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**Research Article**

# SOME PROPERTIES OF $(h,m)$-PREINVEX FUNCTIONS AND HERMITE HADAMARD INEQUALITY

**Erdal ÜNLÜYOL*[1], Mustafa KARADENİZ[2]**

[1]*Ordu University, Department of Mathematics, ORDU;* ORCID:0000-0003-3465-6473
[2]*Ordu University, Department of Mathematics, ORDU*

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**ABSTRACT**

In this paper, firstly it is defined a new class of preinvex, namely, $(\mathrm{h},\mathrm{m})$-preinvex. Secondly it is obtained some algebraic properties of this class, i.e. sum, multiple etc. Finally it is proved the Hermite-Hadamard Type Inequality for $(\mathrm{h},\mathrm{m})$ $-$convex and established some new inequalities.
**Keywords:** Convex, Hermite-Hadamard, preinvex, m-preinvex, $h$-preinvex, $(h,m)$-preinvex.
**MSC 2010:** 26D10, 26D15.

## 1. INTRODUCTION

Invex functions theory was introduced by Hanson [1]. Then Weir and Mond [2] defined the preinvex function. They applied the preinvex function to the establishment of the sufficient optimality conditions and duality in nonlinear programming. After that Noor [3] proved the Hermite-Hadamard inequality for preinvex and log-preinvex functions.

Preinvex functions are an important generalization of convex functions. And if you want to learn more details and resources for invexity and prequasiinvex etc. you can see [4, 6], and reference therein.

Now let we give some basic definitions and theorems.

**Definition 1** : *A function* $f: I \subseteq \mathbb{R} \to \mathbb{R}$ *is said to be convex if*

$$f(tx+(1-t)y) \leqslant tf(x)+(1-t)f(y)$$

*holds for every* $x, y \in I$ *and* $t \in [0,1]$.

**Definition 2** : The following celebrated double inequality

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)dx \leqslant \frac{f(a)+f(b)}{2} \tag{1.1}$$

holds for convex functions and is well-known in the literature as the Hermite-Hadamard inequality. Both the inequalities in (1.1) hold in reversed direction if f is concave.

The inequality (1.1) has been a subject of extensive research since its discovery and a number of paper have been written providing noteworthy extensions, generalizations and refinements.

* Corresponding Author: e-mail: eunluyol@yahoo.com, tel: (452) 226 52 00 / 1822

**Remark 1** : Beckenbach, a leading expert on the history and the theory of convex functions, wrote that this inequality was proven by Hadamard in 1983 [7]. In 1974, Mitrinovic found Hermite's note in Mathesis [8].

**Definition 3 :** [9] Let s be a number, $s \in (0,1]$. A function $f: [0,\infty)] \to [0,\infty)]$ is said to be $s$ −convex(in the second sense), or that f belongs to the class ${K_s}^2$, if

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y)$$

*for all* $x, y \in [0,\infty)]$ *and* $t \in [0,1]$.

**Definition 4 [**10] A function $f: I \subseteq \mathbb{R} \to \mathbb{R}$ is said to belong to the class of $Q(I)$ if it is nonnegative and, for all $x, y \in I$ and $t \in (0,1)$, satisfies the inequality ;

$$f(tx + (1-t)y) \leq \frac{f(x)}{t} + \frac{f(y)}{1-t}.$$

**Definition 5** [11] The function $f: [0, b] \subseteq \mathbb{R} \to \mathbb{R}, b > 0$ , is said to be m-convex , where $m \in [0,1]$ ,if we have

$$f(tx + m(1-t)y) \leqslant tf(x) + m(1-t)f(y)$$

for all $x, y \in [0, b]$ and $t \in [0,1]$. We say that f is m-concave if -f is m-convex.

**Definition 6 [**12] A function,$f: I \subseteq \mathbb{R} \to \mathbb{R}$ is P function or that f belongs to the class of $P(I)$, if it is nonnegative and for all $x, y \in I$ and $t \in [0,1]$, satisfies the following inequality;

$$f(tx + (1-t)y) \leq f(x) + f(y).$$

**Definition 7** [13] Let $h: J \subseteq \mathbb{R} \to \mathbb{R}$ be a positive function. We say that $f: I \subseteq \mathbb{R} \to \mathbb{R}$ is $h$ −convex function, or that f belongs to the class $SX(h, I)$, if is nonnegative and for all $x, y \in I$ and $t \in (0,1)$ we have

$$f(tx + (1-t)y) \leq h(t)f(x) + h(1-t)f(y). \quad (1.2)$$

If inequality (2) is reversed, then f is said to be $h$ −concave, i.e., $f \in SV(h, I)$.

**Remark 2** [13] You can see easily the following results.

- If $h(t) = t$, in( 1.2) then all nonnegative convex functions belong to SX(h,I
- If $h(t) = \frac{1}{t}$, in.(1.2) then $SX(h, I) = Q(I)$
- If $h(t) = 1$, in (1.2) then $SX(h, I) \supseteq P(I)$
- If $h(t) = t^s$, in (1.2) where $s \in (0,1)$, then $SX(h, I) \supseteq {K_s}^2$.

**Definition 8** [13] Let $f, g: I \subseteq \mathbb{R} \to \mathbb{R}$ be any functions. Then f and g are said to be similarly ordered functions, if for all $x, y \in I$

$$0 \leq [f(x) - f(y)][g(x) - g(y)].$$

In other words

$$f(x)g(y) + f(y)g(x) \leq f(x)g(x) + f(y)g(y). \quad (1.3)$$

**Definition 9** [14] Let $h: J \subseteq \mathbb{R} \to \mathbb{R}$ be a non-negative function.We say that $f: [0, b] \to \mathbb{R}$ is a $(h, m)$ −convex function, if f is non-negative and for all $x, y \in [0, b], m \in [0,1]$ and $t \in (0,1)$, we have

$$f(tx + m(1-t)y) \leq h(t)f(x) + mh(1-t)f(y). \quad (1.4)$$

If the inequality (4) is reversed, then f is said to be $(h, m)$ −concave function on [0,b].

**Remark 3** [15] You can see easily the following results.

- If we choose $m = 1$ in (1.4), then we obtain $h$ −convex functions.
- If we choose $h(t) = t$ in (1.4), then we obtain non-negative $m$ −convex functions.
- If we choose $m = 1$ and $h(t) = t$ in (1.4), then we obtain non-negative convex functions.

• If we choose $m = 1$ and $h(t) = 1$ in (1.4), then we obtain P-functions.

• If we choose $m = 1$ and $h(t) = \frac{1}{t}$ in (1.4), then we obtain Godunova-Levin functions.

• If we choose $m = 1$ and $h(t) = t^s$ in (1.4), then we obtain $s$ −convex functions(in the second sense).

**Definition 10** [1] Let K be a subset in $\mathbb{R}^n$ and $\eta: K \times K \to \mathbb{R}^n$ be continuous functions. Let $x \in K$, then the set K is said to be invex at x with respect to $\eta(.,.)$, if for all $x, y \in K$ and $t \in [0,1]$,

$$x + t\eta(y,x) \in K,$$

then K is said to be an invex set with respect to η if K is invex at each $x \in K$. The invex set K is also called an η- connected set.

**Definition 11** [16] The function $f: K \to \mathbb{R}$ on the invex set K is said to be preinvex with respect to η, if

$$f(u + t\eta(v,u)) \leqslant (1-t)f(u) + tf(v), \qquad \forall u, v \in K, \quad t \in [0,1].$$

The function f is said to be preconcave if and only if $-f$ is preinvex.

**Remark 4** It is to be noted that every convex function is preinvex with respect to the map

$$\eta(v,u) = v - u$$

but the converse is not true. (see [2])

**Definition 12** [17] Let $K \subset \mathbb{R}$ be an invex set with respect to bifunction $\eta(.,.)$. Then for any $u, v \in K$ and $t \in [0,1]$,

$$\eta(v, v + t\eta(u,v)) = -t\eta(u,v)$$
$$\eta(u, v + t\eta(u,v)) = (1-t)\eta(u,v)$$

Note that for every $u, v \in K$ , $t_1, t_2 \in [0,1]$ and from Condition C, we have

$$\eta(v, t_2\eta(u,v), v + t_1\eta(u,v)) = (t_2 - t_1)\eta(u,v).$$

**Definition 13** [17] Let K be an invex set in $\mathbb{R}$, and let $h: [0,1] \to \mathbb{R}$ be a nonnegative function. Then, a function $f: \mathbb{R} \to \mathbb{R}$ is said to be $h-$ preinvex function with respect to the bi-function $\eta(\cdot,\cdot)$, if for all $x, y \in K, t \in [0,1]$,

$$f(x + t\eta(y,x)) \leqslant h(1-t)f(x) + h(t)f(y).$$

**Definition 14** [18] The function f on the invex set $K \subseteq [0, b^*], b^* > 0$, is said to be $m$ −preinvex with respect to η if

$$f(u + t\eta(v,u)) \leqslant (1-t)f(u) + mtf(\frac{v}{m})$$

holds for all $u, v \in K, t \in [0,1]$ and $m \in (0,1]$. The function f is said to be $m$ −preconcave if and only if -f is m- preinvex.

## 2. MAIN RESULT

**Definition 15** Let for $b^* > 0$, $[0, b^*] \subseteq \mathbb{R}$ be a invex set with respect to η and $h: (0,1) \subset J \to \mathbb{R}$ *be a non-negative function. Then f is said to* $(h,m)$ *−preinvex function via* $\eta$, *if for all* $x, y \in [0, b^*]$, $m \in [0,1]$ *and* $t \in (0,1)$

$$f(x + t\eta(y,x)) \leq h(1-t)f(x) + mh(t)f(y)$$

the inequality holds.

**Proposition 1** Let f, g are $(h,m)$ −preinvex functions in terms of η. Then for $\lambda > 0$, $\lambda f$ and $f + g$ are $(h,m)$ −preinvex functions.

**Proof.** Since $f, g$ are $(h, m)$ −preinvex functions. Thus we can write for all $x, y \in [0, b^*], b^* > 0$, $h: (0,1) \subset J \to \mathbb{R}$ non-negative function and for all $t \in (0,1), m \in [0,1]$

$$f(x + t\eta(y,x)) \leq h(1-t)f(x) + mh(t)f(y) \tag{2.1}$$

$$g(x + t\eta(y,x)) \leq h(1-t)g(x) + mh(t)g(y) \tag{2.2}$$

If we add (2.1) and (2.2), then we get

$$f(x + t\eta(y,x)) + g(x + t\eta(y,x) \leq h(1-t)f(x) + mh(t)f(y) + h(1-t)g(x) + mh(t)g(y)$$
$$(f+g)(x + t\eta(y,x)) \leq h(1-t)[f(x) + g(x)] + mh(t)[f(y) + g(y)]$$
$$= h(1-t)(f+g)(x) + mh(t)(f+g)(y).$$

Hence $f + g$ are $(h, m)$ −preinvex functions.

Due to $\lambda > 0$, if we multiply $\lambda$ in (5), we have

$$\lambda f(x + t\eta(y,x)) \leq \lambda h(1-t)f(x) + \lambda mh(t)f(y)$$
$$= h(1-t)\lambda(fx) + mh(t)\lambda f(y)$$
$$= h(1-t)(\lambda f)(x) + mh(t)(\lambda f)(y).$$

This completes the proof.

**Proposition 2** Let f and g be two $(h, m)$ −preinvex functions with respect to $\eta$. Thus their product $f.g$ is $(h, m)$ −preinvex function, if f and g are similarly ordered functions and

$$h(1-t) + mh(t) \leq 1.$$

**Proof.** Since f and g are $(h, m)$ −preinvex with respect to $\eta$, (2.1) and (2.2) are hold. If we multiply (2.1) and (2.2), we get

$$f(x + t\eta(y,x))g(x + t\eta(y,x) \leq [h(1-t)f(x) + mh(t)f(y)] * [h(1-t)g(x) + mh(t)g(y)]$$
$$(fg)(x + t\eta(y,x)) \leq h^2(1-t)f(x)g(x) + mh(t)h(1-t)f(x)g(y)$$
$$+mh(t)h(1-t)f(y)g(x) + m^2h^2(t)f(y)g(y)$$
$$= h^2(1-t)f(x)g(x) + m^2h^2(t)f(y)g(y) \tag{2.3}$$
$$+[mh(t)h(1-t)f(x)g(y) + mh(t)h(1-t)f(y)g(x)].$$

Then we can rewrite (2.3) from (1.3)

$$(f.g)(x + t\eta(y,x)) \leq h^2(1-t)f(x)g(x) + mh(t)h(1-t)f(x)g(y)$$
$$+mh(t)h(1-t)f(y)g(x) + m^2h^2(t)f(y)g(y)$$
$$= h(1-t)[h(1-t) + mh(t)]f(x)g(x)$$
$$+mh(t)[h(1-t) + mh(t)]f(y)g(y).$$

*Due to* $h(1-t) + mh(t) \leq 1$, *then*

$$(fg)(x + t\eta(y,x)) \leq h(1-t)(fg)(x) + mh(t)(fg)(y),$$

so the proof completes.

**Proposition 3** Let $h_1$ and $h_2$ be non-negative functions defined on $[0, b^*] \subset \mathbb{R}, b^* > 0$ such that for all $t \in (0,1)$

$$h_1(t) \leq h_2(t).$$

If f is $(h_1, m)$ −preinvex function, then f is $(h_2, m)$ −preinvex function.

**Proof.** Since f is $(h_1, m)$ −preinvex function,we have

$$f(x + t\eta(y,x)) \leq h_1(1-t)f(x) + mh_1(t)f(y).$$

Due to $h_1(t) \leq h_2(t)$, for all $t \in (0,1)$

$$f(x + t\eta(y,x)) \leq h_1(1-t)f(x) + mh_1(t)f(y)$$
$$\leq h_2(1-t)f(x) + mh_2(t)f(y)$$

Thus f is $(h_2, m)$ −preinvex function.

**Proposition 4** Let h be a non-negative function such that for all $t \in (0,1)$

$$t \leq h(t).$$

If f is a non-negative $m$ −preinvex function on $[0,b^*], b^* > 0$ then for all $x,y \in [0,b^*], m \in [0,1]$ and $t \in (0,1)$ f is $(h,m)$ −preinvex function.

**Proof.** Because f is non-negative $m$ −preinvex, we have

$$f(x+t\eta(y,x)) \leq (1-t)f(x) + mtf(y).$$

According to $t \leq h(t)$, we get

$$f(x+t\eta(y,x)) \leq (1-t)f(x) + mtf(y) \\ \leq h(1-t)f(x) + mh(t)f(y).$$

Hence f is a $(h,m)$ −preinvex function. The proof is completed.

**Theorem 1** Let $M \subseteq [0,b^*], b^* > 0$ is an invex set. Let $f: M \to \mathbb{R}$ be a $m$ −preinvex function with $m \in (0,1]$ and $0 < a < a + \eta(b,a)$. Let $\eta$ satisfies condition (C).
If $f \in L_1[a, a+\eta(b,a)]$, then the following inequality holds,

$$f\left(\frac{2a+\eta(b,a)}{2}\right) \leq \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} \frac{f(x)+mf(\frac{x}{m})}{2} dx$$

$$\leq \frac{m+1}{4}\left[\frac{f(a)+f(b)}{2} + m\frac{f(\frac{a}{m})+f(\frac{b}{m})}{2}\right]. \tag{2.4}$$

**Proof.** Firstly we prove the left side of (2.4). Due to $m$ −preinvexity of f we have for all $x,y \in [0,\infty)$ and $t = \frac{1}{2}$

$$f\left(\frac{2x+\eta(y,x)}{2}\right) \leq \frac{f(x)+mf(\frac{y}{m})}{2}$$

If we take $x = a + t\eta(b,a), y = a + (1-t)\eta(b,a)$, we deduce for all $t \in [0,1]$

$$f\left(\frac{2a+\eta(b,a)}{2}\right) \leq \frac{f(a+t\eta(b,a)) + mf(\frac{a}{m} + (1-t)\frac{\eta(b,a)}{m})}{2} \\ = \frac{1}{2}\left[f(a+t\eta(b,a)) + mf\left(\frac{a}{m} + (1-t)\frac{\eta(b,a)}{m}\right)\right]$$

Integrating over $t \in [0,1]$ we get

$$f\left(\frac{2a+\eta(b,a)}{2}\right) \leq \frac{1}{2}\left[\int_0^1 f(a+t\eta(b,a))dt + m\int_0^1 f\left(\frac{a}{m} + (1-t)\frac{\eta(b,a)}{m}\right)dt\right].$$

Taking into account that

$$\int_0^1 f(a+t\eta(b,a))dt = \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx$$

and

$$\int_0^1 f\left(\frac{a}{m} + (1-t)\frac{\eta(b,a)}{m}\right)dt = \frac{m}{\eta(b,a)}\int_{\frac{a}{m}}^{\frac{a}{m}+\frac{\eta(b,a)}{m}} f(x)dx = \frac{1}{\eta(b,a)}\int_a^b f\left(\frac{x}{m}\right)dx.$$

Now, we prove the right side of (2.4).
Due to f $m$ −preinvexity we have also for all $t \in [0,1]$

$$\frac{1}{2}\left[f(a+t\eta(b,a)) + mf\left(\frac{a}{m} + (1-t)\frac{\eta(b,a)}{m}\right)\right] \\ \leq \frac{1}{2}\left[(1-t)f(a) + m(1-t)f\left(\frac{b}{m}\right) + m(1-t)f\left(\frac{a}{m}\right) + m^2tf\left(\frac{b}{m^2}\right)\right]. \tag{2.5}$$

Integrating the inequality (2.5) over t on [0,1], we deduce

$$\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}\frac{f(x)+mf(\frac{x}{m})}{2}dx \le \frac{m+1}{4}[\frac{f(a)+f(b)}{2}+m\frac{f(\frac{a}{m})+f(\frac{b}{m})}{2}].$$

This completes the proof.

**Corollary 1** If we choose $m = 1$ in (2.4), we obtain the following inequality of Hermite-Hadamard type for preinvex functions [19];

$$f(\frac{2a+\eta(b,a)}{2}) \le \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx \le \frac{f(a)+f(b)}{2}.$$

**Corollary 2** If we choose $\eta(b,a) = b - a$ in (2.4), we obtain the following inequality of Hermite-Hadamard type for $m$ −convex functions [20];

$$f(\frac{a+b}{2}) \le \frac{1}{b-a)}\int_a^{b)}\frac{f(x)+mf(\frac{x}{m})}{2}dx \le \frac{m+1}{4}[\frac{f(a)+f(b)}{2}+m\frac{f(\frac{a}{m})+f(\frac{b}{m})}{2}].$$

**Corollary 3** If we choose $\eta(b,a) = b - a, m = 1$ in (2.4), we obtain the following inequality of Hermite-Hadamard type for convex functions;

$$f(\frac{a+b}{2}) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}.$$

**Theorem 2** Let $f: [0,+\infty) \to \mathbb{R}$ be a $(h,m)$ −preinvex function with $m \in (0,1], t \in [0,1]$. Let $\eta$ satisfies condition (C). If $0 < a < a + \eta(b,a)$ and $f \in L_1[a, a + \eta(b,a)]$, then the following inequality holds,

$$f(\frac{2a+\eta(b,a)}{2}) \le \frac{h(\frac{1}{2})}{\eta(b,a)}\int_a^{a+\eta(b,a)}[f(x)+mf(\frac{x}{m})]dx \tag{2.6}$$

$$\le h(\frac{1}{2})[\frac{f(a)+mf(\frac{b}{m})+mf(\frac{a}{m})+m^2f(\frac{b}{m^2})}{2}]\int_0^1 h(t)dt.$$

**Proof.** From the definition of $(h,m)$ −preinvex function, we can write for all $x, y \in [0,\infty)$, and $t = \frac{1}{2}$,

$$f(\frac{2x+\eta(y,x)}{2}) \le h(\frac{1}{2})f(x) + mh(\frac{1}{2})f(\frac{y}{m}).$$

If we choose $x = a + t\eta(b,a)$ and $y = a + (1-t)\eta(b,a)$, we get

$$f(\frac{2a+\eta(b,a)}{2}) \le h(\frac{1}{2})f(a+t\eta(b,a)) + mh(\frac{1}{2})f(\frac{a}{m}+(1-t)\frac{\eta(b,a)}{m})$$

and integrating on $t \in [0,1]$,

$$\int_0^1 f(\frac{2a+\eta(b,a)}{2}) \le h(\frac{1}{2})[\int_0^1 f(a+t\eta(b,a))dt + m\int_0^1 f(\frac{a}{m}+(1-t)\frac{\eta(b,a)}{m})]$$

$$\le h\left(\frac{1}{2}\right)\left[\int_a^{a+\eta(b,a)} f(x)\frac{dx}{\eta(b,a)} + m\int_a^{a+\eta(b,a)} f\left(\frac{x}{m}\right)\frac{dx}{\eta(b,a)}\right] \le \frac{h(\frac{1}{2})}{\eta(b,a)}\int_a^{a+\eta(b,a)}[f(x)+mf(\frac{x}{m})]dx.$$

We proved the left side of inequality.

Now we take the right side of inequality. Let we take $x = a + t\eta(b,a)$ in the last inequality and take its integrating on $t \in [0,1]$,

$$h(\frac{1}{2})\int_{a}^{a+a+\eta(b,a)}[f(x)+mf(\frac{x}{m})]dx \leq \eta(b,a)\int_0^1 [f(a+t\eta(b,a))+f(\frac{a}{m}+t\frac{\eta(b,a)}{m})]dx$$
$$\leq \eta(b,a)\int_0^1 [\left(h(t)f(a)+mh(1-t)f\left(\frac{b}{m}\right)\right)$$
$$+m\int_0^1 (h(t)f(\frac{a}{m})+mh(1-t)f(\frac{b}{m^2}))]$$
$$\leq \eta(b,a)[f(a)+mf(\frac{b}{m})+mf(\frac{a}{m})+m^2f(\frac{b}{m^2})]\int_0^1 h(t)dt$$
$$\frac{h\left(\frac{1}{2}\right)}{\eta(b,a)}\int_a^{a+a+\eta(b,a)}\left[f(x)+mf\left(\frac{x}{m}\right)\right]dx \leq \left[\frac{f(a)+mf\left(\frac{b}{m}\right)+mf\left(\frac{a}{m}\right)+m^2f\left(\frac{b}{m^2}\right)}{2}\right]\int_0^1 h(t)dt.$$

The proof is completed.

**Corollary 4** If we choose $m=1$ in (2.6), we obtain the following inequality of Hermite-Hadamard type for $h$ −preinvex functions [21];

$$\frac{1}{2h(\frac{1}{2})}f(\frac{2a+\eta(b,a)}{2}) \leq \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx \leq [f(a)+f(b)]\int_0^1 h(t)dt.$$

**Corollary 5** If we choose $h(t)=t$ in (2.6) , we obtain the following inequality of Hermite-Hadamard type for $m$ −preinvex functions;

$$f\left(\frac{2a+\eta(b,a)}{2}\right) \leq \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}\frac{f(x)+mf\left(\frac{x}{m}\right)}{2}dx$$
$$\leq \frac{m+1}{4}[\frac{f(a)+f(b)}{2}+m\frac{f(\frac{a}{m})+f(\frac{b}{m})}{2}].$$

**Corollary 6** If we choose $h(t)=t$ and $m=1$ in (2.6, we obtain the following inequality of Hermite-Hadamard type for preinvex functions [19];

$$f(\frac{2a+\eta(b,a)}{2}) \leq \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

**Corollary 7** If we choose $h(t)=t$ and $\eta(b,a)=b-a$ in (2.6) , we obtain the following inequality of Hermite-Hadamard type for $m$ −convex functions [20];

$$f(\frac{a+b}{2}) \leq \frac{1}{b-a)}\int_a^{b)}\frac{f(x)+mf(\frac{x}{m})}{2}dx \leq \frac{m+1}{4}[\frac{f(a)+f(b)}{2}+m\frac{f(\frac{a}{m})+f(\frac{b}{m})}{2}].$$

**Corollary 8** If we choose $m=1$ and $\eta(b,a)=b-a$ in (2.6) , we obtain the following inequality of Hermite-Hadamard type for $h$ −convex functions [22];

$$\frac{1}{2h(\frac{1}{2})}f(\frac{a+b}{2}) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq [f(a)+f(b)]\int_0^1 h(t)dt.$$

**Corollary 9** If we choose $h(t)=t$, $m=1$ and $\eta(b,a)=b-a$ in (2.6), we obtain the following inequality of Hermite-Hadamard type for convex functions;

$$f(\frac{a+b}{2}) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

**Corollary 10** If we choose $\eta(b,a)=b-a$ in (2.6), we obtain the following inequality of Hermite-Hadamard type for $(h,m)-$ convex functions [15];

$$f(\frac{a+b}{2}) \leq \frac{h(\frac{1}{2})}{b-a)}\int_a^b [f(x)+mf(\frac{x}{m})]dx$$
$$\leq h(\frac{1}{2})[\frac{f(a)+mf(\frac{b}{m})+mf(\frac{a}{m})+m^2f(\frac{b}{m^2})}{2}]\int_0^1 h(t)dt.$$

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