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A MORE COMPLETE THERMODYNAMIC FRAMEWORK FOR SOLID CONTINUA

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Dedicated to Professor J.N. Reddy's 70th Birthday

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ABSTRACT

The Jacobian of deformation at a material point can be decomposed into the stretch tensor and the rotation tensor. Thus, varying Jacobians of deformation at the neighboring material points in the deforming volume of solid continua would yield varying stretch and rotation tensors at the material points. Measures of strain, such as Green's strain, at a material point are purely a function of the stretch tensor, i.e. the rotation tensor plays no role in these measures. Alternatively, we could also consider decomposition of displacement gradient tensor into symmetric and skew symmetric tensors. Skew symmetric tensor is also a measure of pure rotations whereas symmetric tensor is a measure of strains, i.e. stretches. The measures of rotations in these two approaches describe the same physics but are in different forms. Polar decomposition gives the rotation matrix and not the rotation angles whereas the skew symmetric part of the displacement gradient tensor yields rotation angles that are explicitly and conveniently defined in terms of the displacement gradients. The varying rotations and rotation rates arise in all deforming solid continua due to varying deformation of the continua at neighboring material points, hence are internal to the volume of solid continua and are explicitly defined by the deformation, therefore do not require additional degrees of freedom to define them. If the internal varying rotations and their rates are resisted by the continua, then there must exist internal moments corresponding to these. The internal rotations and their rates and the corresponding moments can result in additional energy storage and dissipation. This physics is all internal to the deforming continua (hence does not require consideration of additional external degrees of freedom and associated external moments) and is neglected in the presently used continuum theories for isotropic, homogeneous solid continua. The continuum theory presented in this paper considers internal varying rotations and associated conjugate moments in the derivation of the conservation and balance laws, thus the theory presented in this paper is "a polar theory for solid continua" but is different than the micropolar theories published currently in which material points have six external degrees of freedom i.e. rotations are additional external degrees of freedom.

This polar continuum theory only accounts for internal rotations and associated moments that exist as a consequence of deformation but are neglected in the present theories. We call this theory "a polar continuum theory" as it considers rotations and moments as conjugate pairs in a deforming solid continua though these are internal, hence are purely related to the deformation of the solid. It is shown that the polar continuum theory presented in this paper is not the same as the strain gradient theories reported in the literature. The differences are obviously in terms of the physics described by them and the mathematical details associated with conservation and balance laws. In this paper, we only consider polar continuum theory for small deformation and small strain. This polar continuum theory presented here is a more complete thermodynamic framework as it accounts for additional physics of internally varying rotations that is neglected in the currently used thermodynamic framework. This thermodynamic framework is suitable for isotropic, homogeneous solid matter such as thermoelastic and thermoviscoelastic solid continua with and without memory when the deformation is small. The paper also presents preliminary material helpful in consideration of the constitutive theories for polar continua.

1 INTRODUCTION AND LITERATURE REVIEW

In Lagrangian description of deforming matter, the Jacobian of deformation is a fundamental quantity of the measure of deformation of the solid continua. In general, the Jacobian of deformation varies between material points, i.e. it varies between a material point and its neighbors. Polar decomposition of the Jacobian of deformation at material points into stretch (left of right) and pure rotation shows that if the Jacobian of deformation varies between a material point and its neighbors so do the rotations. We could also consider the decomposition of the displacement gradient tensor into symmetric and skew symmetric tensors. The skew symmetric tensor is a measure of pure rotations while the symmetric tensor is a measure of strains. Strain measures (such as Green's strain) are purely a function of stretch tensor or alternatively symmetric part of the displacement gradient tensor. In these measures, rotation tensor plays no role. In non-polar continuum theories, only conjugate stress and strain tensors contribute to the stored energy in the deforming solid continua. Likewise, the dissipation mechanism is purely due to stress tensor and rates of conjugate strain tensor. In such theories, the influence of rotations and the influence of the rates of rotations on the mechanism of energy storage and dissipation is not considered. In the present work, we consider solid continua in which the rotations and the rates of rotations that exist between neighboring material points are resisted by the constitution of the matter, hence result in energy storage and energy dissipation. Thus, the continuum theory presented here for solid continua in Lagrangian description incorporates new physics associated with varying internal rotations and their conjugate moments. This physics is completely absent in the currently used continuum theories for isotropic, homogeneous solid continua. As established in the abstract the theory presented here is indeed a polar continuum theory that incorporates internal varying rotations and conjugate moments in the derivation of conservation and balance laws.

The theory presented here is a continuum theory in Lagrangian description for polar continuum and should not be confused with micropolar continuum theories [1–11] that are designed to accommodate effects at scales smaller than the continuum scale. Micropolar continuum theories require definitions of additional strain measures [6] related to micromechanics. The polar continuum theory presented here uses standard measures of strains as used currently in non-polar continuum theories. In the polar continuum theory presented here, the motivation is to account for the influence of varying rotations at neighboring material points that arises during evolution as these may result in additional energy storage in some solid continua. Polar decomposition of the Jacobian of deformation at neighboring material points clearly substantiates this. An important point to note is that the theory considered here can only account for local rotation effects due to deformation at material points, hence the theory presented here is intrinsically a local polar continuum theory, thus cannot account for nonlocal effects.

In the following we present a brief literature review on micropolar theories, nonlocal theories and stress couple theories. A comprehensive treatment of micropolar theories can be found in the works by Eringen [1–9]. The concept of couple stresses is presented by Koiter [10]. Balance laws for micromorphic materials are presented in reference [11]. The micropolar theories consider micro deformation due to micro constituents in the continuum. In references [12–14] by Reddy et. al. and reference [15] by Zang et. al. nonlocal theories are presented for bending, buckling and vibration of beams, beams with nanocarbon tubes and bending of plates. The nonlocal effects are believed to be incorporated due to the work presented by Eringen [6] in which definition of a nonlocal stress tensor is introduced through integral relationship using the product of macroscopic stress tensor and a distance kernel representing the nonlocal effects. *The polar continuum theory for solid continua presented in this paper is strictly local and non-micropolar*. The concept of couple stresses was introduced by Voigt in 1881 by assuming a couple or moment per unit area on the oblique plane of the deformed tetrahedron in addition to the stress or force per unit area. Since the introduction of this concept many published works have appeared. We cite some recent works, most of which are related to micropolar stress couple theories. Authors in reference [16] report experimental study of micropolar and couple stress elasticity of compact bones in bending. Conservation integrals in couple stress elasticity are reported in reference [17]. A microstructure-dependent Timoshenko beam model based on modified couple stress theories is reported by Ma et. al. [18]. Further account of couple stress theories in conjunction with beams can be found in references [19–21]. Treatment of rotation gradient dependent strain energy and its specialization to Von Kármán plates and beams can be found in reference [22]. Other accounts of micropolar elasticity and Cosserat modeling of cellular solids can be found in references [23–25]. We remark that in references [16–25], Lagrangian description is used for solid matter, however the mathematical descriptions are purely derived using strain energy density functional and principle of virtual work. This approach works well for elastic solids in which mechanical deformation is reversible. Extension of these works to thermoviscoelastic solids with and without memory is not possible. In such materials the thermal field and mechanical deformation are coupled due to the fact that the rate of work results in rate of entropy production. In reference [26] Altenbach and Eremeyev present a linear theory for micropolar plates. Each material point is regarded as a small rigid body with six degrees of freedom. Kinematics of plates is described using the vector of translations and the vector of rotations as dependent variables. Equations of equilibrium are established in $\mathbb{R}^3$ and $\mathbb{R}^2$. Strain energy density function is used to present linear constitutive theory. The mathematical models of reference [27] are extended by the same authors to present strain rate tensors and the constitutive equations for inelastic micropolar materials. In reference [28], authors consider the conditions for the existence of the acceleration waves in thermoelastic micropolar media. The work concludes that the presence of the energy equation with Fourier heat conduction law does not influence the wave physics in thermoelastic micropolar media. Thus, from the point of view of acceleration waves in thermoelastic polar media, thermal effects i.e. temperature can be treated as a parameter. In reference [29], authors present a collection of papers related to the mechanics of continua dealing with micro-macro aspects of the physics (largely related to solid matter). In reference [30] a micro-polar theory is presented for binary media with applications to phase-transitional flow of fiber suspensions. Such flows take place during the filling state of injection molding of short fiber reinforced thermoplastics. A similarity solution for boundary layer flow of a polar fluid is given in reference [31]. In specific the paper borrows constitutive equations that are claimed to be valid for flow behavior of a suspension of very fine particles in a viscous fluid. Kinematics of micropolar continuum is presented in reference [32]. References [33, 34] consider material symmetry groups for linear Cosserat continuum and non-linear polar elastic continuum. Grekova et. al. [35–37] consider various aspects of wave processes in ferromagnetic medium and elastic medium with micro-rotations as well as some aspects of linear reduced Cosserat medium. In references [38–56] various aspects of the kinematics of micropolar theories, stress couple theories, etc. are discussed and presented including some applications to plates and shells.

If the varying rotations and their rates result in energy storage and dissipation, then their energy conjugate moment (shown later in the paper) must exist in the deforming matter. This necessitates the existence of moment (per unit area) on the oblique plane of the deformed tetrahedron. Thus, at the onset, we consider average force per unit area and displacements, and average moment per unit area and the rotations on the oblique plane of the deformed tetrahedron. *The work presented here follows a strictly thermodynamic approach using these* i.e., for polar solid continua we present derivations of: (i) Conservation of mass and present reasons for not deriving conservation of inertia (ii) Balance of linear momenta (iii) Balance of angular momenta (iv) Balance of moments of moments (or couples) (v) First law of thermodynamics and (vi) Second law of thermodynamics in Lagrangian description in which stress and strain, moment and rotations are energy conjugate pairs. *The mathematical description for polar solid continua derived here is applicable to compressible and incompressible polar thermoelastic solids and polar thermoviscoelastic solids with or without memory when augmented with the appropriate constitutive theories.*

2 Notations, Definitions, Measures and Preliminary Considerations

We use an overbar to express quantities in the current configuration, i.e/ all quantities with overbars are functions of deformed coordinates $\bar{x}_i$ and time t . Quantities without an overbar imply Lagrangian description of the quantities in the current configuration, i.e. these are functions of undeformed coordinates x_i and time t. We use the configuration at time $t = t_0 = 0$, commencement of evolution, to be the reference configuration. Thus, x_i ; $i = 1, 2, 3$ and $\bar{\boldsymbol{x}}$ are the coordinates of a material point in the reference and current configurations, respectively, both measured in a fixed Cartesian x-frame. This paper only considers Lagrangian description, hence all measures are expressed in terms of coordinates of the material points in the undeformed configuration (same as reference configuration in the present work) $\boldsymbol{x}$ and time t. We use $[J] = [\frac{\partial\{\bar{x}\}}{\partial\{x\}}]$ to be the Jacobian of deformation. We denote ρ_0 to be the density of the solid matter in the reference configuration, hence it is constant. Φ, θ and η denote the Helmholtz free-energy density, temperature and entropy density.

If the existence of different rotations at the neighboring material points (evident from polar decomposition of the Jacobian of deformation) can result in additional energy storage or dissipation then there must be also coexist moments in the deforming matter. Just like points of application of forces when displaced result in work, the moments through rotations result in work as well. Thus, *in the development of the polar continuum theory presented here we consider existence of internal rotations and moments independent of forces and displacements*. Consider a volume of matter $\underset{\sim}{V}$ in the reference configuration (Figure 1(a)) with closed boundary $\partial\underset{\sim}{V}$. The volume V is isolated from $\underset{\sim}{V}$ by a hypothetical surface ∂V as in cut principle of Cauchy. Consider a tetrahedron T_1 shown in Figure 1(a) such that its oblique plane is part of ∂V and its other three planes are orthogonal to each other and parallel to the planes of the x-frame. Upon deformation $\underset{\sim}{V}$ and $\partial\underset{\sim}{V}$ occupy $\bar{\underset{\sim}{V}}$ and $\partial\bar{\underset{\sim}{V}}$ and likewise V and ∂V deform into $\bar{V}$ and $\partial\bar{V}$. The tetrahedron T_1 deforms into $\bar{T}_1$ whose edges (under finite deformation) are nonorthogonal covariant base vector $\tilde{\boldsymbol{g}}_i$. The plane of the tetrahedron formed by the covariant base vectors are flat but obviously nonorthogonal to each other. We assume the tetrahedron to be the small neighborhood of material point $\bar{o}$ so that assumption of the oblique plane $\bar{A}\bar{B}\bar{C}$ being flat but still part of $\partial\bar{V}$ is valid. When the deformed tetrahedron is isolated from volume $\bar{V}$ it must be in equilibrium under the action of disturbance on the surface of $\bar{A}\bar{B}\bar{C}$ from the volume surrounding $\bar{V}$ and the internal fields that act on the flat faces which equilibrate with the mating faces in volume $\bar{V}$ when the tetrahedron $\bar{T}_1$ is placed back in the volume $\bar{V}$. Consider deformed tetrahedron $\bar{T}_1$. Let $\bar{\boldsymbol{P}}$ be the average stress on plane $\bar{A}\bar{B}\bar{C}$, $\bar{\boldsymbol{M}}$ be the average moment per unit area also on plane $\bar{A}\bar{B}\bar{C}$ henceforth referred to as moment for short and $\bar{\boldsymbol{n}}$ be the normal to the face $\bar{A}\bar{B}\bar{C}$. $\bar{\boldsymbol{P}}$, $\bar{\boldsymbol{M}}$, $\bar{\boldsymbol{n}}$ all have different directions when the deformation is finite.

2.1 Polar Decomposition of the Jacobian of Deformation and Consideration of Local Rotations

Polar decomposition of the Jacobian of deformation is helpful in decomposing deformation into pure stretch and pure rotation. Whether we use left stretch or right stretch, the pure rotation tensor is unique. At each material point with infinitesimal volume surrounding it, the Jacobian of deformation $[J]$ can be decomposed into pure rotation $[R]$ and right stretch tensor $[S_r]$ or left stretch tensor $[S_l]$. $[R]$ is orthogonal and $[S_r]$, $[S_l]$ are symmetric and positive definite. The rotation tensor $[R]$ can equivalently be obtained due to rotations $\boldsymbol{\Theta}$ at the material point. Thus, at every material point, the rotation matrix $[R]$ can be viewed as being due to rotations $\boldsymbol{\Theta}$. If varying rotations at the material points (due to different $[J]$) result in energy storage, then there must be existence of conjugate moments $\boldsymbol{m}$ in the deforming matter, thus the motivation for consideration of $\boldsymbol{\Theta}$ and $\boldsymbol{m}$ in the polar continuum theory presented in this paper.

$$[J] = \left[\frac{\partial\{\bar{x}\}}{\partial\{x\}}\right] = [R][S_r] = [S_l][R] \tag{2.1}$$

$$[R] = [R(\boldsymbol{\Theta})] \tag{2.2}$$

Explicit forms of $\boldsymbol{\Theta}$ that is Θ_{x_1}, Θ_{x_2}, Θ_{x_3} or Θ_1, Θ_2, Θ_3 in terms of antisymmetric part of the displacement gradient tensor establish unique and convenient measures of rotations, hence $[R]$ in

(2.2) based on the gradients of deformation field as shown in section 2.3 is meritorious.

2.2 Rotation Gradients and Strain Gradients

Even though the presence of varying rotations between neighboring material points may influence the energy storage and dissipation in some solid continua, the precise manner in which this occurs is not yet established. All we know at this stage is that just like forces and displacements are work conjugate, the rotations and the moments can also be work conjugate if the deforming matter resists varying rotations between the neighboring material points. Through the derivations of the balance laws presented in section 3 we establish that the symmetric part of the rotation gradient tensor is energy conjugate to the moment tensor. Thus, it is fair to say that the polar part of the theory presented here is due to rotation gradients. The purpose of the material presented in this section is to demonstrate that the polar theory presented here is not the same as the strain gradient theories published in the literature.

In reference [57], the author shows a relationship between the gradients of the rotations in terms of gradients of the strain tensor and the rotation tensor. Based on these and other similar works, it is argued and mostly accepted that the continuum theories that incorporate rotation gradients are same as those that are derived using strain gradients in the conservation and balance laws. In section 1, 2 and 2.1 we have explained the physics we propose to incorporate by using rotations in the continuum theory. The purpose of the material that follows is: (i) first to establish a relationship between the gradients of rotations and the gradients of the strain tensor (similar to reference [57]) and (ii) secondly, to demonstrate, using these relations, that the continuum theories based on rotation gradients and those based on strain gradients are in fact not the same. The resulting theories from the two approaches describe different physics. For simplicity, consider a two dimensional state of deformation in x_1x_2-plane. The displacement gradient tensor $[^dJ]$ in this case is

$$[^dJ] = \frac{\partial\{u_1,u_2\}}{\partial\{x_1,x_2\}} = [^d_sJ] + [^d_aJ] \tag{2.3}$$

$[^d_sJ]$ and $[^d_aJ]$ being symmetric and antisymmetric tensors.

$$[^d_aJ] = \begin{bmatrix} 0 & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) & 0 \end{bmatrix} = \begin{bmatrix} 0 & \Theta_{x_3} \\ -\Theta_{x_3} & 0 \end{bmatrix} \tag{2.4}$$

in which

$$\Theta_{x_3} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right) = \Theta_3 \tag{2.5}$$

is the rotation about the x_3 axis. Gradients of Θ_{x_3} with respect to x_1 and x_2 are

$$\begin{aligned} \Theta_{3,1} &= \frac{1}{2}\left(\frac{\partial^2 u_1}{\partial x_1 \partial x_2} - \frac{\partial^2 u_2}{\partial x_1^2}\right) \\ \Theta_{3,2} &= \frac{1}{2}\left(\frac{\partial^2 u_1}{\partial x_2^2} - \frac{\partial^2 u_2}{\partial x_1 \partial x_2}\right) \end{aligned} \tag{2.6}$$

For small deformation, the strain measures are

$$\begin{aligned} \varepsilon_{11} &= \frac{\partial u_1}{\partial x_1} \\ \varepsilon_{22} &= \frac{\partial u_2}{\partial x_2} \\ \varepsilon_{12} = \varepsilon_{21} &= \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) \end{aligned} \tag{2.7}$$

Substituting from (2.7) into (2.6) we can obtain

$$\begin{aligned} \Theta_{3,1} &= \frac{\partial \varepsilon_{11}}{\partial x_2} - \frac{\partial \varepsilon_{12}}{\partial x_1} \\ \Theta_{3,2} &= \frac{\partial \varepsilon_{12}}{\partial x_2} - \frac{\partial \varepsilon_{22}}{\partial x_1} \end{aligned} \tag{2.8}$$

In (2.8), the gradients $\Theta_{3,1}$ and $\Theta_{3,2}$ of rotation Θ_{x_3} are completely expressed in terms of the gradients of ε_{11} and ε_{22} with respect to x_2 and x_1 and ε_{12} with respect to x_1 as well as x_2.

Remarks

(1) From (2.8) we note that gradients of Θ_{x_3} are functions of $\partial\varepsilon_{11}/\partial x_2$, $\partial\varepsilon_{22}/\partial x_1$, $\partial\varepsilon_{12}/\partial x_1$ and $\partial\varepsilon_{12}/\partial x_2$ but are not dependent on $\partial\varepsilon_{11}/\partial x_1$ and $\partial\varepsilon_{22}/\partial x_2$. This is expected due to the fact that $\partial\varepsilon_{11}/\partial x_1$ and $\partial\varepsilon_{22}/\partial x_2$ are gradients of the elongations per unit length in x_1 and x_2 directions, hence cannot possibly contribute to the gradients of rotations.

(2) Considerations of $\Theta_{3,1}$ and $\Theta_{3,2}$ in the polar theory is identically equivalent to replacing these by the right side of the expressions in (2.8). As long as this condition is satisfied, the polar theory based on rotation gradients is the same as the polar theory based on strain gradients. We keep in mind that $\partial\varepsilon_{11}/\partial x_1$ and $\partial\varepsilon_{22}/\partial x_2$ are not part of the expressions of rotation gradients in (2.8).

(3) A polar theory based on strain gradients must consider $\varepsilon_{ij,k}$, i.e. gradients of all six strains with respect to x_1, x_2 and x_3. Thus, at the onset, it is clear that the strain gradient polar theory for the 2D case will also consider $\partial\varepsilon_{11}/\partial x_1$ and $\partial\varepsilon_{22}/\partial x_2$ in the derivation in addition to the other strain gradients that appear in (2.8). This undoubtedly brings in different physics than what is described by (2.8). If we consider three dimensional case (i.e. $\mathbb{R}^3$) then we would find that additionally $\partial\varepsilon_{33}/\partial x_3$ will appear in this strain gradient polar theory but will be absent in the definitions of the gradients of rotations.

(4) The rotation gradient polar theory resulting due to consideration of local rotations is targeted towards specific physics of rotations and rates of rotations resulting in energy storage and dissipation in a deforming solid. *We have shown that the polar theory based on rotation gradients is clearly not the same as the strain gradient theories*. We remark that equation (2.8) representing rotation gradients as a function of some (and not all) of the strain gradients is a consequence of the mathematical manipulation.

2.3 Stress, Moment and Strain Tensors and Considerations of Rotations

Based on the small deformation assumption, the deformed coordinates $\bar{x}_i$ are approximately same as undeformed coordinates

x_i, thus the deformed tetrahedron $\bar{T}_1$ in the current configuration is close to its map T_1 in the reference configuration. With this assumption all stress measures (first and second Piola-Kirchhoff stress tensors, Cauchy stress tensor) are approximately the same. The same holds for the moment tensors. Thus within the assumption $\bar{\boldsymbol{x}} \simeq \boldsymbol{x}$ we can write

$$\bar{\boldsymbol{P}} = \boldsymbol{P} \quad , \quad \bar{\boldsymbol{M}} = \boldsymbol{M} \tag{2.9}$$

The Cauchy principle for $\boldsymbol{P}$ and $\boldsymbol{M}$ gives

$$\boldsymbol{P} = \boldsymbol{\sigma} \cdot \boldsymbol{n} \quad , \quad \boldsymbol{M} = \boldsymbol{m} \cdot \boldsymbol{n} \tag{2.10}$$

in which $\boldsymbol{\sigma}$ is Cauchy stress tensor and $\boldsymbol{m}$ is Cauchy moment tensor (per unit area). The displacement gradient matrix $[{}^dJ]$ and its decomposition into symmetric and antisymmetric parts $[{}^d_sJ]$ and $[{}^d_aJ]$ gives

$$ {}^dJ_{ij} = \frac{\partial u_i}{\partial x_j} \quad \text{or} \quad [{}^dJ] = \frac{\partial \{u\}}{\partial \{x\}} = [{}^d_sJ] + [{}^d_aJ] \tag{2.11}$$

$$\begin{aligned} [{}^d_sJ] &= \frac{1}{2}\left([{}^dJ] + [{}^dJ]^T\right) \\ [{}^d_aJ] &= \frac{1}{2}\left([{}^dJ] - [{}^dJ]^T\right) \end{aligned} \tag{2.12}$$

Let $\{\Theta\} = \begin{bmatrix} \Theta_{x_1} & \Theta_{x_2} & \Theta_{x_3} \end{bmatrix}^T$ or $\boldsymbol{\Theta}$ be the *rotation about* ox_1*,* ox_2 *and* ox_3 *axes of the x-frame*, then we have

$$[{}^d_aJ] = \begin{bmatrix} 0 & \Theta_{x_3} & \Theta_{x_2} \\ -\Theta_{x_3} & 0 & \Theta_{x_1} \\ -\Theta_{x_2} & -\Theta_{x_1} & 0 \end{bmatrix} \tag{2.13}$$

in which

$$\begin{aligned} \Theta_{x_1} &= \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2}\right) \\ \Theta_{x_2} &= \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) \\ \Theta_{x_3} &= \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right) \end{aligned} \tag{2.14}$$

We define the *gradients of rotation* $\boldsymbol{\Theta}$ by

$$[{}^\Theta J] = \frac{\partial \{\Theta\}}{\partial \{x\}} \quad \text{or} \quad {}^\Theta J_{ij} = \frac{\partial \Theta_i}{\partial x_j} \tag{2.15}$$

We also decompose $[{}^\Theta J]$ into symmetric and antisymmetric parts $[{}^\Theta_sJ]$ and $[{}^\Theta_aJ]$

$$[{}^\Theta J] = [{}^\Theta_sJ] + [{}^\Theta_aJ] \tag{2.16}$$

in which

$$\begin{aligned} [{}^\Theta_sJ] &= \frac{1}{2}\left([{}^\Theta J] + [{}^\Theta J]^T\right) \\ [{}^\Theta_aJ] &= \frac{1}{2}\left([{}^\Theta J] - [{}^\Theta J]^T\right) \end{aligned} \tag{2.17}$$

For finite deformation, Green's strain tensor is a suitable choice for measure of strain.

$$[\varepsilon] = \frac{1}{2}\left([J]^T[J] - [I]\right) \tag{2.18}$$

and since

$$[J] = [I] + [{}^dJ] \tag{2.19}$$

then $[\varepsilon]$ can be expressed in terms of $[{}^dJ]$

$$[\varepsilon] = \frac{1}{2}\left([{}^dJ] + [{}^dJ]^T + [{}^dJ]^T[{}^dJ]\right) \tag{2.20}$$

For small deformation, we approximate $[\varepsilon]$ by

$$[\varepsilon] \simeq \frac{1}{2}\left([{}^dJ] + [{}^dJ]^T\right) = [{}^d_sJ] \tag{2.21}$$

and correspondingly, due to rotation, we define $[{}^\Theta\varepsilon]$ as

$$[{}^\Theta\varepsilon] = \frac{1}{2}\left([{}^\Theta J] + [{}^\Theta J]^T\right) = [{}^\Theta_sJ] \tag{2.22}$$

We also define the *gradients of velocities* as

$$\frac{\partial \{v\}}{\partial \{x\}} = [L] = [D] + [W] \tag{2.23}$$

in which

$$\begin{aligned} [D] &= \frac{1}{2}\left([L] + [L]^T\right) \\ [W] &= \frac{1}{2}\left([L] - [L]^T\right) \end{aligned} \tag{2.24}$$

Likewise, the *gradients of the rates of rotation* are defined as

$$\frac{\partial \{\dot{\Theta}\}}{\partial \{x\}} = [{}^\Theta L] = [{}^\Theta D] + [{}^\Theta W] \tag{2.25}$$

in which

$$\begin{aligned} [{}^\Theta D] &= \frac{1}{2}\left([{}^\Theta L] + [{}^\Theta L]^T\right) \\ [{}^\Theta W] &= \frac{1}{2}\left([{}^\Theta L] - [{}^\Theta L]^T\right) \end{aligned} \tag{2.26}$$

3 Conservation and Balance Laws

We remark that the polar continuum theory considered here incorporates new physics due to internal varying rotations between the material points. This physics is absent in the currently used thermodynamic framework for isotropic, homogeneous solid continua. This new physics due to rotations may influence some or all conservation and balance laws. In order to determine the precise influence of the new physics (or lack of it) on the conservation and balance laws, we must initiate the derivations of the conservation and balance laws at a fundamental stage as we do for the non-polar case [58] so that the resulting equations can be compared with the non-polar case to determine how these laws are modified or influenced by the physics due to internal varying rotations. We caution that after the derivation of conservation and balance laws we may find that some laws are not influenced by this new physics in which case the corresponding equations will obviously be the same as those for the non-polar case. Nonetheless the derivation of all conservation and balance laws must be presented in completeness otherwise we can not determine whether a particular law is influenced by this new physics when compared to the non-polar case. We wish to remark that in the following sections even if some derivations yield the same equations as for the non-polar case, their derivations are essential to keep in the paper as these are necessary to establish this conclusion compared to the non-polar case.

In a polar continuum theory with displacements, displacement gradients, rotations, and rotation gradients as field variables, we must consider the following conservation and balance laws based on the assumption of thermodynamic equilibrium during the evolution: (1) conservation of mass and conservation of inertia, (2) balance of linear momenta, (3) balance of angular momenta, (4) balance of moments of moments (i.e., couples), (5) first law of thermodynamics (i.e. balance of energy), and (6) second law of thermodynamics (i.e. entropy inequality). We consider details of the derivations of these in the following sections.

3.1 Conservation of Mass and Inertia

The continuity equation resulting from the principle of conservation of mass remains for non-polar continuum remains the same as for the polar case. We obtain the following continuity equation in Lagrangian description [58–61]:

$$\rho_0(\boldsymbol{x}) = |J|\rho(\boldsymbol{x},t) \tag{3.1}$$

where $\rho_0(\boldsymbol{x})$ is the density in the reference configuration and $\rho(\boldsymbol{x},t)$ is the Lagrangian description of the density of a material point at $\bar{\boldsymbol{x}}$ in the current configuration. In micropolar continuum theories we consider continuum with microfibers. In a deforming volume of matter these microfibers (considered inextensible in micro-polar continuum theories) will have inertial effects due to rotation. Conservation of inertia refers to such inertial effects. *In the polar continuum theory considered here we do not consider the inertial effects at present*. Thus, we assume that in the polar continuum theory considered here there is only one conservation law leading to same continuity equation (3.1) as in case of non-polar continuum theory.

3.2 Balance of Linear Momenta

For a deforming volume of matter the rate of change of linear momenta must be equal to the sum of all other forces acting on it. This is Newton's second law applied to a volume of matter. The derivation is same as that for non-polar continuum theory. Thus, we can write (for small deformation) the following [58]:

$$\rho_0 \frac{D\boldsymbol{v}}{Dt} - \rho_0 \boldsymbol{F}^b - \boldsymbol{\nabla}\cdot\boldsymbol{\sigma} = 0$$
$$\text{or} \tag{3.2}$$
$$\rho_0 \frac{D\{v\}}{Dt} - \rho_0\{F^b\} - [\sigma]^T\{\nabla\} = 0$$

In Lagrangian description $\frac{D}{Dt} = \frac{\partial}{\partial t}$ and $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{x},t)$ are velocities, $\boldsymbol{F}^b$ are body forces per unit mass and $\boldsymbol{\sigma}$ is the stress tensor. Equations (3.2) are momentum equations in x_1, x_2, and x_3 directions.

3.3 Balance of Angular Momenta

The principle of balance of angular momentum for a polar continuum can be stated as follows: *The time rate of change of total moment of momenta for a polar continuum is equal to the vector sum of the moments of external forces and the moments*. Thus, due to the surface stress $\bar{\boldsymbol{P}}$, surface moment $\bar{\boldsymbol{M}}$ (per unit area), body force $\bar{\boldsymbol{F}}^b$ (per unit mass) and the momentum $\bar{\rho}\bar{\boldsymbol{v}}d\bar{V}$ for an elemental mass $\bar{\rho}d\bar{V}$ in the current configuration (using the Eulerian description) we can write the following:

$$\frac{D}{Dt}\int_{\bar{V}(t)} \bar{\boldsymbol{x}}\times\bar{\rho}\bar{\boldsymbol{v}}\,d\bar{V} = \int_{\partial\bar{V}(t)} \left(\bar{\boldsymbol{x}}\times\bar{\boldsymbol{P}} - \bar{\boldsymbol{M}}\right) d\bar{A} + \int_{\bar{V}(t)} \bar{\boldsymbol{x}}\times\bar{\rho}\bar{\boldsymbol{F}}^b\,d\bar{V} \tag{3.3}$$

We consider each terms in (3.3) individually.

$$\begin{aligned}
\frac{D}{Dt}\int_{\bar{V}(t)} \bar{\boldsymbol{x}}\times\bar{\rho}\bar{\boldsymbol{v}}d\bar{V} &= \frac{D}{Dt}\int_{\bar{V}(t)} \epsilon_{ijk}\bar{x}_i\bar{v}_j\bar{\rho}d\bar{V} \\
&= \frac{D}{Dt}\int_V \epsilon_{ijk}x_iv_j\rho_0 dV \\
&= \int_V \rho_0\epsilon_{ijk}\frac{D}{Dt}(x_iv_j)\,dV \\
&= \int_V \rho_0\epsilon_{ijk}\left(v_iv_j + x_i\frac{Dv_j}{Dt}\right) dV
\end{aligned} \tag{3.4}$$

The first term on the right hand side is

$$\begin{aligned}
\int_{\partial\bar{V}(t)} \left(\bar{\boldsymbol{x}}\times\bar{\boldsymbol{P}} - \bar{\boldsymbol{M}}\right) d\bar{A} &= \int_{\partial\bar{V}(t)} \left(\bar{\boldsymbol{x}}\times(\bar{\boldsymbol{\sigma}})^T\cdot\bar{\boldsymbol{n}} - (\bar{\boldsymbol{m}})^T\cdot\bar{\boldsymbol{n}}\right) d\bar{A} \\
&= \int_{\partial\bar{V}(t)} \left(\bar{\boldsymbol{x}}\times(\bar{\boldsymbol{\sigma}})^T\cdot\bar{\boldsymbol{n}}d\bar{A} - (\bar{\boldsymbol{m}})^T\cdot\bar{\boldsymbol{n}}d\bar{A}\right) \\
&= \int_{\partial V} \left(\boldsymbol{x}\times(\boldsymbol{\sigma})^T\cdot\boldsymbol{n}dA - (\boldsymbol{m})^T\cdot\boldsymbol{n}dA\right) \\
&= \int_{\partial V} \left(\epsilon_{ijk}x_i\sigma_{mj}n_m - m_{mk}n_m\right) dA
\end{aligned} \tag{3.5}$$

in which $\bar{\boldsymbol{\sigma}}$ is the *Cauchy stress tensor* and $\bar{\boldsymbol{m}}$ is the *Cauchy moment tensor*. Using divergence theorem yields

$$\begin{aligned}
\int_{\partial\bar{V}(t)} \left(\bar{\boldsymbol{x}}\times\bar{\boldsymbol{P}} - \bar{\boldsymbol{M}}\right) d\bar{A} &= \int_V \left(\epsilon_{ijk}\left(x_i\sigma_{mj}\right)_{,m} - m_{mk,m}\right) dV \\
&= \int_V \left(\epsilon_{ijk}\left(\delta_{im}\sigma_{mj} + x_i(\sigma_{mj})_{,m}\right) - m_{mk,m}\right) dV \\
&= \int_V \left(\epsilon_{ijk}\left(\sigma_{ij} + x_i(\sigma_{mj})_{,m}\right) - m_{mk,m}\right) dV
\end{aligned} \tag{3.6}$$

The second term on the right hand side is

$$\int_{\bar{V}(t)} \bar{\boldsymbol{x}}\times\bar{\rho}\bar{\boldsymbol{F}}^b d\bar{V} = \int_{\bar{V}(t)} \epsilon_{ijk}\bar{x}_i\bar{F}_j^b\bar{\rho}d\bar{V} = \int_V \epsilon_{ijk}x_iF_j{}^b\rho_0 dV \tag{3.7}$$

Substituting from (3.4), (3.5) and (3.7) into (3.3)

$$\int_V \rho_0 \epsilon_{ijk} \left(v_i v_j + x_i \frac{Dv_j}{Dt} \right) dV = \int_V \left(\epsilon_{ijk}(\sigma_{ij} + x_i(\sigma_{mj})_{,m}) - m_{mk,m} \right) dV + \int_V \epsilon_{ijk} x_i F_j^b \rho_0 \, dV \tag{3.8}$$

We note that

$$\epsilon_{ijk} v_i v_j = 0 \tag{3.9}$$

hence, (3.8) reduces to

$$\int_V \epsilon_{ijk} \left(x_i \left(\rho_0 \frac{Dv_j}{Dt} - \rho_0 F_j^b - \sigma_{mj,m} \right) \right) dV + \int_V \left(m_{mk,m} - \epsilon_{ijk}\sigma_{ij} \right) dV = 0 \tag{3.10}$$

Using balance of linear momenta (3.2) in (3.10) we obtain

$$\int_V \left(m_{mk,m} - \epsilon_{ijk}\sigma_{ij} \right) dV = 0 \tag{3.11}$$

and since the volume V is arbitrary

$$m_{mk,m} - \epsilon_{ijk}\sigma_{ij} = 0 \tag{3.12}$$

$$\text{or} \quad \boldsymbol{\nabla} \cdot \boldsymbol{m} - \boldsymbol{\epsilon} : \boldsymbol{\sigma} = 0 \tag{3.13}$$

$$\text{or} \quad [m]^T \{\nabla\} - \boldsymbol{\epsilon} : \boldsymbol{\sigma} = 0 \tag{3.14}$$

Equation (3.12) represents balance of angular momenta. We note that *the Cauchy stress tensor* $\boldsymbol{\sigma}$ *is non-symmetric*. It is instructive to expand (3.12) into three equations

$$\begin{aligned} \frac{\partial m_{i1}}{\partial x_i} - (\sigma_{23} - \sigma_{32}) &= 0 \\ \frac{\partial m_{i2}}{\partial x_i} - (\sigma_{31} - \sigma_{13}) &= 0 \\ \frac{\partial m_{i3}}{\partial x_i} - (\sigma_{12} - \sigma_{21}) &= 0 \end{aligned} \tag{3.15}$$

From (3.15) we note that *off-diagonal elements of the stress tensor* $\boldsymbol{\sigma}$ *are balanced by the gradients of the Cauchy moment tensor*.

Remarks

(a) In the balance of angular momenta, the rate of change of angular momenta is balanced by the vector sum of the moments of the forces. Thus, this balance law naturally contains moments due to components of the stress tensor acting on the faces of the deformed tetrahedron. Normal stress components obviously do not contribute to this. Hence, the moments contained in this balance law due to stresses are only caused by the shear stresses.

(b) In the case of non-polar solid continua, the balance of angular momenta is a statement of self equilibrating moments due to shear stresses that yields

$$\boldsymbol{\varepsilon} : \boldsymbol{\sigma} = 0 \tag{3.16}$$

which implies that $\boldsymbol{\sigma}$ is symmetric. An important point to note is that (3.16) is a result of stress couples due to shear stresses.

(c) In the case of polar continua, the existence of moments $[m]$ due to the material constitution resisting the rotations results in the shear stress couples being balanced by the internal moments. Thus, for polar continua, the balance of angular momenta yields (3.14) instead of (3.16), i.e.

$$[m]^T \{\nabla\} - \boldsymbol{\epsilon} : \boldsymbol{\sigma} = 0 \tag{3.17}$$

We note that (3.17) is also a result of stress couples caused by shear stresses.

(d) Thus, both non-polar and polar continuum theories use stress couples in the angular momenta balance law. *Referring to the polar continuum theory as stress couple theory is inappropriate as the non-polar theory also makes use of the stress couples.*

(e) From (3.14) or (3.15) we note that gradients of $[m]$ equilibrate with the antisymmetric components of the stress tensor $\boldsymbol{\sigma}$ as the symmetric components cancel each other in each of the three equations in (3.15).

(f) Lastly, we emphasize that *appearance of equation* (3.14) *in other theories published in the literature does not necessarily make the polar continuum theory presented here same as those in the literature*. In this work, we begin by demonstrating that the varying rotations at the neighboring material points, when resisted by the deforming matter, require existence of internal moment tensor $[m]$. The balance of angular momenta establishes relationship between $[m]$ and $[\sigma]$ (equations (3.14) or (3.15)).

3.4 Balance of Moments of Moments (or Couples)

Forces, moments, moments of moments ... are progressively higher order effects or terms, hence must satisfy appropriate balance laws to ensure absence of rigid rotation or rigid translation of the deforming volume of continua. Balance of angular momenta (moments of forces) must be considered for couples created by forces and the moments. Likewise, since moment is similar to force, but is a higher order effect or term than force, a balance law similar to balance of angular momentum i.e. balance of moment of couples or moments must be considered to ensure lack of rigid motion of the deforming continua. Just like in the case of non-polar, isotropic, homogeneous fluent continua balance of angular momenta (moments of forces) restricts the Cauchy stress tensor to be symmetric, we can expect this balance law to impose some restrictions on the Cauchy moment tensor. See reference [56] for additional information. Many published works use moment of moments but this is not specifically stated as a balance law for the polar case, hence we do not point these out here. However, reference [56] explicitly states this as a balance law and uses it to derive relations similar to those presented here.

For the deforming volume of matter to be in equilibrium, the moments of moments (or couples) must vanish. In the moment of moments we must consider $\bar{\boldsymbol{M}}$ and also the shear components of the stress tensor $\bar{\boldsymbol{\sigma}}$, i.e., $\boldsymbol{\epsilon} : \bar{\boldsymbol{\sigma}}$. Thus, we can write (neglecting inertial terms) in Eulerian description

$$\int_{\bar{V}} \bar{\boldsymbol{x}} \times (\boldsymbol{\epsilon} : \bar{\boldsymbol{\sigma}})\, d\bar{V} - \int_{\partial\bar{V}} \bar{\boldsymbol{x}} \times \bar{\boldsymbol{M}}\, d\bar{A} = 0 \tag{3.18}$$

We expand the second term in (3.18) and then convert the integral over $\partial\bar{V}$ to the integral over $\bar{V}$ using the divergence theorem.

$$\begin{aligned}
\int_{\partial\bar{V}} \bar{\boldsymbol{x}} \times \bar{\boldsymbol{M}}\, d\bar{A} &= \int_{\partial\bar{V}} \epsilon_{ijk} x_i \bar{M}_j\, d\bar{A} \\
&= \int_{\partial\bar{V}} \epsilon_{ijk} \bar{x}_i \bar{m}_{mj} \bar{n}_m\, d\bar{A} \\
&= \int_{\bar{V}} (\epsilon_{ijk} \bar{x}_i \bar{m}_{mj})_{,m}\, d\bar{V} \\
&= \int_{\bar{V}} \epsilon_{ijk} (\bar{x}_{i,m} \bar{m}_{mj} + \bar{x}_i \bar{m}_{mj,m})\, d\bar{V} \\
&= \int_{\bar{V}} \epsilon_{ijk} (\delta_{im} \bar{m}_{mj} + \bar{x}_i \bar{m}_{mj,m})\, d\bar{V} \\
&= \int_{\bar{V}} \epsilon_{ijk} (\bar{m}_{ij} + \bar{x}_i \bar{m}_{mj,m})\, d\bar{V} \\
&= \int_{\bar{V}} \epsilon_{ijk} \bar{m}_{ij}\, d\bar{V} + \int_{\bar{V}} \epsilon_{ijk} \bar{x}_i \bar{m}_{mj,m}\, d\bar{V} \\
&= \int_{\bar{V}} \epsilon_{ijk} \bar{m}_{ij}\, d\bar{V} + \int_{\bar{V}} \bar{\boldsymbol{x}} \times \left(\bar{\boldsymbol{\nabla}} \cdot \bar{\boldsymbol{m}}\right) d\bar{V}
\end{aligned} \tag{3.19}$$

Using equation (3.19) in (3.18) and collecting terms

$$\int_{\bar{V}} \bar{\boldsymbol{x}} \times \left(-\bar{\boldsymbol{\nabla}} \cdot \bar{\boldsymbol{m}} + \boldsymbol{\epsilon} : \bar{\boldsymbol{\sigma}}\right) d\bar{V} - \int_{\bar{V}} \epsilon_{ijk} \bar{m}_{ij}\, d\bar{V} = 0 \tag{3.20}$$

The first term in (3.20) vanishes due to balance of angular momenta (3.12) and we obtain

$$\int_{\bar{V}} \epsilon_{ijk} \bar{m}_{ij}\, d\bar{V} = 0 \tag{3.21}$$

and since $\bar{V}$ is arbitrary, (3.21) implies

$$\epsilon_{ijk} \bar{m}_{ij} = 0 \qquad \text{and} \qquad \epsilon_{ijk} m_{ij} = 0 \tag{3.22}$$

Equation (3.22) implies that *the Cauchy moment tensor* $\boldsymbol{m}$ *is symmetric*. Thus, we can see that the consequence of this balance law is to impose the restriction of symmetry on the Cauchy moment tensor. We note that in the polar theory presented here, the Cauchy moment tensor is symmetric, but the Cauchy stress tensor is nonsymmetric, whereas in the corresponding non-polar theory, Cauchy stress tensor is symmetric and Cauchy moment tensor is null as the internal rotations are ignored in the theory. Symmetry of the Cauchy moment tensor is a restriction placed on the Cauchy moment tensor due to this balance law.

3.5 First Law of Thermodynamics

The sum of work and heat added to a deforming volume of matter must result in increase of the energy of the system. Expressing this as a rate equation in Eulerian description we can write

$$\frac{D\bar{E}_t}{Dt} = \frac{D\bar{Q}}{Dt} + \frac{D\bar{W}}{Dt} \tag{3.23}$$

$\bar{E}_t$, $\bar{Q}$ and $\bar{W}$ are total energy, heat added and work done. These can be written as

$$\frac{D\bar{E}_t}{Dt} = \frac{D}{Dt} \int_{\bar{V}(t)} \bar{\rho} \left(\bar{e} + \frac{1}{2} \bar{\boldsymbol{v}} \cdot \bar{\boldsymbol{v}} - \bar{\boldsymbol{F}}^b \cdot \bar{\boldsymbol{u}}\right) d\bar{V} \tag{3.24}$$

$$\frac{D\bar{Q}}{Dt} = - \int_{\partial\bar{V}(t)} \bar{\boldsymbol{q}} \cdot \bar{\boldsymbol{n}}\, d\bar{A} \tag{3.25}$$

$$\frac{D\bar{W}}{Dt} = \int_{\partial\bar{V}(t)} (\bar{\boldsymbol{P}} \cdot \bar{\boldsymbol{v}} + \bar{\boldsymbol{M}} \cdot \dot{\bar{\boldsymbol{\Theta}}})\, d\bar{A} \tag{3.26}$$

where $\bar{e}$ is specific internal energy, $\bar{\boldsymbol{F}}^b$ is body force vector per unit mass, $\bar{\boldsymbol{q}}$ is rate of heat. *In* (3.24) *we have neglected rotary inertia*. This is consistent with the assumption used in the derivation of the conservation law in section 3.1. Note that the additional term $\bar{\boldsymbol{M}} \cdot \dot{\bar{\boldsymbol{\Theta}}}$ in $\frac{D\bar{W}}{Dt}$ contributes additional rate of work due to rates of rotations. We expand integrals in (3.24)-(3.26). Following reference [58], we can show the following.

$$\begin{aligned}
&\frac{D}{Dt} \int_{\bar{V}(t)} \bar{\rho} \left(\bar{e} + \frac{1}{2} \bar{\boldsymbol{v}} \cdot \bar{\boldsymbol{v}} - \bar{\boldsymbol{F}}^b \cdot \bar{\boldsymbol{u}}\right) d\bar{V} \\
&\qquad = \int_{V} \left(\rho_0 \frac{De}{Dt} + \rho_0 \boldsymbol{v} \cdot \frac{D\boldsymbol{v}}{Dt} - \rho_0 \boldsymbol{F}^b \cdot \boldsymbol{v}\right) dV
\end{aligned} \tag{3.27}$$

Using

$$\begin{aligned}
\bar{\boldsymbol{q}} \cdot \bar{\boldsymbol{n}}\, d\bar{A} &= \boldsymbol{q} \cdot \boldsymbol{n} dA \\
\bar{\rho} d\bar{V} &= \rho_0 dV \\
d\bar{V} &= |J| dV
\end{aligned} \tag{3.28}$$

then, applying divergence theorem

$$- \int_{\partial\bar{V}(t)} \bar{\boldsymbol{q}} \cdot \bar{\boldsymbol{n}}\, d\bar{A} = - \int_{\partial V} \boldsymbol{q} \cdot \boldsymbol{n}\, dA = - \int_{V} \boldsymbol{\nabla} \cdot \boldsymbol{q}\, dV \tag{3.29}$$

Using stress tensor $\boldsymbol{\sigma}$ and moment tensor $\boldsymbol{m}$ and following reference [58] we can show

$$\bar{\boldsymbol{P}} \cdot \bar{\boldsymbol{v}} d\bar{A} = \boldsymbol{v} \cdot (\boldsymbol{\sigma})^T \cdot \boldsymbol{n} dA = \left(\boldsymbol{v} \cdot (\boldsymbol{\sigma})^T\right) \cdot d\boldsymbol{A} \tag{3.30}$$

$$\bar{\boldsymbol{M}} \cdot \dot{\bar{\boldsymbol{\Theta}}} d\bar{A} = \left(\dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{m})^T\right) \cdot \boldsymbol{n} dA = \left(\dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{m})^T\right) \cdot d\boldsymbol{A} \tag{3.31}$$

Thus, we can write the following for (3.23).

$$\begin{aligned}
&\int_{V} \left(\rho_0 \frac{De}{Dt} + \rho_0 \boldsymbol{v} \cdot \frac{D\boldsymbol{v}}{Dt} - \rho_0 \boldsymbol{F}^b \cdot \boldsymbol{v}\right) dV \\
&\qquad = - \int_{V} \boldsymbol{\nabla} \cdot \boldsymbol{q} dV + \int_{\partial V} \left(\boldsymbol{v} \cdot (\boldsymbol{\sigma})^T\right) \cdot d\boldsymbol{A} + \int_{\partial V} \left(\dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{m})^T\right) \cdot d\boldsymbol{A}
\end{aligned} \tag{3.32}$$

and using divergence theorem for the integrals over ∂V

$$\int_V \left(\rho_0 \frac{De}{Dt} + \rho_0 \boldsymbol{v} \cdot \frac{D\boldsymbol{v}}{Dt} - \rho_0 \boldsymbol{F}^b \cdot \boldsymbol{v}\right) dV = -\int_V \boldsymbol{\nabla} \cdot \boldsymbol{q} dV + \int_V \boldsymbol{\nabla} \cdot \left(\boldsymbol{v} \cdot (\boldsymbol{\sigma})^T\right) dV + \int_V \boldsymbol{\nabla} \cdot \left(\dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{m})^T\right) dV \tag{3.33}$$

Following reference [58] we can also show that

$$\boldsymbol{\nabla} \cdot \left(\boldsymbol{v} \cdot (\boldsymbol{\sigma})^T\right) = \boldsymbol{v} \cdot (\boldsymbol{\nabla} \cdot \boldsymbol{\sigma}) + \sigma_{ji} \frac{\partial v_i}{\partial x_j} \tag{3.34}$$

$$\boldsymbol{\nabla} \cdot \left(\dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{m})^T\right) = \dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{\nabla} \cdot \boldsymbol{m}) + m_{ji} \frac{\partial \dot{\Theta}_i}{\partial x_j} \tag{3.35}$$

and substituting from (3.34) and (3.35) into (3.33)

$$\int_V \left(\rho_0 \frac{De}{Dt} + \rho_0 \boldsymbol{v} \cdot \frac{D\boldsymbol{v}}{Dt} - \rho_0 \boldsymbol{F}^b \cdot \boldsymbol{v}\right) dV = -\int_V \boldsymbol{\nabla} \cdot \boldsymbol{q} dV + \int_V \left(\boldsymbol{v} \cdot (\boldsymbol{\nabla} \cdot \boldsymbol{\sigma}) + \sigma_{ji} \frac{\partial v_i}{\partial x_j} + \dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{\nabla} \cdot \boldsymbol{m}) + m_{ji} \frac{\partial \dot{\Theta}_i}{\partial x_j}\right) dV \tag{3.36}$$

Moving all terms to the left of the equality and regrouping

$$\int_V \boldsymbol{v} \cdot \left(\rho_0 \frac{D\boldsymbol{v}}{Dt} - \rho_0 \boldsymbol{F}^b - \boldsymbol{\nabla} \cdot \boldsymbol{\sigma}\right) dV + \int_V \left(\rho_0 \frac{De}{Dt} + \boldsymbol{\nabla} \cdot \boldsymbol{q} - \sigma_{ji} \frac{\partial v_i}{\partial x_j} - m_{ji} \frac{\partial \dot{\Theta}_i}{\partial x_j} - \dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{\nabla} \cdot \boldsymbol{m})\right) dV = 0 \tag{3.37}$$

Using (3.2) (balance of linear momenta), (3.37) reduces to

$$\int_V \left(\rho_0 \frac{De}{Dt} + \boldsymbol{\nabla} \cdot \boldsymbol{q} - \sigma_{ji} \frac{\partial v_i}{\partial x_j} - m_{ji} \frac{\partial \dot{\Theta}_i}{\partial x_j} - \dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{\nabla} \cdot \boldsymbol{m})\right) dV = 0 \tag{3.38}$$

Since volume V is arbitrary, we have

$$\rho_0 \frac{De}{Dt} + \boldsymbol{\nabla} \cdot \boldsymbol{q} - \sigma_{ji} \frac{\partial v_i}{\partial x_j} - \left(m_{ji} \frac{\partial \dot{\Theta}_i}{\partial x_j} + \dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{\nabla} \cdot \boldsymbol{m})\right) = 0 \tag{3.39}$$

We note that in $\dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{\nabla} \cdot \boldsymbol{m})$, the term $\boldsymbol{\nabla} \cdot \boldsymbol{m}$ can be substituted from (3.13) thereby eliminating gradients of $\boldsymbol{m}$ but introducing $\boldsymbol{\sigma}$ in its place.

3.6 Second Law of Thermodynamics

If $\bar{\eta}$ is entropy density in volume $\bar{V}(t)$, $\bar{h}$ is the entropy flux between $\bar{V}(t)$ and the volume of matter surrounding it and $\bar{s}$ is the source of entropy in $\bar{V}(t)$ due to non-contacting bodies, then the rate of increase of entropy in volume $\bar{V}(t)$ is at least equal to that supplied to $\bar{V}(t)$ from all contacting and non-contacting sources [58]. Thus

$$\frac{D}{Dt} \int_{\bar{V}(t)} \bar{\eta} \bar{\rho} \, d\bar{V} \geq \int_{\partial \bar{V}(t)} \bar{h} \, d\bar{A} + \int_{\bar{V}(t)} \bar{s} \bar{\rho} \, d\bar{V} \tag{3.40}$$

Using Cauchy's postulate for $\bar{h}$ i.e.,

$$\bar{h} = -\bar{\boldsymbol{\psi}} \cdot \bar{\boldsymbol{n}} \tag{3.41}$$

Using (3.41) in (3.40)

$$\frac{D}{Dt} \int_{\bar{V}(t)} \bar{\eta} \bar{\rho} d\bar{V} \geq -\int_{\partial \bar{V}(t)} \bar{\boldsymbol{\psi}} \cdot \bar{\boldsymbol{n}} \, d\bar{A} + \int_{\bar{V}(t)} \bar{s} \bar{\rho} \, d\bar{V} \tag{3.42}$$

We need to transform (3.42) to Lagrangian description. This can be done using

$$\begin{aligned} d\bar{V} &= |J| dV \\ \rho_0 &= |J| \bar{\rho} \\ \bar{\boldsymbol{\psi}} \cdot \bar{\boldsymbol{n}} d\bar{A} &= \boldsymbol{\psi} \cdot \boldsymbol{n} \, dA \end{aligned} \tag{3.43}$$

Using (3.43) in (3.42)

$$\frac{D}{Dt} \int_V \eta \rho_0 \, dV \geq -\int_{\partial V} \boldsymbol{\psi} \cdot \boldsymbol{n} \, dA + \int_V s \rho_0 \, dV \tag{3.44}$$

Using Gauss's divergence theorem for the terms over ∂V gives (noting that $\boldsymbol{\psi}$ is a tensor of rank one)

$$\frac{D}{Dt} \int_V \eta \rho_0 \, dV \geq -\int_V \boldsymbol{\nabla} \cdot \boldsymbol{\psi} \, dV + \int_V s \rho_0 \, dV \tag{3.45}$$

or

$$\int_v \left(\rho_0 \frac{D\eta}{Dt} + \boldsymbol{\nabla} \cdot \boldsymbol{\psi} - \rho_0 s\right) dV \geq 0 \tag{3.46}$$

and since volume V is arbitrary

$$\rho_0 \frac{D\eta}{Dt} + \boldsymbol{\nabla} \cdot \boldsymbol{\psi} - \rho_0 s \geq 0 \tag{3.47}$$

Equation (3.47) is entropy inequality and is the most fundamental form resulting from the second law of thermodynamics. A more useful form can be derived if we assume

$$\boldsymbol{\psi} = \frac{\boldsymbol{q}}{\theta} \quad , \quad s = \frac{r}{\theta} \tag{3.48}$$

where θ is absolute temperature, $\boldsymbol{q}$ is the heat vector and r is a suitable potential, then

$$\boldsymbol{\nabla} \cdot \boldsymbol{\psi} = \psi_{i,i} = \frac{q_{i,i}}{\theta} - \frac{q_i \theta_{,i}}{\theta^2} = \frac{q_{i,i}}{\theta} - \frac{q_i g_i}{\theta^2} = \frac{\boldsymbol{\nabla} \cdot \boldsymbol{q}}{\theta} - \frac{\boldsymbol{q} \cdot \boldsymbol{g}}{\theta^2} \tag{3.49}$$

Substituting from (3.49) into (3.47) and multiplying throughout by θ yields

$$\rho_0 \frac{D\eta}{Dt} + (\boldsymbol{\nabla} \cdot \boldsymbol{q} - \rho_0 r) - \frac{1}{\theta} \boldsymbol{q} \cdot \boldsymbol{g} \geq 0 \tag{3.50}$$

From energy equation (3.39) (after inserting $\rho_0 r$ term)

$$\boldsymbol{\nabla} \cdot \boldsymbol{q} - \rho_0 r = -\rho_0 \frac{De}{Dt} + \sigma_{ji} \frac{\partial v_i}{\partial x_j} + m_{ji} \frac{\partial \dot{\Theta}_i}{\partial x_j} + \dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{\nabla} \cdot \boldsymbol{m}) \tag{3.51}$$

Substituting from (3.51) into (3.50)

$$\rho_0 \theta \frac{D\eta}{Dt} - \rho_0 \frac{De}{Dt} + \sigma_{ji}\frac{\partial v_i}{\partial x_j} + m_{ji}\frac{\partial \dot{\Theta}_i}{\partial x_j} + \dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{\nabla} \cdot \boldsymbol{m}) - \frac{1}{\theta}\boldsymbol{q} \cdot \boldsymbol{g} \geq 0 \tag{3.52}$$

or

$$\rho_0 \left(\frac{De}{Dt} - \theta \frac{D\eta}{Dt} \right) + \frac{1}{\theta}\boldsymbol{q} \cdot \boldsymbol{g} - \sigma_{ji}\frac{\partial v_i}{\partial x_j} - m_{ji}\frac{\partial \dot{\Theta}_i}{\partial x_j} - \dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{\nabla} \cdot \boldsymbol{m}) \leq 0 \tag{3.53}$$

Let Φ be the Helmholtz free energy density defined by

$$\Phi = e - \eta\theta \tag{3.54}$$

$$\therefore \qquad \frac{De}{Dt} - \theta\frac{D\eta}{Dt} = \frac{D\Phi}{Dt} + \eta\frac{D\theta}{Dt} \tag{3.55}$$

Substituting from (3.55) into (3.53) we obtain

$$\rho_0 \left(\frac{D\Phi}{Dt} + \eta \frac{D\theta}{Dt} \right) + \frac{\boldsymbol{q} \cdot \boldsymbol{g}}{\theta} - \sigma_{ji}\frac{\partial v_i}{\partial x_j} - m_{ji}\frac{\partial \dot{\Theta}_i}{\partial x_j} - \dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{\nabla} \cdot \boldsymbol{m}) \leq 0 \tag{3.56}$$

We note that

$$\sigma_{ji}\frac{\partial v_i}{\partial x_j} = \mathrm{tr}\left([\sigma]^T [\dot{J}]^T \right) = \mathrm{tr}\left([\sigma][\dot{J}] \right) \tag{3.57}$$

$$m_{ji}\frac{\partial \dot{\Theta}_i}{\partial x_j} = \mathrm{tr}\left([m]^T [{}^{\Theta}\dot{J}]^T \right) = \mathrm{tr}\left([m][{}^{\Theta}\dot{J}] \right) \tag{3.58}$$

3.7 Complete Mathematical Model and Stress Decomposition

The mathematical model derived using conservation of mass, balance of linear and angular momenta, balance of moments of moments (or couples) and first and second laws of thermodynamics is summarized as follows (for small deformation):

$$\rho_0 = |J|\rho \tag{3.59}$$

$$\rho_0 \frac{D\boldsymbol{v}}{Dt} - \rho_0 \boldsymbol{F}^b - \boldsymbol{\nabla} \cdot \boldsymbol{\sigma} = 0 \tag{3.60}$$

$$m_{mk,m} - \epsilon_{ijk}\sigma_{ij} = 0 \tag{3.61}$$

$$\epsilon_{ijk} m_{ij} = 0 \tag{3.62}$$

$$\rho_0 \frac{De}{Dt} + \boldsymbol{\nabla} \cdot \boldsymbol{q} - \sigma_{ji}\frac{\partial v_i}{\partial x_j} - \left(m_{ji}\frac{\partial \dot{\Theta}_i}{\partial x_j} + \dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{\nabla} \cdot \boldsymbol{m}) \right) = 0 \tag{3.63}$$

$$\rho_0 \left(\frac{D\Phi}{Dt} + \eta \frac{D\theta}{Dt} \right) + \frac{\boldsymbol{q} \cdot \boldsymbol{g}}{\theta} - \sigma_{ji}\frac{\partial v_i}{\partial x_j} - \left(m_{ji}\frac{\partial \dot{\Theta}_i}{\partial x_j} + \dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{\nabla} \cdot \boldsymbol{m}) \right) \leq 0 \tag{3.64}$$

The Cauchy stress tensor $\boldsymbol{\sigma}$ is non-symmetric (due to (3.61)) whereas the Cauchy moment tensor $\boldsymbol{m}$ is symmetric (due to (3.62)). We decompose Cauchy stress tensor $\boldsymbol{\sigma}$ into symmetric and antisymmetric tensors ${}_s\boldsymbol{\sigma}$ and ${}_a\boldsymbol{\sigma}$.

$$\boldsymbol{\sigma} = {}_s\boldsymbol{\sigma} + {}_a\boldsymbol{\sigma} \tag{3.65}$$

and we note that

$$\boldsymbol{\epsilon} : \boldsymbol{\sigma} = \boldsymbol{\epsilon} : ({}_s\boldsymbol{\sigma} + {}_a\boldsymbol{\sigma}) = \boldsymbol{\epsilon} : {}_a\boldsymbol{\sigma} \tag{3.66}$$

since

$$\boldsymbol{\epsilon} : {}_s\boldsymbol{\sigma} = 0 \tag{3.67}$$

Using

$$\frac{\partial\{v\}}{\partial\{x\}} = [L] = [D] + [W] \tag{3.68}$$

$$[D] = \frac{1}{2}([L] + [L]^T) \tag{3.69}$$

$$[W] = \frac{1}{2}([L] - [L]^T) \tag{3.70}$$

we obtain

$$\begin{aligned} \sigma_{ji}\frac{\partial v_i}{\partial x_j} &= \sigma_{ji}L_{ij} = ({}_s\sigma_{ji} + {}_a\sigma_{ji})(D_{ij} + W_{ij}) \\ &= {}_s\sigma_{ji}(D_{ij}) + {}_a\sigma_{ji}(W_{ij}) \end{aligned} \tag{3.71}$$

since

$${}_s\sigma_{ji}(W_{ij}) = {}_a\sigma_{ji}(D_{ij}) = 0 \tag{3.72}$$

due to symmetry of ${}_s\boldsymbol{\sigma}$ and $\boldsymbol{D}$. Thus, from (3.71), we can write

$$\mathrm{tr}\left([\sigma]^T[L]^T\right) = \mathrm{tr}\left([\sigma][L]\right) = \mathrm{tr}\left([{}_s\sigma][D]\right) + \mathrm{tr}\left([{}_a\sigma][W]\right) \tag{3.73}$$

Likewise, using

$$\frac{\partial\{\dot{\Theta}\}}{\partial\{x\}} = [{}^{\Theta}L] = [{}^{\Theta}D] + [{}^{\Theta}W] \tag{3.74}$$

$$[{}^{\Theta}D] = \frac{1}{2}([{}^{\Theta}L] + [{}^{\Theta}L]^T) \tag{3.75}$$

$$[{}^{\Theta}W] = \frac{1}{2}([{}^{\Theta}L] - [{}^{\Theta}L]^T) \tag{3.76}$$

we obtain

$$\begin{aligned} m_{ji}\frac{\partial \dot{\Theta}_i}{\partial x_j} &= m_{ji}({}^{\Theta}L_{ij}) = m_{ji}({}^{\Theta}D_{ij} + {}^{\Theta}W_{ij}) \\ &= m_{ji}({}^{\Theta}D_{ij}) \end{aligned} \tag{3.77}$$

since

$$m_{ji}\left({}^{\Theta}W_{ij}\right) = 0 \tag{3.78}$$

due to symmetry of $\boldsymbol{m}$. Thus, from (3.77), we can write

$$\mathrm{tr}\left([m]^T[{}^{\Theta}L]^T\right) = \mathrm{tr}\left([m][{}^{\Theta}L]\right) = \mathrm{tr}\left([m][{}^{\Theta}D]\right) \tag{3.79}$$

We also note that by using (3.61), (3.66) and (3.67) we can show that

$$\dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{\nabla} \cdot \boldsymbol{m}) = -\dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{\epsilon} : {}_a\boldsymbol{\sigma}) \tag{3.80}$$

From (3.80), we can substitute in (3.63) and (3.64) if we wish to do so . This substitution eliminates appearance of the last term in the energy equation (3.63) and the entropy inequality (3.64) but introduce ${}_a\boldsymbol{\sigma}$ instead. Using relations (3.65), (3.66), (3.71) and (3.77), the mathematical model can be written as

$$\rho_0 = |J|\rho \tag{3.81}$$

$$\rho_0 \frac{D\boldsymbol{v}}{Dt} - \rho_0 \boldsymbol{F}^b - \boldsymbol{\nabla} \cdot ({}_s\boldsymbol{\sigma} + {}_a\boldsymbol{\sigma}) = 0 \tag{3.82}$$

$$m_{mk,m} - \epsilon_{ijk}({}_a\sigma_{ij}) = 0 \tag{3.83}$$

$$\epsilon_{ijk} m_{ij} = 0 \tag{3.84}$$

$$\begin{aligned} \rho_0 \frac{De}{Dt} &+ \boldsymbol{\nabla} \cdot \boldsymbol{q} - {}_s\sigma_{ji}(D_{ij}) - m_{ji}({}^{\Theta}D_{ij}) \\ &- {}_a\sigma_{ji}(W_{ij}) - \dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{\epsilon} : ({}_a\boldsymbol{\sigma})) = 0 \end{aligned} \tag{3.85}$$

$$\begin{aligned} \rho_0 &\left(\frac{D\Phi}{Dt} + \eta \frac{D\theta}{Dt} \right) + \frac{q_i g_i}{\theta} - {}_s\sigma_{ji}(D_{ij}) \\ &- m_{ji}({}^{\Theta}D_{ij}) - {}_a\sigma_{ji}(W_{ij}) - \dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{\epsilon} : ({}_a\boldsymbol{\sigma})) \leq 0 \end{aligned} \tag{3.86}$$

A simple calculation by expanding the terms shows that

$$\dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{\epsilon} : {}_a\boldsymbol{\sigma}) = -\mathrm{tr}\left([{}_a\sigma][W]\right) \tag{3.87}$$

By substituting (3.87) in (3.85) and (3.86), the energy equation and entropy inequality reduce to

$$\rho_0 \frac{De}{Dt} + \boldsymbol{\nabla}\cdot\boldsymbol{q} - \mathrm{tr}\left([{}_s\sigma][D]\right) - \mathrm{tr}\left([m][{}^{\Theta}D]\right) = 0 \tag{3.88}$$

$$\rho_0\left(\frac{D\Phi}{Dt} + \eta\frac{D\theta}{Dt}\right) + \frac{q_i g_i}{\theta} - \mathrm{tr}\left([{}_s\sigma][D]\right) - \mathrm{tr}\left([m][{}^{\Theta}D]\right) \leq 0 \tag{3.89}$$

Generally we denote

$$\psi_d = \mathrm{tr}\left([{}_s\sigma][D]\right) + \mathrm{tr}\left([m][{}^{\Theta}D]\right) = {}^s\psi_d + {}^m\psi_d \tag{3.90}$$

where

$$\begin{aligned} {}^s\psi_d &= \mathrm{tr}\left([{}_s\sigma][D]\right) \\ {}^m\psi_d &= \mathrm{tr}\left([m][{}^{\Theta}D]\right) \end{aligned} \tag{3.91}$$

in which ψ_d is the dissipation function which is sum of ${}^s\psi_d$ and ${}^m\psi_d$, the dissipation functions due to ${}_s\boldsymbol{\sigma}$ and $\boldsymbol{m}$. *Equations* (3.81)-(3.84)*,* (3.88) *and* (3.89) *constitute the complete and final mathematical model*. From the energy equation (3.88) and entropy inequality (3.89) *we clearly observe that* $([{}_s\sigma],[D])$ *and* $([m],[{}^{\Theta}D])$ *are conjugate pairs*. This conclusion is important in the derivation of the constitutive theories for $[{}_s\sigma]$ and $[m]$.

4 Closure of Mathematical Model and Comments on the Constitutive Theories

In this mathematical model, the dependent variables are (numbers in the lower case brackets indicate the number of variables):

$$\begin{gathered} v_i(3)\,,\quad {}_s\boldsymbol{\sigma}(6)\,,\quad {}_a\boldsymbol{\sigma}(3)\,,\quad \boldsymbol{m}(6)\,,\quad e(1)\,,\quad \boldsymbol{q}(3) \\ \Phi(1)\,,\quad \eta(1)\,,\quad \theta(1)\quad ;\quad \text{a total of 25} \end{gathered} \tag{4.1}$$

In these, Φ and η will be eliminated from the list of variables. The specific internal energy is a function of ρ and θ, that is $e(\rho,\theta)$ for most general case of compressible matter, hence e is also eliminated from the list of dependent variables. This leaves us with remaining 22 dependent variables in the mathematical model. We have linear momentum equation (3), angular momentum equation (3), energy equation (1) and, from entropy inequality we have constitutive theories for ${}_s\boldsymbol{\sigma}$ (6), $\boldsymbol{m}$ (6) and $\boldsymbol{q}$ (3), a total of 22 equations, hence this mathematical model will have closure once we have constitutive theories for ${}_s\boldsymbol{\sigma}$, $\boldsymbol{m}$ and $\boldsymbol{q}$. Development of the constitutive theories is clearly treatment of matter specific physics. This mathematical model is suited for solid matter experiencing small to moderate deformation both compressible and incompressible. The derivation of constitutive theories must consider: (i) Thermoelastic solid in which deformation is reversible, that is, for such solids rate of mechanical work does not result in entropy production, hence cannot influence inertial energy. (ii) Thermoviscoelastic solids without memory. Such solids have dissipation mechanism, that is, in such materials a part of the rate of work results in entropy production, thus influences internal energy. However, such solids do not have memory. (iii) Thermoviscoelastic solids with fading memory. Such solids obviously have dissipation mechanism as well as memory. Derivation of the constitutive theories for these solid continua, the model problems and their solutions are given in forth coming papers.

5 Summary and Conclusions

The development of the continuum theory (polar continuum theory) presented in this paper for isotropic, homogeneous solid continua is motivated by the fact that polar decomposition of the changing Jacobian of deformation at a material point and its neighbors with different Jacobians of deformation can result in different rotations which if resisted by the solid continua can result in conjugate internal moments. These conjugate internally varying rotations and moments can result in additional energy storage. The currently used thermodynamic framework for isotropic, homogeneous solid continua completely ignores this physics in the derivation of the conservation and balance laws. The theory resulting from incorporating the new physics considered here is in fact 'a polar theory' as it considers rotations and moments as a conjugate pair. The rotations are internal and are completely defined by the skew symmetric part of the displacement gradient tensor, thus this theory does not require rotations as external degrees of freedom. The thermodynamic framework resulting from the new theory is obviously a more complete thermodynamic framework for isotropic, homogeneous solid continua as it incorporates additional physics due to internal rotations in the derivation of the conservation and balance laws that is completely ignored in the presently used thermodynamic framework.

Derivation of the balance laws have been presented for polar solid continua for small strain and small deformation using Cauchy stress tensor, Cauchy moment tensor, heat vector, specific entropy, and Helmholtz free energy density. Derivations show that: (i) Cauchy stress tensor is non-symmetric; (ii) Cauchy moment tensor is symmetric due to the balance of moment of moments (or couples); (iii) the symmetric part of the Cauchy stress tensor and symmetric part of the displacement gradient tensor are conjugate (due to first and second laws of thermodynamics); (iv) the antisymmetric part of the Cauchy stress tensor is balanced by the gradient of the Cauchy moment tensor (due to balance of angular momenta); (v) Cauchy moment tensor and the symmetric part of the gradient of the rotation tensor are conjugate (due to first and second laws of thermodynamics); and (vi) it is shown that the constitutive theories for the symmetric Cauchy stress tensor, Cauchy moment tensor, heat vector, and thermodynamics relations for specific internal energy and others provide closure to the mathematical model presented here. The constitutive theories for polar solid continua (thermoelastic solids and thermoviscoelastic solids with and without memory) are subjects of forthcoming papers.

We emphasize that the polar continuum theory presented here is not micropolar theory (as discussed in section 1). This theory is for isotropic, homogeneous solid continua in which varying rotations, their gradients and their rates can result in energy storage and dissipation. This theory is inherently local and hence not capable of capturing nonlocal effects. We remark that the polar theory presented here is also not to be labeled as "stress couple theory" (see remarks in section 3.3). The existence of the rate of work and the rate of dissipation due to rotations and their rates necessitates existence of conjugate moment tensor. It is only after the balance of angular momenta we realize that only the antisymmetric part of the stress tensor is balanced by the gradients of the moment tensor. We note that existence of the moment tensor is established long before we realize a relationship between its gradients and the antisymmetric part of the stress tensor. The polar theory based on internal varying rotations presented in this paper is not the same as strain gradient theory (see section 2.2). Extensions of this work for finite deformation is in progress. Since the theory presented here accounts for the deformation physics resulting in internal varying rotations and the conjugate moment tensor, it is perhaps fitting to call this theory "an internal polar theory for solid continua" so that this theory can be clearly distinguished from the micropolar theories. In forthcoming publications related to the constitutive theories we refer to this polar theory as "internal polar theory".

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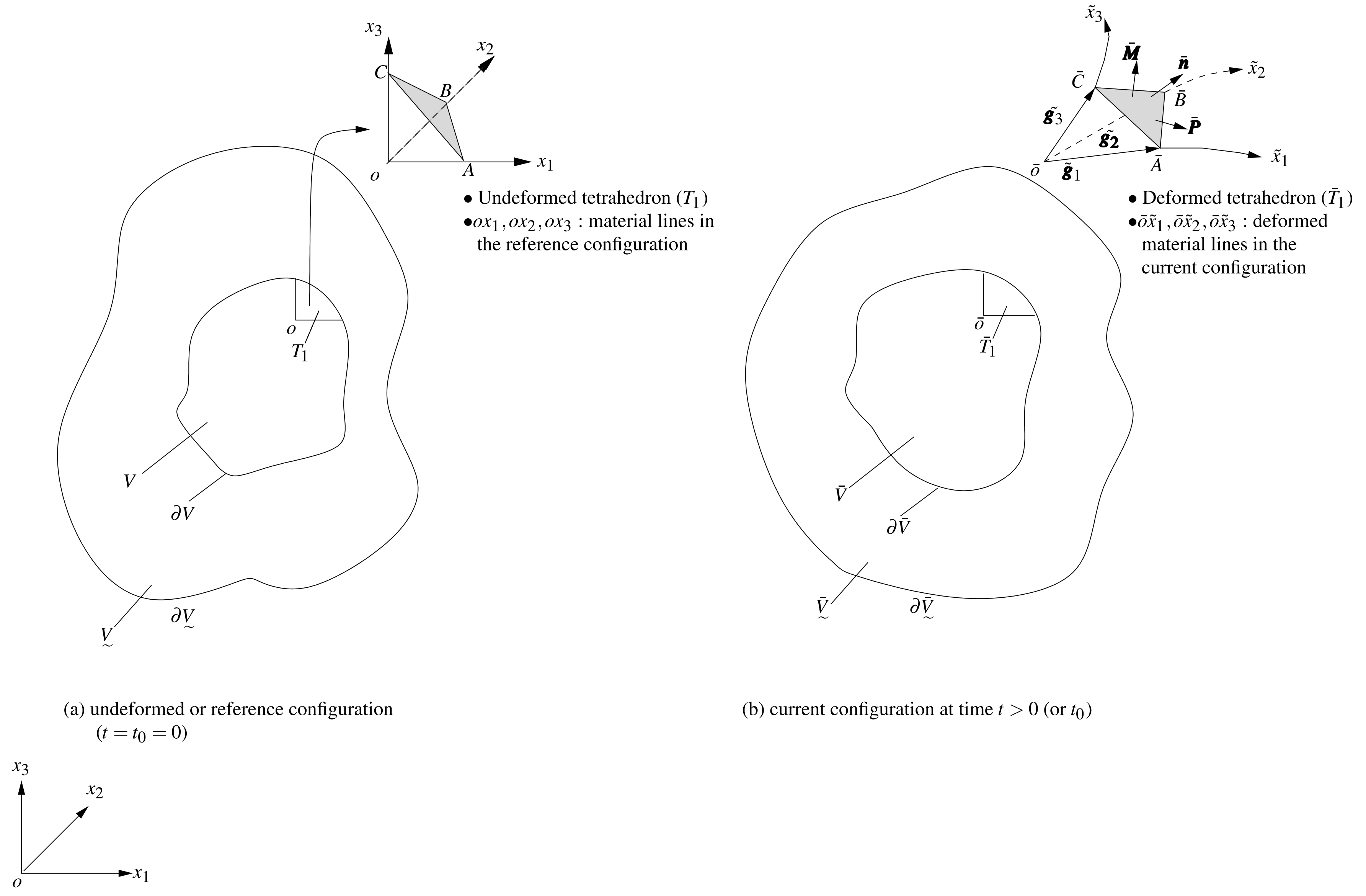

(a) undeformed or reference configuration ($t = t_0 = 0$)

(b) current configuration at time $t > 0$ (or t_0)

Fig. 1: Reference and current configurations for a deforming volume of matter