theta}_3$ are functions of $\frac{\partial \gamma_{11}}{\partial \bar{x}_2}$, $\frac{\partial \gamma_{22}}{\partial \bar{x}_1}$, $\frac{\partial \gamma_{12}}{\partial \bar{x}_1}$, and $\frac{\partial \gamma_{12}}{\partial \bar{x}_2}$ but are not functions of $\frac{\partial \gamma_{11}}{\partial \bar{x}_1}$ and $\frac{\partial \gamma_{22}}{\partial \bar{x}_2}$. This is expected due to the fact that $\frac{\partial \gamma_{11}}{\partial \bar{x}_1}$ and $\frac{\partial \gamma_{22}}{\partial \bar{x}_2}$ are gradients of elongation rates per unit length in $\bar{x}_1$ and $\bar{x}_2$ directions, hence can not possibly contribute to the gradients of the rotation rates.
2. Consideration of ${}^t\bar{\Theta}_{3,1}$ and ${}^t\bar{\Theta}_{3,2}$ in polar theory is identically equivalent to replacing these by the right sides of the expressions in (2.15). As long as this condition is satisfied the polar theory based on the gradients of rotation rates is the same as the polar theory based on gradients of the strain rates. We keep in mind that $\frac{\partial \gamma_{11}}{\partial \bar{x}_1}$ and $\frac{\partial \gamma_{22}}{\partial \bar{x}_2}$ are not part of the expressions of the gradients of rotation rates in (2.15).
3. A polar theory based on strain rate gradients must consider $\gamma_{ij,k}$ i.e. gradients of all six strain rates with respect to $\bar{x}_k$. Thus at the onset it is clear that the strain rate gradient theory for 2D cases will also consider $\frac{\partial \gamma_{11}}{\partial \bar{x}_1}$ and $\frac{\partial \gamma_{22}}{\partial \bar{x}_2}$ in the derivation in addition to the other strain rate gradients that appear in (2.15). If we consider three dimensional case (i.e. $\mathbb{R}^3$) then we would find that additionally $\frac{\partial \gamma_{22}}{\partial \bar{x}_2}$ will appear in the strain rate gradient theory but will be absent in the definitions of the gradients of the rotation rates.
4. The rotation rate polar theory resulting due to consideration of local rotation rates is targeted towards specific physics of rotation rates resulting in additional dissipation in a deforming fluent continua. We have shown that a polar theory based on gradients of rates of rotations is not the same as the theories derived using gradients of strain rates. We remark that equation (2.15) representing gradients of rotation rates as a function of some (and not all) of the gradients of strain rates is a consequence of mathematical manipulation.

2.3 Covariant and Contravariant bases

The edges of the deformed tetrahedron $\bar{T}_1$ are covariant base vectors $\tilde{\boldsymbol{g}}_i$ that are tangent to the deformed material lines at $\bar{o}$. The faces of the tetrahedron are formed by the covariant base vectors $\tilde{\boldsymbol{g}}_2, \tilde{\boldsymbol{g}}_3$, $\tilde{\boldsymbol{g}}_3, \tilde{\boldsymbol{g}}_1$ and $\tilde{\boldsymbol{g}}_1, \tilde{\boldsymbol{g}}_2$. Following [67, 70, 71] we can define

$$\tilde{\boldsymbol{g}}_i = \frac{\partial \bar{x}_k}{\partial x_i} \mathbf{e}_k \tag{2.16}$$

x_i and $\bar{x}_k$ being coordinates of a material point in the reference configuration and current configuration respectively. If $[J]$ is the Jacobian of deformation

$$[J] = \frac{\partial \{\bar{x}\}}{\partial \{x\}} \quad \text{or} \quad J_{ij} = \frac{\partial \bar{x}_i}{\partial x_j} \tag{2.17}$$

then the columns of $[J]$ are covariant base vectors $\tilde{\boldsymbol{g}}_i$. The contravariant basis are reciprocal to the covariant basis [67, 70, 71] are defined by the base vectors $\tilde{\boldsymbol{g}}^i$

$$\tilde{\boldsymbol{g}}^j = \frac{\partial x_j}{\partial \bar{x}_l} \boldsymbol{e}_l \tag{2.18}$$

We note that

$$\tilde{\boldsymbol{g}}_i \cdot \tilde{\boldsymbol{g}}^j = \delta_{ij} \tag{2.19}$$

Alternatively to (2.18) we can also define $\tilde{\boldsymbol{g}}^i$ as

$$\begin{aligned} \tilde{\boldsymbol{g}}^1 &= \frac{\tilde{\boldsymbol{g}}_2 \times \tilde{\boldsymbol{g}}_3}{\tilde{\boldsymbol{g}}_1 \cdot (\tilde{\boldsymbol{g}}_2 \times \tilde{\boldsymbol{g}}_3)} \\ \tilde{\boldsymbol{g}}^2 &= \frac{\tilde{\boldsymbol{g}}_3 \times \tilde{\boldsymbol{g}}_1}{\tilde{\boldsymbol{g}}_2 \cdot (\tilde{\boldsymbol{g}}_3 \times \tilde{\boldsymbol{g}}_1)} \\ \tilde{\boldsymbol{g}}^3 &= \frac{\tilde{\boldsymbol{g}}_1 \times \tilde{\boldsymbol{g}}_2}{\tilde{\boldsymbol{g}}_3 \cdot (\tilde{\boldsymbol{g}}_1 \times \tilde{\boldsymbol{g}}_2)} \end{aligned} \tag{2.20}$$

The volume of the parallelepiped framed by $\tilde{\boldsymbol{g}}_i$ in the current configuration is given by (same as denominators in 2.20)

$$\bar{V} = \tilde{\boldsymbol{g}}_1 \cdot (\tilde{\boldsymbol{g}}_2 \times \tilde{\boldsymbol{g}}_3) = \tilde{\boldsymbol{g}}_2 \cdot (\tilde{\boldsymbol{g}}_3 \times \tilde{\boldsymbol{g}}_1) = \tilde{\boldsymbol{g}}_3 \cdot (\tilde{\boldsymbol{g}}_1 \times \tilde{\boldsymbol{g}}_2) \tag{2.21}$$

We note that $\tilde{\boldsymbol{g}}^i$ in (2.18) as well as $\tilde{\boldsymbol{g}}^j$ in (2.20) satisfy (2.19). Thus definitions of $\tilde{\boldsymbol{g}}^j$ in (2.18) and (2.20) are exactly the same, as both definitions with (2.16) satisfy (2.19). We note that $\tilde{\boldsymbol{g}}^1, \tilde{\boldsymbol{g}}^2, \tilde{\boldsymbol{g}}^3$ are normal to the faces of the deformed tetrahedron formed by $\tilde{\boldsymbol{g}}_2, \tilde{\boldsymbol{g}}_3$; $\tilde{\boldsymbol{g}}_3, \tilde{\boldsymbol{g}}_1$; $\tilde{\boldsymbol{g}}_1, \tilde{\boldsymbol{g}}_2$ covariant base vectors. Covariant and contravariant directions are important in defining and choosing the correct measures of strains, stresses, moment intensities, etc. Under the action of $\bar{\boldsymbol{P}}$ and $\bar{\boldsymbol{M}}$ on surface $\bar{A}\bar{B}\bar{C}$ and the stress and moment intensities on the faces of the tetrahedron formed by $\tilde{\boldsymbol{g}}_2, \tilde{\boldsymbol{g}}_3$; $\tilde{\boldsymbol{g}}_3, \tilde{\boldsymbol{g}}_1$; and $\tilde{\boldsymbol{g}}_1, \tilde{\boldsymbol{g}}_2$ base vectors, the tetrahedron $\bar{T}_1$ is in equilibrium.

2.4 Definition of stress measures

2.4.1 Contravariant Cauchy stress tensor The definition of the stresses on the non-oblique faces of the tetrahedron in the contravariant directions is the most natural way to define stress. Let $\bar{\boldsymbol{\sigma}}^{(0)}$ or $\underset{\sim}{\boldsymbol{\sigma}}^{(0)}$ be the contravariant stress tensor with components $\bar{\sigma}_{ij}^{(0)}$ or $\underset{\sim}{\sigma}_{ij}^{(0)}$ and dyads $\tilde{\boldsymbol{g}}_i \otimes \tilde{\boldsymbol{g}}_j$. Component $\bar{\sigma}_{11}^{(0)}$ or $\underset{\sim}{\sigma}_{11}^{(0)}$ is in the $\tilde{\boldsymbol{g}}^1$ direction on a face of the tetrahedron with unit exterior normal $\tilde{\boldsymbol{g}}^1$ i.e. on the $\tilde{\boldsymbol{g}}^1$ face. Likewise $\bar{\sigma}_{12}^{(0)}$ or $\underset{\sim}{\sigma}_{12}^{(0)}$ and $\bar{\sigma}_{31}^{(0)}$ or $\underset{\sim}{\sigma}_{31}^{(0)}$ act on the $\tilde{\boldsymbol{g}}^1$ and $\tilde{\boldsymbol{g}}^3$ faces in the $\tilde{\boldsymbol{g}}^2$ and $\tilde{\boldsymbol{g}}^1$ directions. Using the dyads $\tilde{\boldsymbol{g}}_i \otimes \tilde{\boldsymbol{g}}_j$ or contravariance law of transformation we can write

$$\boldsymbol{\sigma}^{(0)} = \tilde{\boldsymbol{g}}_i \otimes \tilde{\boldsymbol{g}}_j \underset{\sim}{\sigma}_{ij}^{(0)} \tag{2.22}$$

using (2.16) we can write

$$\begin{aligned} \boldsymbol{\sigma}^{(0)} &= \boldsymbol{e}_i \otimes \boldsymbol{e}_j \sigma_{ij}^{(0)} \\ \sigma_{ij}^{(0)} &= J_{ik} \underset{\sim}{\sigma}_{kl}^{(0)} J_{jl} \\ \text{or} \quad [\sigma^{(0)}]^T &= [J]\,[\underset{\sim}{\sigma}^{(0)}]\,[J]^T \end{aligned} \tag{2.23}$$

$\boldsymbol{\sigma}^{(0)}$ is the contravariant Cauchy stress tensor (Lagrangian) from which $\bar{\boldsymbol{\sigma}}^{(0)}$ can be easily obtained by replacing $[J]$ with $[\bar{J}]^{-1}$ and $\boldsymbol{\sigma}^{(0)}$ with $\bar{\boldsymbol{\sigma}}^{(0)}$ in (2.23). Since the dyads of $\boldsymbol{\sigma}^{(0)}$ or $\bar{\boldsymbol{\sigma}}^{(0)}$ are $\boldsymbol{e}_i \otimes \boldsymbol{e}_j$, the Cauchy principle holds between $\bar{\boldsymbol{P}}$ and $\bar{\boldsymbol{\sigma}}^{(0)}$ i.e.

$$\bar{\boldsymbol{P}} = \left(\bar{\boldsymbol{\sigma}}^{(0)}\right)^T \cdot \bar{\boldsymbol{n}} \tag{2.24}$$

2.4.2 Covariant Cauchy stress tensor Instead of using contravariant directions and stress components $\underset{\sim}{\boldsymbol{\sigma}}^{(0)}$ and covariant basis $\tilde{\boldsymbol{g}}_i$ we could use covariant stress components $(\underset{\sim}{\sigma}_{(0)})_{ij}$ or $(\bar{\underset{\sim}{\sigma}}_{(0)})_{ij}$ and contravariant basis $\tilde{\boldsymbol{g}}^i$. Consideration of $(\underset{\sim}{\sigma}_{(0)})_{ij}$ of course will require a different deformed tetrahedron such that covariant vectors $\tilde{\boldsymbol{g}}_i$ are normal to its non-oblique faces. The adverse consequences of choosing this measure of stress for finite deformation are discussed in references [67, 72]. Here we proceed using this measure as an alternative to the contravariant stress measure. Using dyads $\tilde{\boldsymbol{g}}^i \otimes \tilde{\boldsymbol{g}}^j$ and components $(\underset{\sim}{\sigma}_{(0)})_{ij}$ we can write

$$\bar{\boldsymbol{\sigma}}_{(0)} = \tilde{\boldsymbol{g}}^i \otimes \tilde{\boldsymbol{g}}^j (\underset{\sim}{\sigma}_{(0)})_{ij} \tag{2.25}$$

using (2.18) in (2.25) we can write

$$\begin{aligned} \bar{\boldsymbol{\sigma}}_{(0)} &= \boldsymbol{e}_i \otimes \boldsymbol{e}_j \left(\bar{\sigma}_{(0)}\right)_{ij} \\ \left(\bar{\sigma}_{(0)}\right)_{ij} &= \bar{J}_{ki} \left(\underset{\sim}{\sigma}_{(0)}\right)_{kl} \bar{J}_{lj} \\ \text{or} \quad [\bar{\sigma}_{(0)}] &= [\bar{J}]^T\,[\underset{\sim}{\sigma}_{(0)}][\bar{J}] \end{aligned} \tag{2.26}$$

$\bar{\boldsymbol{\sigma}}_{(0)}$ is the covariant Cauchy stress tensor (Eulerian) from which $\boldsymbol{\sigma}_{(0)}$ can be obtained by replacing $[\bar{J}]$ with $[J]^{-1}$ and $\bar{\boldsymbol{\sigma}}_{(0)}$ with $\boldsymbol{\sigma}^{(0)}$ in (2.26). Since the dyads of $\bar{\boldsymbol{\sigma}}_{(0)}$ are $\boldsymbol{e}_i \otimes \boldsymbol{e}_j$, the Cauchy principle holds between $\bar{\boldsymbol{P}}$ and $\bar{\boldsymbol{\sigma}}_{(0)}$ i.e.

$$\bar{\boldsymbol{P}} = \left(\bar{\boldsymbol{\sigma}}_{(0)}\right)^T \cdot \bar{\boldsymbol{n}} \tag{2.27}$$

RemarkThe Cauchy stress tensors $\boldsymbol{\sigma}^{(0)}$ or $\bar{\boldsymbol{\sigma}}^{(0)}$ and $\boldsymbol{\sigma}_{(0)}$ or $\bar{\boldsymbol{\sigma}}_{(0)}$ are nonsymmetric at this stage and so are stress tensors $\underset{\sim}{\boldsymbol{\sigma}}^{(0)}$ and $\underset{\sim}{\boldsymbol{\sigma}}_{(0)}$. Following the details in reference [67] we can also define Jaumann stress tensor ${}^{(0)}\bar{\boldsymbol{\sigma}}^J$ using $\bar{\boldsymbol{\sigma}}^{(0)}$ and $\bar{\boldsymbol{\sigma}}_{(0)}$ stress measures.

2.5 Definitions of moment tensors

2.5.1 Contravariant Cauchy moment tensor When the deformed tetrahedron with moment $\bar{\boldsymbol{M}}$ (per unit area) on its oblique face $\bar{A}\bar{B}\bar{C}$ is isolated from volume $\bar{V}$, its non-oblique face will have existence of moments (per unit area) on them. As in the case of stress, contravariant basis is the most natural way to define these. Let $\underset{\sim}{\boldsymbol{m}}^{(0)}$ or $\bar{\underset{\sim}{\boldsymbol{m}}}^{(0)}$ be the contravariant moment tensors with components $\underset{\sim}{m}_{ij}^{(0)}$ or $\bar{\underset{\sim}{m}}_{ij}^{(0)}$ and dyads $\tilde{\boldsymbol{g}}_i \otimes \tilde{\boldsymbol{g}}_j$. Component $\underset{\sim}{m}_{11}^{(0)}$ or $\bar{\underset{\sim}{m}}_{11}^{(0)}$ is along $\tilde{\boldsymbol{g}}^1$ direction on a face of the tetrahedron with unit exterior normal $\tilde{\boldsymbol{g}}^1$ i.e. on $\tilde{\boldsymbol{g}}^1$ face. Likewise $\underset{\sim}{m}_{12}^{(0)}$ or $\bar{\underset{\sim}{m}}_{12}^{(0)}$ and $\underset{\sim}{m}_{31}^{(0)}$ or $\bar{\underset{\sim}{m}}_{31}^{(0)}$ act on $\tilde{\boldsymbol{g}}^1$ and $\tilde{\boldsymbol{g}}^3$ faces in the $\tilde{\boldsymbol{g}}^2$ and $\tilde{\boldsymbol{g}}^1$ directions. Using the dyads $\tilde{\boldsymbol{g}}_i \otimes \tilde{\boldsymbol{g}}_j$ or contravariance law of transformation we can write

$$\boldsymbol{m}^{(0)} = \tilde{\boldsymbol{g}}_i \otimes \tilde{\boldsymbol{g}}_j \underset{\sim}{m}_{ij}^{(0)} \tag{2.28}$$

Using (2.16) we can write

$$\begin{aligned} \boldsymbol{m}^{(0)} &= \boldsymbol{e}_i \otimes \boldsymbol{e}_j m_{ij}^{(0)} \\ m_{ij}^{(0)} &= J_{ik} \underset{\sim}{m}_{kl}^{(0)} J_{jl} \\ \text{or} \quad [m^{(0)}]^T &= [J]\,[\underset{\sim}{m}^{(0)}]\,[J]^T \end{aligned} \tag{2.29}$$

$\boldsymbol{m}^{(0)}$ is contravariant Cauchy moment tensor (Lagrangian) from which $\bar{\boldsymbol{m}}^{(0)}$ can be obtained by replacing $[J]$ with $[\bar{J}]^{-1}$ and $\boldsymbol{m}^{(0)}$ with $\bar{\boldsymbol{m}}^{(0)}$. Since the dyads of $\boldsymbol{m}^{(0)}$ or $\bar{\boldsymbol{m}}^{(0)}$ are $\boldsymbol{e}_i \otimes \boldsymbol{e}_j$, based on Koiter [17], the Cauchy principle is assumed to hold between $\bar{\boldsymbol{M}}$ and $\bar{\boldsymbol{m}}^{(0)}$ i.e.

$$\bar{\boldsymbol{M}} = \left(\bar{\boldsymbol{m}}^{(0)}\right)^T \cdot \bar{\boldsymbol{n}} \tag{2.30}$$

We need to establish whether $\bar{\boldsymbol{m}}^{(0)}$ is symmetric or not, hence at this stage $\bar{\boldsymbol{m}}^{(0)}$ is not symmetric.

2.5.2 Covariant Cauchy moment tensor Instead of using contravariant directions we could instead use covariant directions with moment tensor components $(\underset{\sim}{m}_{(0)})_{ij}$ and contravariant basis with dyads $\tilde{\boldsymbol{g}}^i \otimes \tilde{\boldsymbol{g}}^j$. Consideration of $(\underset{\sim}{m}_{(0)})_{ij}$ will of course require a different deformed tetrahedron such that covariant vectors $\tilde{\boldsymbol{g}}_i$ are normal to its non-oblique faces. The adverse consequences of choosing this measure are similar to those for the choice of $(\underset{\sim}{\sigma}_{(0)})_{ij}$ for the stress measure. Using the dyads $\tilde{\boldsymbol{g}}^i \otimes \tilde{\boldsymbol{g}}^j$ with components $(\underset{\sim}{m}_{(0)})_{ij}$ we can write

$$\bar{\boldsymbol{m}}_{(0)} = \tilde{\boldsymbol{g}}^i \otimes \tilde{\boldsymbol{g}}^j \left(\underset{\sim}{m}_{(0)}\right)_{ij} \tag{2.31}$$

Using (2.17) we can write

$$\begin{aligned} \bar{\boldsymbol{m}}_{(0)} &= \boldsymbol{e}_i \otimes \boldsymbol{e}_j \left(\bar{m}_{(0)}\right)_{ij} \\ \left(\bar{m}_{(0)}\right)_{ij} &= \bar{J}_{ki} \left(\underset{\sim}{m}_{(0)}\right)_{kl} \bar{J}_{lj} \\ \text{or} \quad [\bar{m}_{(0)}] &= [\bar{J}]^T \left[\underset{\sim}{m}_{(0)}\right] [\bar{J}] \end{aligned} \tag{2.32}$$

$\bar{\boldsymbol{m}}_{(0)}$ is a covariant Cauchy moment tensor (Eulerian) from which $\boldsymbol{m}_{(0)}$ can be obtained by replacing $[\bar{J}]$ with $[J]^{-1}$ and $\bar{\boldsymbol{m}}_{(0)}$ with $\boldsymbol{m}_{(0)}$. Following Koiter and since the dyads of $\bar{\boldsymbol{m}}_{(0)}$ are $\bar{\boldsymbol{e}}_i \otimes \bar{\boldsymbol{e}}_j$, the Cauchy principle holds between $\bar{\boldsymbol{M}}$ and $\bar{\boldsymbol{m}}_{(0)}$ i.e.

$$\bar{\boldsymbol{M}} = \left(\bar{\boldsymbol{m}}_{(0)}\right)^T \cdot \bar{\boldsymbol{n}} \tag{2.33}$$

As in the case of the contravariant moment tensor, $\bar{\boldsymbol{m}}_{(0)}$ is also a non-symmetric Cauchy moment tensor in covariant basis unless established otherwise.

2.6 Velocity and rotation rate gradient tensors

The velocity gradient tensor $\bar{\boldsymbol{L}}$ and its decomposition into symmetric and skew symmetric parts $\bar{\boldsymbol{D}}$ and $\bar{\boldsymbol{W}}$ gives

$$\bar{L}_{ij} = \frac{\partial \bar{v}_i}{\partial \bar{x}_j} \quad \text{or} \quad [\bar{L}] = \left[\frac{\partial \{\bar{v}\}}{\partial \{\bar{x}\}}\right] = [\bar{D}] + [\bar{W}] \tag{2.34}$$

$$[\bar{D}] = \frac{1}{2}\left([\bar{L}] + [\bar{L}]^T\right) \quad ; \quad [\bar{W}] = \frac{1}{2}\left([\bar{L}] - [\bar{L}]^T\right) \tag{2.35}$$

Let $\{{}^t\bar{\Theta}\} = [{}^t\bar{\Theta}_{x_1}\ {}^t\bar{\Theta}_{x_2}\ {}^t\bar{\Theta}_{x_3}]^T$ be the rates of rotation about ox_1, ox_2, and ox_3 axes of the x-frame, then we have

$$[\bar{W}] = \begin{bmatrix} 0 & {}^t\bar{\Theta}_{x_3} & {}^t\bar{\Theta}_{x_2} \\ -{}^t\bar{\Theta}_{x_3} & 0 & {}^t\bar{\Theta}_{x_1} \\ -{}^t\bar{\Theta}_{x_2} & -{}^t\bar{\Theta}_{x_1} & 0 \end{bmatrix} \tag{2.36}$$

in which

$$\begin{aligned} {}^t\bar{\Theta}_1 = {}^t\bar{\Theta}_{x_1} &= \frac{1}{2}\left(\frac{\partial \bar{v}_2}{\partial \bar{x}_3} - \frac{\partial \bar{v}_3}{\partial \bar{x}_2}\right) \\ {}^t\bar{\Theta}_2 = {}^t\bar{\Theta}_{x_2} &= \frac{1}{2}\left(\frac{\partial \bar{v}_1}{\partial \bar{x}_3} - \frac{\partial \bar{v}_3}{\partial \bar{x}_1}\right) \\ {}^t\bar{\Theta}_3 = {}^t\bar{\Theta}_{x_3} &= \frac{1}{2}\left(\frac{\partial \bar{v}_1}{\partial \bar{x}_2} - \frac{\partial \bar{v}_2}{\partial \bar{x}_1}\right) \end{aligned} \tag{2.37}$$

We define gradients of ${}^t\bar{\boldsymbol{\Theta}}$ by

$$\begin{aligned} {}^{\bar{\Theta}}\bar{L}_{ij} &= \frac{\partial({}^t\bar{\Theta}_i)}{\partial \bar{x}_j} \\ \left[{}^{\bar{\Theta}}\bar{L}\right] &= \frac{\partial\{{}^t\bar{\Theta}\}}{\partial\{\bar{x}\}} = \left[{}^{\bar{\Theta}}\bar{D}\right] + \left[{}^{\bar{\Theta}}\bar{W}\right] \end{aligned} \tag{2.38}$$

Symmetric and skew symmetric tensors $[{}^{\bar{\Theta}}\bar{D}]$ and $[{}^{\bar{\Theta}}\bar{W}]$ are defined by

$$\begin{aligned} [{}^{\bar{\Theta}}\bar{D}] &= \frac{1}{2}\left(\left[{}^{\bar{\Theta}}\bar{L}\right] + \left[{}^{\bar{\Theta}}\bar{L}\right]^T\right) \\ \left[{}^{\bar{\Theta}}\bar{W}\right] &= \frac{1}{2}\left(\left[{}^{\bar{\Theta}}\bar{L}\right] - \left[{}^{\bar{\Theta}}\bar{L}\right]^T\right) \end{aligned} \tag{2.39}$$

3 CONSERVATION AND BALANCE LAWS

We remark that the continuum theory considered here incorporates new physics due to rates of rotations. This physics is absent in the currently used thermodynamic framework for isotropic, homogeneous fluent continua. This new physics due to rates of rotations may influence some or all conservation and balance laws. In order to determine the precise influence of the new physics (or lack of it) on the conservation and balance laws, we must initiate the derivations of the conservation and balance laws at a fundamental stage as we do for the non-polar case [67] so that the resulting equations can be compared with the non-polar case to determine how these laws are modified or influenced by the physics due to rates of rotations. We caution that after the derivation of conservation and balance laws we may find that some laws are not influenced by this new physics in which case the corresponding equations will obviously be the same as those for the non-polar case. Nonetheless the derivation of all conservation and balance laws must be presented in completeness otherwise we can not determine whether a particular law is influenced by this new physics when compared to the non-polar case. We wish to remark that in the following sections even if some derivations yield the same equations as for the non-polar case, their derivations are essential to keep in the paper as these are necessary to establish this conclusion compared to the non-polar case.

In polar continuum theory we must consider velocity gradient tensor and rate of rotation gradient tensor in the derivations of the following conservation and balance laws based on the assumption of thermodynamic equilibrium during evolution: (i) Conservation of mass and conservation of inertia (ii) Balance of linear momenta (iii) Balance of angular momenta (iv) Balance of moments of moments (v) First law of thermodynamics, balance of energy (vi) Second law of thermodynamics, entropy inequality. We consider the derivations in the following.

3.1 Conservation of mass and inertia

The derivation of the continuity equation based on conservation of mass remains the same as for non-polar continuum, Following reference [67] we can derive the following continuity equation in Eulerian description.

$$\frac{\partial \bar{\rho}}{\partial t} + \bar{\boldsymbol{\nabla}} \cdot (\bar{\rho}\bar{\boldsymbol{v}}) = 0 \tag{3.1}$$

$$\text{or} \quad \frac{D\bar{\rho}}{Dt} + \bar{\rho}\,\text{div}\,(\bar{\boldsymbol{v}}) = 0 \tag{3.2}$$

in which $\bar{\rho}(\bar{\boldsymbol{x}},t)$ is the density of a material point at $\bar{\boldsymbol{x}}$ in the current configuration. Micro-polar continuum theories consider continua with micro-fibers. In a deforming volume of matter these micro-fibers (considered inextensible in micro-polar continuum theory) will have inertial effects due to rotation. Conservation of inertia refers to such inertial effects. In the polar continuum theory presented here this inertial effect is neglected at present but can be included if desired. Thus, we assume that in the polar continuum theory considered here there is only one conservation law leading to the same continuity equation (3.1) or (3.2) as in the case of non-polar continuum theory.

3.2 Balance of linear momenta

For a deforming volume of matter, the rate change of linear momenta must be equal to the sum of all other forces acting on it. This is Newton's second law applied to a volume of matter. This derivation also is exactly the same as that for non-polar continuum theory. Following reference [67] we can write the following in Eulerian description (using contravariant Cauchy stress tensor).

$$\bar{\rho}\frac{D\bar{\boldsymbol{v}}}{Dt} - \bar{\rho}\bar{\boldsymbol{F}}^b - \bar{\boldsymbol{\nabla}} \cdot \bar{\boldsymbol{\sigma}}^{(0)} = 0 \tag{3.3}$$

$$\text{or} \quad \bar{\rho}\frac{\partial \bar{v}_i}{\partial t} + \bar{\rho}\bar{v}_j\frac{\partial \bar{v}_i}{\partial \bar{x}_j} - \bar{\rho}\bar{F}_i^b - \frac{\partial \bar{\sigma}_{ji}^{(0)}}{\partial \bar{x}_j} = 0 \tag{3.4}$$

in which $\bar{\boldsymbol{F}}^b$ are body forces per unit mass and $\bar{\boldsymbol{\sigma}}^{(0)}$ is the contravariant Cauchy stress tensor (See reference [67] for using covariant Cauchy stress tensor $\bar{\boldsymbol{\sigma}}^{(0)}$ and Jaumann stress tensor ${}^{(0)}\bar{\boldsymbol{\sigma}}^J$ in place of $\bar{\boldsymbol{\sigma}}^{(0)}$ and the consequences of doing so). Equations (3.3) or (3.4) are the momentum equations in x_1, x_2, and x_3 directions.

3.3 Balance of angular momenta

The principle of balance of angular momenta for a polar continuum can be stated as follows. The time rate of change of total moment of momentum for a polar continuum is equal to the vector sum of the moments of external forces and the moments. Thus, due to the surface stress $\bar{\boldsymbol{P}}$, surface moment $\bar{\boldsymbol{M}}$ (per unit area), body force $\bar{\boldsymbol{F}}^b$ (per unit mass), and the momentum $\bar{\rho}\bar{\boldsymbol{v}}d\bar{V}$ for an elemental mass $\bar{\rho}d\bar{V}$ in the current configuration (using Eulerian description) we can write the following

$$\frac{D}{Dt}\int\limits_{\bar{V}(t)} \bar{\boldsymbol{x}} \times \bar{\rho}\bar{\boldsymbol{v}} d\bar{V} = \int\limits_{\partial\bar{V}(t)} \left(\bar{\boldsymbol{x}} \times \bar{\boldsymbol{P}} - \bar{\boldsymbol{M}}\right) d\bar{A} + \int\limits_{\bar{V}(t)} \bar{\boldsymbol{x}} \times \bar{\rho}\bar{\boldsymbol{F}}^b d\bar{V} \tag{3.5}$$

In the following derivation we consider contravariant basis. We use Cauchy principle $\bar{\boldsymbol{P}} = (\bar{\boldsymbol{\sigma}}^{(0)})^T \cdot \bar{\boldsymbol{n}}$ or $\bar{P}_j = \bar{\sigma}^{(0)}_{mj}\bar{n}_m$ and express cross products using permutation symbol $\boldsymbol{\epsilon}$. We also use Cauchy principle $\bar{\boldsymbol{M}} = (\bar{\boldsymbol{m}}^{(0)})^T \cdot \bar{\boldsymbol{n}}$ or $\bar{M}_k = \bar{m}^{(0)}_{mk}\bar{n}_m$. Substituting into (3.5).

$$\begin{aligned}\frac{D}{Dt}\int\limits_{\bar{V}(t)} \bar{\rho}\epsilon_{ijk}\bar{x}_i\bar{v}_j d\bar{V} &= \int\limits_{\partial\bar{V}(t)} \left(\epsilon_{ijk}\bar{x}_i\bar{\sigma}^{(0)}_{mj}\bar{n}_m - \bar{m}^{(0)}_{mk}\bar{n}_m\right) d\bar{A} \\ &\quad + \int\limits_{\bar{V}(t)} \bar{\rho}\epsilon_{ijk}\bar{x}_i\bar{F}^b_j d\bar{V}\end{aligned} \tag{3.6}$$

Using transport theorem for the left side of (3.6), Gauss's divergence theorem for the first term on the right side of (3.6) and using $\frac{D\bar{x}_i}{Dt} = \bar{v}_i$

$$\begin{aligned}\int\limits_{\bar{V}(t)} \bar{\rho}\epsilon_{ijk}\left(\bar{v}_i\bar{v}_j + \bar{x}_i\frac{D\bar{v}_j}{Dt}\right) d\bar{V} &= \int\limits_{\bar{V}(t)} \left(\epsilon_{ijk}\left(\bar{x}_i\bar{\sigma}^{(0)}_{mj}\right)_{,m} - \left(\bar{m}^{(0)}_{mk}\right)_{,m}\right) d\bar{V} \\ &\quad + \int\limits_{\bar{V}(t)} \bar{\rho}\epsilon_{ijk}\bar{x}_i\bar{F}^b_j d\bar{V}\end{aligned} \tag{3.7}$$

We note that

$$\epsilon_{ijk}\bar{v}_i\bar{v}_j = 0 \tag{3.8}$$

and

$$\begin{aligned}\left(\bar{x}_i\bar{\sigma}^{(0)}_{mj}\right)_{,m} &= \bar{x}_{i,m}\bar{\sigma}^{(0)}_{mj} + \bar{x}_i\bar{\sigma}^{(0)}_{mj,m} \\ &= \delta_{im}\bar{\sigma}^{(0)}_{mj} + \bar{x}_i\bar{\sigma}^{(0)}_{mj,m} \\ &= \bar{\sigma}^{(0)}_{ij} + \bar{x}_i\bar{\sigma}^{(0)}_{mj,m}\end{aligned} \tag{3.9}$$

Using (3.8) and (3.9) in (3.7) and regrouping

$$\int\limits_{\bar{V}(t)} \epsilon_{ijk}\left(\bar{x}_i\left(\bar{\rho}\frac{D\bar{v}_j}{Dt} - \bar{\rho}\bar{F}^b_j - \bar{\sigma}^{(0)}_{mj,m}\right)\right) d\bar{V} = \int\limits_{\bar{V}(t)} \left(-\bar{m}^{(0)}_{mk,m} + \epsilon_{ijk}\bar{\sigma}^{(0)}_{ij}\right) d\bar{V} \tag{3.10}$$

Using momentum equations (3.4) in (3.10), we obtain

$$\int\limits_{\bar{V}(t)} \left(-\bar{m}^{(0)}_{mk,m} + \epsilon_{ijk}\bar{\sigma}^{(0)}_{ij}\right) d\bar{V} = 0 \tag{3.11}$$

Since $\bar{V}(t)$ is arbitrary, (3.10) implies

$$\bar{m}^{(0)}_{mk,m} - \epsilon_{ijk}\bar{\sigma}^{(0)}_{ij} = 0 \tag{3.12}$$

Equations (3.12) represents balance of angular momenta. We note that $\bar{\boldsymbol{\sigma}}^{(0)}$ is a nonsymmetric Cauchy stress tensor. It is instructive to expand (3.12) into three equations

$$\begin{aligned}\frac{\partial\bar{m}^{(0)}_{i1}}{\partial\bar{x}_i} - \left(\bar{\sigma}^{(0)}_{23} - \bar{\sigma}^{(0)}_{32}\right) &= 0 \\ \frac{\partial\bar{m}^{(0)}_{i2}}{\partial\bar{x}_i} - \left(\bar{\sigma}^{(0)}_{31} - \bar{\sigma}^{(0)}_{13}\right) &= 0 \\ \frac{\partial\bar{m}^{(0)}_{i3}}{\partial\bar{x}_i} - \left(\bar{\sigma}^{(0)}_{12} - \bar{\sigma}^{(0)}_{21}\right) &= 0\end{aligned} \tag{3.13}$$

From (3.13), we note that the off diagonal elements of stress tensor $\bar{\boldsymbol{\sigma}}^0$ are balanced by the gradients of the Cauchy moment tensor. Equations (3.13) can also be obtained in covariant basis and Jaumann rates by replacing $\bar{\boldsymbol{m}}^{(0)}, \bar{\boldsymbol{\sigma}}^{(0)}$ with $\bar{\boldsymbol{m}}_{(0)}, \bar{\boldsymbol{\sigma}}_{(0)}$ and ${}^{(0)}\bar{\boldsymbol{m}}^J, {}^{(0)}\bar{\boldsymbol{\sigma}}^J$.

Remarks

1. In the balance of angular momenta, the rate of change of angular momenta is balanced by the vector sum of the moments of the forces. Thus this balance law naturally contains moments due to components of the stress tensor acting on the faces of the deformed tetrahedron. Normal stress components obviously do not contribute to this. Hence, the moments contained in this balance law due to stresses are only caused by shear stresses.
2. In the case of non-polar fluent continua, the balance of angular momenta is a statement of self equilibrating moments due to shear stresses that yields
$$\boldsymbol{\epsilon} : \bar{\boldsymbol{\sigma}}^{(0)} = 0 \tag{3.14}$$
which implies that $\bar{\boldsymbol{\sigma}}^{(0)}$ is symmetric. An important point to note is that (3.14) is a result of stress couples due to shear stresses.
3. In the case of polar continua, the existence of moments $[\bar{m}^{(0)}]$ due to the material constitution resisting the rotations results in the shear stress couples being balanced by the internal moments. Thus, for polar continua, the balance of angular momenta yields (3.13) instead of (3.14), i.e.
$$[\bar{m}^{(0)}]^T\left\{\bar{\nabla}\right\} - \boldsymbol{\epsilon} : \bar{\boldsymbol{\sigma}}^{(0)} = 0 \tag{3.15}$$
We note that (3.15) is also a result of stress couples caused by shear stresses.

4. Thus, both non-polar and polar continuum theories use stress couples in the angular momenta balance law. *Referring to the polar continuum theory presented here as stress couple theory is inappropriate as the non-polar theory also make use of stress couples.*
5. From (3.12) or (3.13) we note that gradients of $[\bar{m}^{(0)}]$ equilibrate with the antisymmetric components of the stress tensor $\bar{\boldsymbol{\sigma}}^{(0)}$ as the symmetric components cancel each other in each of the three equations in (3.13).
6. Lastly, we emphasize that *appearance of equation (3.12) in other theories published in the literature does not necessarily make the polar continuum theory presented here the same as those in the literature.* In this work, we begin by demonstrating that the varying rotation rates at neighboring locations, when resisted by the deforming fluent continua, require existence of internal moment tensor $[\bar{m}^{(0)}]$. The balance of angular momenta establishes a relationship between $[\bar{m}^{(0)}]$ and $[\bar{\sigma}^{(0)}]$ (equations (3.12) or 3.13).

3.4 Balance of moments of the moments (or couples)

Forces, moments, moments of moments . . . are progressively higher order effects or terms, hence must satisfy appropriate balance laws to ensure absence of rigid rotation or rigid translation of the deforming volume of continua. Balance of angular momenta (moments of forces) must be considered for couples created by forces and the moments. Likewise, since moment is similar to force, but is a higher order effect or term than force, a balance law similar to balance of angular momentum i.e. balance of moment of couples or moments must be considered to ensure lack of rigid motion of the deforming continua. Just like in the case of non-polar, isotropic, homogeneous fluent continua balance of angular momenta (moments of forces) restricts the Cauchy stress tensor to be symmetric, we can expect this balance law to impose some restrictions on the Cauchy moment tensor. See reference [66] for additional information. Many published works use moment of moments but this is not specifically stated as a balance law for the polar case, hence we do not cite these references here. However, reference [66] explicitly states this as a balance law and uses it to derive relations similar to those presented here.

For the deformed tetrahedron to be in equilibrium the moments of the moments (or couples) must vanish. In the moments of the moments we must consider $\bar{\boldsymbol{M}}$ and also shear components of $\bar{\boldsymbol{\sigma}}^{(0)}$ i.e. $\boldsymbol{\epsilon} : \bar{\boldsymbol{\sigma}}^{(0)}$ (in contravariant basis). Thus, we can write (neglecting inertial terms)

$$\int_{\bar{V}} \bar{\boldsymbol{x}} \times \left(\boldsymbol{\epsilon} : \bar{\boldsymbol{\sigma}}^{(0)}\right) d\bar{V} - \int_{\partial\bar{V}} \bar{\boldsymbol{x}} \times \bar{\boldsymbol{M}} d\bar{A} = 0 \tag{3.16}$$

We expand the second term in (3.16) and then convert the integral over $\partial\bar{V}$ to the integral over $\bar{V}$ using divergence theorem.

$$\begin{aligned}
\int_{\partial\bar{V}} \bar{\boldsymbol{x}} \times \bar{\boldsymbol{M}} d\bar{A} &= \int_{\partial\bar{V}} \epsilon_{ijk}\bar{x}_i\bar{M}_j = \int_{\partial\bar{V}} \epsilon_{ijk}\bar{x}_i\bar{m}_{mj}^{(0)}\bar{n}_m d\bar{A} \\
&= \int_{\bar{V}} \left(\epsilon_{ijk}\bar{x}_i\bar{m}_{mj}^{(0)}\right)_{,m} d\bar{V} \\
&= \int_{\bar{V}} \epsilon_{ijk}\left(\bar{x}_{i,m}\bar{m}_{mj}^{(0)} + \bar{x}_i\bar{m}_{mj,m}^{(0)}\right) d\bar{V} \\
&= \int_{\bar{V}} \epsilon_{ijk}\left(\delta_{im}\bar{m}_{mj}^{(0)} + \bar{x}_i\bar{m}_{mj,m}^{(0)}\right) d\bar{V} \\
&= \int_{\bar{V}} \epsilon_{ijk}\left(\bar{m}_{ij}^{(0)} + \bar{x}_i\bar{m}_{mj,m}^{(0)}\right) d\bar{V} \\
&= \int_{\bar{V}} \epsilon_{ijk}\bar{m}_{ij}^{(0)} d\bar{V} + \int_{\bar{V}} \epsilon_{ijk}\bar{x}_i\bar{m}_{mj,m}^{(0)} d\bar{V} \\
&= \int_{\bar{V}} \epsilon_{ijk}\bar{m}_{ij}^{(0)} d\bar{V} + \int_{\bar{V}} \bar{\boldsymbol{x}} \times \left(\bar{\boldsymbol{m}}^{(0)} \cdot \bar{\boldsymbol{\nabla}}\right) d\bar{V}
\end{aligned} \tag{3.17}$$

Using (3.17) in (3.16) and collecting terms

$$\int_{\bar{V}} \bar{\boldsymbol{x}} \times \left(-\bar{\boldsymbol{m}}^{(0)} \cdot \bar{\boldsymbol{\nabla}} + \boldsymbol{\epsilon} : \bar{\boldsymbol{\sigma}}^{(0)}\right) d\bar{V} - \int_{\bar{V}} \epsilon_{ijk}\bar{m}_{ij}^{(0)} d\bar{V} = 0 \tag{3.18}$$

The first term in (3.18) vanishes due to (3.12) (balance of angular momenta) and we obtain

$$\int_{\bar{V}} \epsilon_{ijk}\bar{m}_{ij}^{(0)} d\bar{V} = 0 \tag{3.19}$$

Since $\bar{V}$ is arbitrary, (3.19) implies

$$\epsilon_{ijk}\bar{m}_{ij}^{(0)} = 0 \tag{3.20}$$

That is $\bar{m}_{ij}^{(0)}$, the Cauchy moment tensor, is symmetric. Relation (3.20) also holds in covariant basis and Jaumann rates by replacing $\bar{\boldsymbol{m}}^{(0)}$ with $\bar{\boldsymbol{m}}_{(0)}$ and ${}^{(0)}\bar{\boldsymbol{m}}^{J}$. Thus, we can see that the consequence of this balance law is to impose the restriction of symmetry on the Cauchy moment tensor.

We note that in the polar theory presented here, the Cauchy moment tensor is symmetric, but the Cauchy stress tensor is nonsymmetric, whereas in the corresponding non-polar theory, Cauchy stress tensor is symmetric and Cauchy moment tensor is null as rates of rotations are ignored in the theory. Symmetry of the Cauchy moment tensor is a restriction placed on the Cauchy moment tensor due to this balance law.

3.5 First law of thermodynamics

The sum of work and heat added to a deforming volume of matter must result in the increase in energy of the system. Expressing this as a rate statement we can write [67]

$$\frac{D\bar{E}_t}{Dt} = \frac{D\bar{Q}}{Dt} + \frac{D\bar{W}}{Dt} \tag{3.21}$$

$\bar{E}_t$, $\bar{Q}$, and $\bar{W}$ are total energy, heat added, and work done. These can be written as

$$\frac{D\bar{E}_t}{Dt} = \frac{D}{Dt}\int_{\bar{V}(t)} \bar{\rho}\left(\bar{e} + \frac{1}{2}\bar{\boldsymbol{v}}\cdot\bar{\boldsymbol{v}} - \bar{\boldsymbol{F}}^b\cdot\bar{\boldsymbol{u}}\right)d\bar{V} \tag{3.22}$$

$$\frac{D\bar{Q}}{Dt} = -\int_{\partial\bar{V}(t)} \bar{\boldsymbol{q}}\cdot\bar{\boldsymbol{n}}d\bar{A} \tag{3.23}$$

$$\frac{D\bar{W}}{Dt} = \int_{\partial\bar{V}(t)} \left(\bar{\boldsymbol{P}}\cdot\bar{\boldsymbol{v}} + \bar{\boldsymbol{M}}\cdot{}^t\bar{\boldsymbol{\Theta}}\right)d\bar{A} \tag{3.24}$$

Where $\bar{e}$ is specific internal energy, $\bar{\boldsymbol{F}}^b$ is body force per unit mass, $\bar{\boldsymbol{u}}$ are displacement, and $\bar{\boldsymbol{q}}$ is rate of heat. Note the additional term $\bar{\boldsymbol{M}}\cdot{}^t\bar{\boldsymbol{\Theta}}$ in $\frac{D\bar{W}}{Dt}$ contributes additional rate of work due to rates of rotation. In (3.22), we have neglected the energy due to rotary inertia. This is consistent with the assumption used in the conservation law in section 3.1. We expand each of the integrals in (3.22)–(3.24). Following reference [67], it is straight forward to show that:

$$\frac{D}{Dt}\int_{\bar{V}(t)} \bar{\rho}\left(\bar{e} + \frac{1}{2}\bar{\boldsymbol{v}}\cdot\bar{\boldsymbol{v}} - \bar{\boldsymbol{F}}^b\cdot\bar{\boldsymbol{u}}\right)d\bar{V} = \int_{\bar{V}(t)} \bar{\rho}\left(\frac{D\bar{e}}{Dt} + \bar{\boldsymbol{v}}\cdot\frac{D\bar{\boldsymbol{v}}}{Dt} - \bar{\boldsymbol{F}}^b\cdot\bar{\boldsymbol{v}}\right)d\bar{V} \tag{3.25}$$

$$-\int_{\partial\bar{V}(t)} \bar{\boldsymbol{q}}\cdot\bar{\boldsymbol{n}}d\bar{A} = -\int_{\bar{V}(t)} \bar{\boldsymbol{\nabla}}\cdot\bar{\boldsymbol{q}}d\bar{V} \tag{3.26}$$

$$\frac{D\bar{W}}{Dt} = \int_{\partial\bar{V}(t)} \bar{\boldsymbol{P}}\cdot\bar{\boldsymbol{v}}d\bar{A} + \int_{\partial\bar{V}(t)} \bar{\boldsymbol{M}}\cdot{}^t\bar{\boldsymbol{\Theta}}d\bar{A} \tag{3.27}$$

Using contravariant Cauchy stress tensor $\bar{\boldsymbol{\sigma}}^{(0)}$, Cauchy principle, and following the details in reference [67] we can write

$$\int_{\partial\bar{V}(t)} \bar{\boldsymbol{P}}\cdot\bar{\boldsymbol{v}}d\bar{A} = \int_{\bar{V}(t)} \left(\bar{\boldsymbol{v}}\cdot\left(\bar{\boldsymbol{\nabla}}\cdot\bar{\boldsymbol{\sigma}}^{(0)}\right) + \bar{\sigma}^{(0)}_{ji}\frac{\partial\bar{v}_i}{\partial\bar{x}_j}\right)d\bar{V} \tag{3.28}$$

Likewise using contravariant moment tensor (per unit area) $\bar{\boldsymbol{m}}^{(0)}$, Cauchy principle, and following the details similar to these used in deriving (3.28), we can write

$$\int_{\partial\bar{V}(t)} \bar{\boldsymbol{M}}\cdot{}^t\bar{\boldsymbol{\Theta}}d\bar{A} = \int_{\bar{V}(t)} \left({}^t\bar{\boldsymbol{\Theta}}\cdot\left(\bar{\boldsymbol{\nabla}}\cdot\bar{\boldsymbol{m}}^{(0)}\right) + \bar{m}^{(0)}_{ji}\frac{\partial({}^t\bar{\Theta}_i)}{\partial\bar{x}_j}\right)d\bar{V} \tag{3.29}$$

Using (3.25)–(3.29) in (3.21)

$$\begin{aligned}\int_{\bar{V}(t)} \bar{\rho}\left(\frac{D\bar{e}}{Dt} + \bar{\boldsymbol{v}}\cdot\frac{D\bar{\boldsymbol{v}}}{Dt} - \bar{\boldsymbol{F}}^b\cdot\bar{\boldsymbol{v}}\right)d\bar{V} &= -\int_{\bar{V}(t)} \bar{\boldsymbol{\nabla}}\cdot\bar{\boldsymbol{q}}d\bar{V} \\ &+ \int_{\bar{V}(t)} \left(\bar{\boldsymbol{v}}\cdot\left(\bar{\boldsymbol{\nabla}}\cdot\bar{\boldsymbol{\sigma}}^{(0)}\right) + \bar{\sigma}^{(0)}_{ji}\frac{\partial\bar{v}_i}{\partial\bar{x}_j}\right)d\bar{V} \\ &+ \int_{\bar{V}(t)} \left({}^t\bar{\boldsymbol{\Theta}}\cdot\left(\bar{\boldsymbol{\nabla}}\cdot\bar{\boldsymbol{m}}^{(0)}\right) + \bar{m}^{(0)}_{ji}\frac{\partial({}^t\bar{\Theta}_i)}{\partial\bar{x}_j}\right)d\bar{V}\end{aligned} \tag{3.30}$$

Transferring all terms to left of equality and regrouping

$$\begin{aligned}&\int_{\bar{V}(t)} \bar{\rho}\left(\bar{\boldsymbol{v}}\cdot\left(\frac{D\bar{\boldsymbol{v}}}{Dt} - \bar{\boldsymbol{F}}^b - \bar{\boldsymbol{\nabla}}\cdot\bar{\boldsymbol{\sigma}}^{(0)}\right)\right)d\bar{V} \\ &+ \int_{\bar{V}(t)} \left(\frac{D\bar{e}}{Dt} + \bar{\boldsymbol{\nabla}}\cdot\bar{\boldsymbol{q}} - \bar{\sigma}^{(0)}_{ji}\frac{\partial\bar{v}_i}{\partial\bar{x}_j} \right. \\ &\left. - \bar{m}^{(0)}_{ji}\frac{\partial({}^t\bar{\Theta}_i)}{\partial\bar{x}_j} - {}^t\bar{\boldsymbol{\Theta}}\cdot\left(\bar{\boldsymbol{\nabla}}\cdot\bar{\boldsymbol{m}}^{(0)}\right)\right)d\bar{V} = 0\end{aligned} \tag{3.31}$$

Using (3.3) (balance of linear momenta) and (3.12) balance of angular momenta, (3.30) reduces to

$$\begin{aligned}\int_{\bar{V}(t)} &\left(\bar{\rho}\frac{D\bar{e}}{Dt} + \bar{\boldsymbol{\nabla}}\cdot\bar{\boldsymbol{q}} - \bar{\sigma}^{(0)}_{ji}\frac{\partial\bar{v}_i}{\partial\bar{x}_j} \right. \\ &\left. - \bar{m}^{(0)}_{ji}\frac{\partial({}^t\bar{\Theta}_i)}{\partial\bar{x}_j} - {}^t\bar{\boldsymbol{\Theta}}\cdot\left(\boldsymbol{\epsilon}:\bar{\boldsymbol{\sigma}}^{(0)}\right)\right)d\bar{V} = 0\end{aligned} \tag{3.32}$$

Since $\bar{V}(t)$ is arbitrary, (3.32) implies that

$$\bar{\rho}\frac{D\bar{e}}{Dt} + \bar{\boldsymbol{\nabla}}\cdot\bar{\boldsymbol{q}} - \bar{\sigma}^{(0)}_{ji}\frac{\partial\bar{v}_i}{\partial\bar{x}_j} - \bar{m}^{(0)}_{ji}\frac{\partial({}^t\bar{\Theta}_i)}{\partial\bar{x}_j} - {}^t\bar{\boldsymbol{\Theta}}\cdot\left(\boldsymbol{\epsilon}:\bar{\boldsymbol{\sigma}}^{(0)}\right) = 0 \tag{3.33}$$

Equation (3.33) is the final form of the energy equation in which $\bar{\boldsymbol{\sigma}}^{(0)}$ is a nonsymmetric Cauchy stress tensor and $\bar{\boldsymbol{m}}^{(0)}$ is a symmetric Cauchy moment tensor. Thus in (3.33) we can use

$$\begin{aligned}\bar{m}^{(0)}_{mj}\frac{\partial({}^t\bar{\Theta}_i)}{\partial\bar{x}_j} &= \bar{m}^{(0)}_{mj}\left({}^{\bar{\Theta}}\bar{D}_{ij} + {}^{\bar{\Theta}}\bar{W}_{ij}\right) = \bar{m}^{(0)}_{mj}\left({}^{\bar{\Theta}}\bar{D}_{ij}\right) \\ &\text{as}\quad \bar{m}^{(0)}_{mj}\left({}^{\bar{\Theta}}\bar{W}_{ij}\right) = 0\end{aligned} \tag{3.34}$$

Equation (3.33) representing balance of energy can also be derived in covariant basis or in Jaumann rates. In (3.33) we replace $\bar{\boldsymbol{\sigma}}^{(0)}$, $\bar{\boldsymbol{m}}^{(0)}$ by $\bar{\boldsymbol{\sigma}}_{(0)}$, $\bar{\boldsymbol{m}}_{(0)}$ and ${}^{(0)}\bar{\boldsymbol{\sigma}}^J$, ${}^{(0)}\bar{\boldsymbol{m}}^J$ to obtain its corresponding form in covariant basis and in Jaumann rates.

3.6 Second law of thermodynamics

If $\bar{\eta}$ is the entropy density in volume $\bar{V}(t)$, $\bar{h}$ is the entropy flux between $\bar{V}(t)$ and the volume of matter surrounding it and $\bar{s}$ is the source of entropy in $\bar{V}$ due to non-contacting bodies, then the rate of increase in entropy in volume $\bar{V}(t)$ is at least equal to that supplied to $\bar{V}(t)$ from all contacting and non-contacting sources [67, 70, 71]. Thus

$$\frac{D}{Dt}\int_{\bar{V}(t)} \bar{\eta}\bar{\rho}d\bar{V} \geq \int_{\partial\bar{V}(t)} \bar{h}d\bar{A} + \int_{\bar{V}(t)} \bar{s}\bar{\rho}d\bar{V} \tag{3.35}$$

Using Cauchy's postulate for $\bar{h}$ i.e.

$$\bar{h} = -\bar{\boldsymbol{\Psi}}\cdot\bar{\boldsymbol{n}} \tag{3.36}$$

Using (3.36) in (3.35)

$$\frac{D}{Dt}\int_{\bar{V}(t)} \bar{\eta}\bar{\rho}d\bar{V} \geq -\int_{\partial\bar{V}(t)} \bar{\boldsymbol{\Psi}}\cdot\bar{\boldsymbol{n}}d\bar{A} + \int_{\bar{V}(t)} \bar{s}\bar{\rho}d\bar{V} \tag{3.37}$$

We recall that [67]

$$\frac{D}{Dt}\int\limits_{\bar V(t)} \bar\eta\bar\rho d\bar V = \int\limits_{\bar V(t)} \bar\rho\frac{D\bar\eta}{Dt}d\bar V \tag{3.38}$$

and

$$-\int\limits_{\partial\bar V(t)} \bar{\boldsymbol{\Psi}}\cdot\bar{\boldsymbol{n}}d\bar A = -\int\limits_{\bar V(t)} \bar{\boldsymbol{\nabla}}\cdot\bar{\boldsymbol{\Psi}}d\bar V = -\int\limits_{\bar V(t)} \bar\Psi_{i,i}d\bar V \tag{3.39}$$

Substituting from (3.38) and (3.39) in (3.37) and transferring all terms to the left of inequality

$$\int\limits_{\bar V(t)}\left(\bar\rho\frac{D\bar\eta}{Dt}+\bar\Psi_{i,i}-\bar s\bar\rho\right)d\bar V \geq 0 \tag{3.40}$$

Since volume $\bar V(t)$ is arbitrary, (3.40) implies

$$\bar\rho\frac{D\bar\eta}{Dt}+\bar\Psi_{i,i}-\bar s\bar\rho \geq 0 \tag{3.41}$$

Equation (3.41) is the entropy inequality and is the most fundamental form resulting from the second law of thermodynamics. A more useful form of (3.41) can be derived if we assume

$$\bar{\boldsymbol{\Psi}} = \frac{\bar{\boldsymbol{q}}}{\bar\theta} \quad ; \quad \bar s = \frac{\bar r}{\bar\theta} \tag{3.42}$$

Where $\bar\theta$ is absolute temperature, $\bar{\boldsymbol{q}}$ is heat vector, and $\bar r$ is a suitable potential. Using (3.42)

$$\bar\Psi_{i,i} = \frac{\bar q_{i,i}}{\bar\theta} - \frac{\bar q_i}{(\bar\theta)^2}\bar\theta_{,i} = \frac{\bar q_{i,i}}{\bar\theta} - \frac{\bar q_i}{(\bar\theta)^2}\bar g_i \quad ; \quad \bar g_i = \bar\theta_{,i} \tag{3.43}$$

Substituting for $\bar s$ from (3.42) and for $\bar\Psi_{i,i}$ from (3.43) into (3.41) and multiplying by $\bar\theta$.

$$\bar\rho\theta\frac{D\bar\eta}{Dt} + (\bar q_{i,i} - \bar\rho\bar r) - \frac{\bar q_i\bar g_i}{\bar\theta} \geq 0 \tag{3.44}$$

From energy equation (3.33) (after inserting $\bar\rho\bar r$ term) in contravariant basis

$$\begin{aligned}\bar{\boldsymbol{\nabla}}\cdot\bar{\boldsymbol{q}} - \bar\rho\bar r = \bar q_{i,i} - \bar\rho\bar r = -\bar\rho\frac{D\bar e}{Dt} + \bar\sigma_{ji}^{(0)}\frac{\partial\bar v_i}{\partial\bar x_j}\\ + \bar m_{ji}^{(0)}\frac{\partial({}^t\bar\Theta_i)}{\partial\bar x_j} + {}^t\bar{\boldsymbol{\Theta}}\cdot\left(\boldsymbol{\epsilon}:\bar{\boldsymbol{\sigma}}^{(0)}\right)\end{aligned} \tag{3.45}$$

Substituting from (3.45) into (3.44)

$$\begin{aligned}\bar\rho\theta\frac{D\bar\eta}{Dt} - \bar\rho\frac{D\bar e}{Dt} + \bar\sigma_{ji}^{(0)}\frac{\partial\bar v_i}{\partial\bar x_j} + \bar m_{ji}^{(0)}\frac{\partial({}^t\bar\Theta_i)}{\partial\bar x_j}\\ + {}^t\bar{\boldsymbol{\Theta}}\cdot\left(\boldsymbol{\epsilon}:\bar{\boldsymbol{\sigma}}^{(0)}\right) - \frac{\bar q_i\bar g_i}{\bar\theta} \geq 0\end{aligned} \tag{3.46}$$

or

$$\begin{aligned}\bar\rho\left(\frac{D\bar e}{Dt} - \theta\frac{D\bar\eta}{Dt}\right) - \bar\sigma_{ji}^{(0)}\frac{\partial\bar v_i}{\partial\bar x_j} - \bar m_{ji}^{(0)}\frac{\partial({}^t\bar\Theta_i)}{\partial\bar x_j}\\ - {}^t\bar{\boldsymbol{\Theta}}\cdot\left(\boldsymbol{\epsilon}:\bar{\boldsymbol{\sigma}}^{(0)}\right) + \frac{\bar q_i\bar g_i}{\bar\theta} \leq 0\end{aligned} \tag{3.47}$$

Let $\bar\Phi$ be Helmholtz free energy density (specific Helmholtz free energy) defined by

$$\bar\Phi = \bar e - \bar\eta\bar\theta \tag{3.48}$$

Hence

$$\frac{D\bar e}{Dt} - \bar\theta\frac{D\bar\eta}{Dt} = \left(\frac{D\bar\Phi}{Dt} + \bar\eta\frac{D\bar\theta}{Dt}\right) \tag{3.49}$$

Substituting from (3.49) into (3.47)

$$\begin{aligned}\bar\rho\left(\frac{D\bar\Phi}{Dt} + \bar\eta\frac{D\bar\theta}{Dt}\right) + \frac{\bar q_i\bar g_i}{\bar\theta} - \bar\sigma_{ji}^{(0)}\frac{\partial\bar v_i}{\partial\bar x_j}\\ - \bar m_{ji}^{(0)}\frac{\partial({}^t\bar\Theta_i)}{\partial\bar x_j} - {}^t\bar{\boldsymbol{\Theta}}\cdot\left(\boldsymbol{\epsilon}:\bar{\boldsymbol{\sigma}}^{(0)}\right) \leq 0\end{aligned} \tag{3.50}$$

or

$$\begin{aligned}\bar\rho\left(\frac{D\bar\Phi}{Dt} + \bar\eta\frac{D\bar\theta}{Dt}\right) + \frac{\bar q_i\bar g_i}{\bar\theta} - \mathrm{tr}\left([\bar\sigma^{(0)}]^T[\bar L]^T\right)\\ - \mathrm{tr}\left([\bar m^{(0)}][{}^{\bar\Theta}\bar L]\right) - {}^t\bar{\boldsymbol{\Theta}}\cdot\left(\boldsymbol{\epsilon}:\bar{\boldsymbol{\sigma}}^{(0)}\right) \leq 0\end{aligned} \tag{3.51}$$

$\bar{\boldsymbol{m}}^{(0)}$ is symmetric but $\bar{\boldsymbol{\sigma}}^{(0)}$ is not symmetric. Since $\bar{\boldsymbol{m}}^{(0)}$ is symmetric, we can use the following in (3.51).

$$\mathrm{tr}\left([\bar m^{(0)}][{}^{\bar\Theta}\bar L]\right) = \mathrm{tr}\left([\bar m^{(0)}][{}^{\bar\Theta}\bar D]\right) \tag{3.52}$$

The entropy inequality (3.51) in covariant basis and in Jaumann rates can be obtained by replacing $\bar{\boldsymbol{\sigma}}^{(0)}$, $\bar{\boldsymbol{m}}^{(0)}$ with $\bar{\boldsymbol{\sigma}}_{(0)}$, $\bar{\boldsymbol{m}}_{(0)}$ and ${}^{(0)}\bar{\boldsymbol{\sigma}}^J$, ${}^{(0)}\bar{\boldsymbol{m}}^J$.

3.7 Stress decomposition and balance laws

It is instructive to decompose stress tensor $\bar{\boldsymbol{\sigma}}^{(0)}$ (considering contravariant basis) into symmetric ${}_s\bar{\boldsymbol{\sigma}}^{(0)}$ and antisymmetric ${}_a\bar{\boldsymbol{\sigma}}^{(0)}$ tensors

$$\bar{\boldsymbol{\sigma}}^{(0)} = {}_s\bar{\boldsymbol{\sigma}}^{(0)} + {}_a\bar{\boldsymbol{\sigma}}^{(0)} \tag{3.53}$$

where

$$\begin{aligned}{}_s\bar{\boldsymbol{\sigma}}^{(0)} &= \frac{1}{2}\left(\bar{\boldsymbol{\sigma}}^{(0)} + \left(\bar{\boldsymbol{\sigma}}^{(0)}\right)^T\right)\\ {}_a\bar{\boldsymbol{\sigma}}^{(0)} &= \frac{1}{2}\left(\bar{\boldsymbol{\sigma}}^{(0)} - \left(\bar{\boldsymbol{\sigma}}^{(0)}\right)^T\right)\end{aligned} \tag{3.54}$$

We substitute these in the balance of linear momenta (3.4), balance of angular momenta (3.12), energy equation (3.33), and entropy inequality (3.51). First we note that

$$\boldsymbol{\epsilon}:\bar{\boldsymbol{\sigma}}^{(0)} = \boldsymbol{\epsilon}:\left({}_s\bar{\boldsymbol{\sigma}}^{(0)} + {}_a\bar{\boldsymbol{\sigma}}^{(0)}\right) = \boldsymbol{\epsilon}:\left({}_a\bar{\boldsymbol{\sigma}}^{(0)}\right) \tag{3.55}$$

as

$$\boldsymbol{\epsilon}:\left({}_s\bar{\boldsymbol{\sigma}}^{(0)}\right) = 0 \tag{3.56}$$

$$\bar{\sigma}_{ji}^{(0)} \frac{\partial \bar{v}_i}{\partial \bar{x}_j} = \left({}_s\bar{\sigma}_{ji}^{(0)} + {}_a\bar{\sigma}_{ji}^{(0)} \right) (\bar{D}_{ij} + \bar{W}_{ij}) = \left({}_s\bar{\sigma}_{ji}^{(0)} \right) \bar{D}_{ij} + \left({}_a\bar{\sigma}_{ji}^{(0)} \right) \bar{W}_{ij} \tag{3.57}$$

as

$$\left({}_s\bar{\sigma}_{ji}^{(0)} \right) \bar{W}_{ij} = \left({}_a\bar{\sigma}_{ji}^{(0)} \right) \bar{D}_{ij} = 0 \tag{3.58}$$

we can write (3.57) as

$$\operatorname{tr}\left([\bar{\sigma}^{(0)}][\bar{L}] \right) = \operatorname{tr}\left([{}_s\bar{\sigma}^{(0)}][\bar{D}] \right) + \operatorname{tr}\left([{}_a\bar{\sigma}^{(0)}][\bar{W}] \right) \tag{3.59}$$

Using (3.55)–(3.59) in (3.4), (3.12), (3.33), and (3.51) we can obtain

$$\bar{\rho} \frac{\partial \bar{v}_i}{Dt} + \bar{\rho} \bar{v}_j \frac{\partial \bar{v}_i}{\partial \bar{x}_j} - \bar{\rho} \bar{F}_i^b - \frac{\partial_s \bar{\sigma}_{ji}^{(0)}}{\partial \bar{x}_j} - \frac{\partial_a \bar{\sigma}_{ji}^{(0)}}{\partial \bar{x}_j} = 0 \tag{3.60}$$

$$\bar{m}_{mk,m}^{(0)} - \epsilon_{ijk} \left({}_a\bar{\sigma}_{ij}^{(0)} \right) = 0 \tag{3.61}$$

$$\bar{\rho} \frac{D\bar{e}}{Dt} + \bar{\boldsymbol{\nabla}} \cdot \bar{\boldsymbol{q}} - \operatorname{tr}\left([{}_s\bar{\sigma}^{(0)}][\bar{D}] \right) - \operatorname{tr}\left([{}_a\bar{\sigma}^{(0)}][\bar{W}] \right) - \operatorname{tr}\left([\bar{m}^{(0)}][{}^{\bar{\Theta}}\bar{D}] \right) - {}^t\bar{\boldsymbol{\Theta}} \cdot \left(\boldsymbol{\epsilon} : {}_a\bar{\boldsymbol{\sigma}}^{(0)} \right) = 0 \tag{3.62}$$

$$\bar{\rho} \left(\frac{D\bar{\Phi}}{Dt} + \bar{\eta} \frac{D\bar{\theta}}{Dt} \right) + \frac{\bar{q}_i \bar{g}_i}{\bar{\theta}} - \operatorname{tr}\left([{}_s\bar{\sigma}^{(0)}][\bar{D}] \right) - \operatorname{tr}\left([{}_a\bar{\sigma}^{(0)}][\bar{W}] \right) - \operatorname{tr}\left([\bar{m}^{(0)}][{}^{\bar{\Theta}}\bar{D}] \right) - {}^t\bar{\boldsymbol{\Theta}} \cdot \left(\boldsymbol{\epsilon} : {}_a\bar{\boldsymbol{\sigma}}^{(0)} \right) \leq 0 \tag{3.63}$$

A simple calculation by expanding the terms shows that

$$\operatorname{tr}\left([{}_a\bar{\sigma}^{(0)}][\bar{W}] \right) = -{}^t\bar{\boldsymbol{\Theta}} \cdot \left(\boldsymbol{\epsilon} : {}_a\bar{\boldsymbol{\sigma}}^{(0)} \right) \tag{3.64}$$

If we substitute (3.64) in (3.62) and (3.63) then the energy equation and entropy inequality simplify.

$$\bar{\rho} \frac{D\bar{e}}{Dt} + \bar{\boldsymbol{\nabla}} \cdot \bar{\boldsymbol{q}} - \operatorname{tr}\left([{}_s\bar{\sigma}^{(0)}][\bar{D}] \right) - \operatorname{tr}\left([\bar{m}^{(0)}][{}^{\bar{\Theta}}\bar{D}] \right) = 0 \tag{3.65}$$

$$\bar{\rho} \left(\frac{D\bar{\Phi}}{Dt} + \bar{\eta} \frac{D\bar{\theta}}{Dt} \right) + \frac{\bar{q}_i \bar{g}_i}{\bar{\theta}} - \operatorname{tr}\left([{}_s\bar{\sigma}^{(0)}][\bar{D}] \right) - \operatorname{tr}\left([\bar{m}^{(0)}][{}^{\bar{\Theta}}\bar{D}] \right) \leq 0 \tag{3.66}$$

Remarks

(1) Equations (3.60), (3.61), (3.65), and (3.66) can also be expressed in covariant basis and using Jaumann rates.
(2) Equations (3.1) (Continuity), (3.60), (3.61), (3.65), and (3.66) constitute a complete mathematical model for fluent media in Eulerian description.
(3) From (3.65) and (3.66) we can conclude that ${}_s\bar{\boldsymbol{\sigma}}^{(0)}, \bar{\boldsymbol{D}}$ and $\bar{\boldsymbol{m}}^{(0)}, {}^{\bar{\Theta}}\bar{\boldsymbol{D}}$ are conjugate pairs, hence are responsible for conversion of mechanical energy into heat or entropy. The conjugate pairs are instrumental in deciding the dependent variables in the constitutive theories and some of their argument tensors. These conjugate pairs suggest that ${}_s\bar{\boldsymbol{\sigma}}^{(0)}$ can be expressed as a function of $\bar{\boldsymbol{D}}$ and $\bar{\boldsymbol{m}}^{(0)}$ as a function of ${}^{\bar{\Theta}}\bar{\boldsymbol{D}}$. We note that $\bar{\boldsymbol{q}}$ and $\bar{\boldsymbol{g}}$ are also conjugate, thus $\bar{\boldsymbol{q}}$ can be expressed as a function of $\bar{\boldsymbol{g}}$. These details will be considered in the separate papers on constitutive theories.

4 Closure of mathematical model and comments on constitutive theories

In this mathematical model the dependent variables are (numbers in lower case brackets indicate the count i.e. number of variables):

$\bar{\rho}(1)$, $\bar{v}_i(3)$, ${}_s\bar{\boldsymbol{\sigma}}^{(0)}(6)$, ${}_a\bar{\boldsymbol{\sigma}}^{(0)}(3)$, $\bar{\boldsymbol{m}}^{(0)}(6)$, $\bar{e}(1)$, $\bar{\theta}(1)$, $\bar{\boldsymbol{q}}(3)$, $\bar{\Phi}(1)$, $\bar{\eta}(1)$, a total of 26.

In these, $\bar{\Phi}$ and $\bar{\eta}$ will be eliminated, $\bar{e}(\bar{\rho}, \bar{\theta})$ i.e. $\bar{e}$ is a function of $\bar{\rho}$ and $\bar{\theta}$ for the most general case of compressible matter, hence $\bar{e}$ is also eliminated. This leaves us with the remaining 23 dependent variables in the mathematical model. We have continuity equation (1), linear momentum equations (3), angular momentum equations (3), energy equation (1) and, from the entropy inequality, we have constitutive theories for ${}_s\bar{\boldsymbol{\sigma}}^{(0)}$ (6), $\bar{\boldsymbol{m}}^{(0)}$ (6), and $\bar{\boldsymbol{q}}$, (3), a total of 23 equations, hence this mathematical model will have closure once we have constitutive theories for ${}_s\bar{\boldsymbol{\sigma}}^{(0)}$ (6), $\bar{\boldsymbol{m}}^{(0)}$ (6), and $\bar{\boldsymbol{q}}$ (3).

Development of the constitutive theory is clearly treatment of matter specific physics. The mathematical model derived here is valid for compressible as well as incompressible fluids as well as hypo-elastic solids. In the case of fluids, the derivations of the constitutive theory must consider: (i) polar thermofluids that have mechanism of dissipation such as Newtonian and generalized Newtonian fluids (Power law, Carreau-Yasuda fluids, etc.) (ii) polar thermoviscoelastic fluids that have mechanisms of dissipation as well as memory. The general constitutive theories for such polar fluids must naturally yield constitutive theories for Maxwell model, Oldroyd-B model, and Giesekus model for polar case as well as currently used constitutive theories within nonpolar continuum mechanics theories. The derivations of these constitutive theories in contravariant and covariant bases and Jaumann rates are presented in companion papers.

5 Summary and Conclusions

The development of the continuum theory (polar continuum theory) presented in this paper for isotropic, homogeneous fluent continua is motivated by the fact that the polar decomposition of changing velocity gradient tensor at a location and its neighbors with different velocity gradient tensors can result in different rates of rotations which if resisted by the fluent continua result in conjugate internal moments. These conjugate internal rates of rotations and the internal moments can result in additional energy dissipation. The currently used thermodynamic framework for isotropic, homogeneous fluent continua completely ignores this physics in the derivation of the conservation and balance laws. The theory resulting from incorporating the new physics considered here is in fact 'a polar theory' as it considers rates of rotations and moments as a conjugate pair. The rates of rotations are internal and are completely defined using skew symmetric part of the velocity gradient tensor, thus this theory does not require rotations as external degrees of freedom. The thermodynamic framework resulting from this new theory is obviously a more complete thermodynamic framework for isotropic, homogeneous fluent continua as it incorporates additional physics due to internal rates of rotations in the derivation of conservation and balance laws that is completely ignored in the presently used framework. In fact, the currently used thermodynamic framework is a subset of the more complete thermodynamic framework presented in this paper resulting from the polar theory.

Derivation of conservation and balance laws have been presented for polar fluent continua in contravariant and covariant bases and in Jaumann rates using Cauchy stress tensor, Cauchy moment tensor, heat vector, Helmholtz free energy density, and entropy density. Derivations show that (i) Cauchy stress tensor is nonsymmetric (ii) Cauchy moment tensor is symmetric due to moment of moments (or couples) balance law (iii) Decomposition of Cauchy stress tensor into symmetric and antisymmetric tensors shows that (a) symmetric Cauchy stress tensor and symmetric part of the velocity gradient tensor are conjugate (due to energy equation and entropy inequality) (b) antisymmetric part of the Cauchy stress tensor is balanced by the gradients of the Cauchy moment tensor (due to balance of angular momenta). (iv) Cauchy moment tensor and symmetric part of the gradient of rate of rotation tensor are conjugate (due to energy equation and entropy inequality) (v) It is shown that the constitutive theories for symmetric Cauchy stress tensor, Cauchy moment tensor, heat vector and the thermodynamic relations for specific internal energy and others provide closure to the mathematical model presented here. Details of the constitutive theories for polar thermofluids and polar thermoviscoelastic fluids are presented in separate papers that follow.

We emphasize that the polar theory presented here is not a micropolar theory (as mentioned in section 1). The theory presented here is for isotropic, homogeneous fluent continua in which internal varying rates of rotations and their gradients can result in additional energy dissipation. The polar theory presented in this paper is inherently local and hence not capable of capturing nonlocal effects. We remark that the polar continuum theory presented in this paper is not to be labeled as "stress couple theory" (see remarks in section 3.3). Rate of dissipation due to rates of rotations necessitates existence of conjugate moment tensor. It is only after the balance of angular momenta we realize that only the antisymmetric part of the Cauchy stress tensor is balanced by the gradients of the Cauchy moment tensor. We note that the existence of the Cauchy moment tensor is established long before we realize a relationship between its gradients and the antisymmetric part of the Cauchy stress tensor. We note that the continuum theory presented in this paper is a more complete thermodynamic framework for isotropic, homogeneous fluent continua compared to what is being used now as it incorporates additional physics due to rates of rotations which is completely ignored in the current thermodynamic framework. The polar continuum theory presented here based on rates of internal varying rotations is not the same as the strain rate gradient theory (see section 2.2). Since the theory presented here accounts for the deformation physics resulting in internal varying rotation rates and the conjugate moment tensor, it is perhaps fitting to call this theory "an internal polar theory for fluent continua" so that this theory can be clearly distinguished from the micropolar theories. In forthcoming publications related to the constitutive theories we refer to this polar theory as "internal polar theory".

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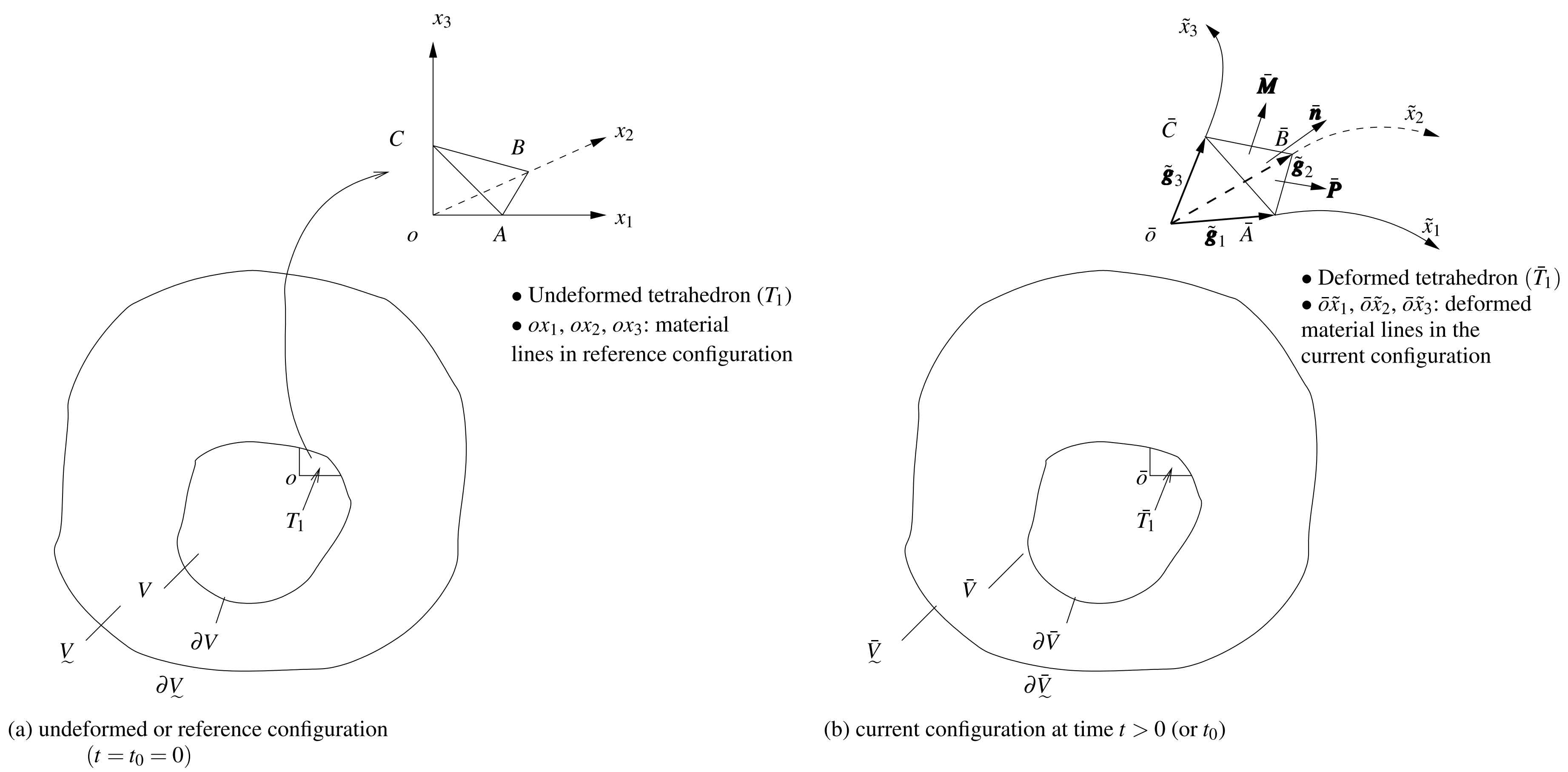

Fig. 1: Reference and current configurations for a deforming volume of matter