Research Article
SOME PROPERTIES OF \((h,m)\)-PREINVEX FUNCTIONS AND HERMITE HADAMARD INEQUALITY

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ABSTRACT
In this paper, firstly it is defined a new class of preinvex, namely, \((h,m)\)-preinvex. Secondly it is obtained some algebraic properties of this class, i.e. sum, multiple etc. Finally it is proved the Hermite-Hadamard Type Inequality for \((h,m)\) —convex and established some new inequalities.

Keywords: Convex, Hermite-Hadamard, preinvex, m-preinvex, \(h\)-preinvex, \((h,m)\)-preinvex.
MSC 2010: 26D10, 26D15.

1. INTRODUCTION

Invex functions theory was introduced by Hanson [1]. Then Weir and Mond [2] defined the preinvex function. They applied the preinvex function to the establishment of the sufficient optimality conditions and duality in nonlinear programming. After that Noor [3] proved the Hermite-Hadamard inequality for preinvex and log-preinvex functions.

Preinvex functions are an important generalization of convex functions. And if you want to learn more details and resources for invexity and prequasiinvex etc. you can see [4, 6], and reference therein.

Now let us give some basic definitions and theorems.

**Definition 1**: A function \(f: I \subseteq \mathbb{R} \rightarrow \mathbb{R}\) is said to be convex if
\[
f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)
\]
holds for every \(x, y \in I\) and \(t \in [0,1]\).

**Definition 2**: The following celebrated double inequality
\[
f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_{a}^{b} f(x)dx \leq \frac{f(a)+f(b)}{2}
\]
(1.1)
holds for convex functions and is well-known in the literature as the Hermite-Hadamard inequality. Both the inequalities in (1.1) hold in reversed direction if \(f\) is concave.

The inequality (1.1) has been a subject of extensive research since its discovery and a number of paper have been written providing noteworthy extensions, generalizations and refinements.

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Remark 1: Beckenbach, a leading expert on the history and the theory of convex functions, wrote that this inequality was proven by Hadamard in 1983 [7]. In 1974, Mitroinovic found Hermite’s note in Mathesis [8].

Definition 3: [9] Let s be a number, s ∈ (0,1]. A function f: [0,∞) → [0,∞)) is said to be s-convex (in the second sense), or that f belongs to the class $K_s^2$, if

\[ f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y) \]

for all $x, y \in [0, \infty)$ and $t \in [0, 1]$. 

Definition 4 [10] A function $f: I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is said to belong to the class of Q(I) if it is nonnegative and, for all $x, y \in I$ and $t \in (0,1)$, satisfies the inequality:

\[ f(tx + (1-t)y) \leq \frac{f(x)}{t} + \frac{f(y)}{1-t} \]

Definition 5 [11] The function $f: [0, b] \subseteq \mathbb{R} \rightarrow \mathbb{R}, b > 0$, is said to be $m$-convex, where $m \in [0,1]$ if we have

\[ f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y) \]

for all $x, y \in [0, b]$ and $t \in [0, 1]$. We say that $f$ is $m$-concave if $-f$ is $m$-convex.

Definition 6 [12] A function $f: I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is P function or that $f$ belongs to the class of P(I), if it is nonnegative and for all $x, y \in I$ and $t \in (0,1)$ satisfies the following inequality:

\[ f(tx + (1-t)y) \leq f(x) + f(y) \]

Definition 7 [13] Let $h: I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a positive function. We say that $f: I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is $h$-convex function, or that $f$ belongs to the class $SX(h, I)$, if it is nonnegative and for all $x, y \in I$ and $t \in (0,1)$ we have

\[ f(tx + (1-t)y) \leq h(t)f(x) + h(1-t)f(y) \]

(1.2)

If inequality (2) is reversed, then $f$ is said to be $h$-concave, i.e., $f \in SV(h, I)$.

Remark 2 [13] You can see easily the following results.

• If $h(t) = t$, in (1.2) then all nonnegative convex functions belong to $SX(h, I)$
• If $h(t) = \frac{t^3}{t^3}$, in (1.2) then $SX(h, I) = Q(I)$
• If $h(t) = 1$, in (1.2) then $SX(h, I) \supseteq P(I)$
• If $h(t) = t^3$, in (1.2) where $s \in (0,1)$, then $SX(h, I) \supseteq K_s^2$.

Definition 8 [13] Let $f, g: I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be any functions. Then $f$ and $g$ are said to be similarly ordered functions, if for all $x, y \in I$

\[ 0 \leq [f(x) - f(y)][g(x) - g(y)] \]

(1.3)

In other words

\[ f(x)g(y) + f(y)g(x) \leq f(x)g(x) + f(y)g(y) \]

Definition 9 [14] Let $h: I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a non-negative function. We say that $f: [0, b] \rightarrow \mathbb{R}$ is a $(h, m)$-convex function, if $f$ is non-negative and for all $x, y \in [0, b], m \in [0,1]$ and $t \in (0,1)$, we have

\[ f(tx + (1-t)y) \leq h(t)f(x) + mh(1-t)f(y) \]

(1.4)

If the inequality (4) is reversed, then $f$ is said to be $(h, m)$-concave function on $[0, b]$.

Remark 3 [15] You can see easily the following results.

• If we choose $m = 1$ in (1.4), then we obtain $h$-convex functions.
• If we choose $h(t) = t$ in (1.4), then we obtain non-negative $m$-convex functions.
• If we choose $m = 1$ and $h(t) = t$ in (1.4), then we obtain non-negative convex functions.
• If we choose \( m = 1 \) and \( h(t) = 1 \) in (1.4), then we obtain P-functions.
• If we choose \( m = 1 \) and \( h(t) = \frac{t}{0} \) in (1.4), then we obtain Godunova-Levin functions.
• If we choose \( m = 1 \) and \( h(t) = t^s \) in (1.4), then we obtain \( s \) -convex functions (in the second sense).

**Definition 10** [1] Let \( K \) be a subset in \( \mathbb{R}^n \) and \( \eta: K \times K \rightarrow \mathbb{R}^n \) be continuous functions. Let \( x \in K \), then the set \( K \) is said to be invex at \( x \) with respect to \( \eta(,,) \), if for all \( x, y \in K \) and \( t \in [0,1] \),

\[
x + t \eta(y, x) \in K,
\]

then \( K \) is said to be an invex set with respect to \( \eta \) if \( K \) is invex at each \( x \in K \). The invex set \( K \) is also called an \( \eta \)-connected set.

**Definition 11** [16] The function \( f: K \rightarrow \mathbb{R} \) on the invex set \( K \) is said to be preinvex with respect to \( \eta \), if

\[
f(u + t\eta(v, u)) \leq (1 - t)f(u) + tf(v), \quad \forall u, v \in K, \ t \in [0,1].
\]

The function \( f \) is said to be preconcave if and only if \(-f\) is preinvex.

**Remark 4** It is to be noted that every convex function is preinvex with respect to the map

\[
\eta(v, u) = v - u
\]

but the converse is not true. (see [2])

**Definition 12** [17] Let \( K \subset \mathbb{R} \) be an invex set with respect to bifunction \( \eta(,,) \). Then for any \( u, v \in K \) and \( t \in [0,1] \),

\[
\eta(v, v + t\eta(u, v)) = -t\eta(u, v) \quad \eta(u, v + t\eta(u, v)) = (1 - t)\eta(u, v)
\]

Note that for every \( u, v \in K \), \( t_1, t_2 \in [0,1] \) and from Condition C, we have

\[
\eta(v, t_2\eta(u, v), v + t_1\eta(u, v)) = (t_2 - t_1)\eta(u, v).
\]

**Definition 13** [17] Let \( K \) be an invex set in \( \mathbb{R} \), and let \( h: [0,1] \rightarrow \mathbb{R} \) be a nonnegative function. Then, a function \( f: \mathbb{R} \rightarrow \mathbb{R} \) is said to be \( h - \) preinvex function with respect to the bi-function \( \eta(,,) \), if for all \( x, y \in K, t \in [0,1] \),

\[
f(x + t\eta(y, x)) \leq h(1 - t)f(x) + h(t)f(y).
\]

**Definition 14** [18] The function \( f \) on the invex set \( K \subseteq [0, b^*], b^* > 0 \), is said to be \( m - \) preinvex with respect to \( \eta \) if

\[
f(u + t\eta(v, u)) \leq (1 - t)f(u) + mtf\left(\frac{v}{m}\right)
\]

holds for all \( u, v \in K, t \in [0,1] \) and \( m \in (0,1) \). The function \( f \) is said to be \( m - \) preconcave if and only if \(-f\) is \( m \)-preinvex.

2. MAIN RESULT

**Definition 15** Let for \( b^* > 0, [0, b^*] \subseteq \mathbb{R} \) be a invex set with respect to \( \eta \) and \( h: (0,1) \subset J \rightarrow \mathbb{R} \) be a non-negative function. Then \( f \) is said to be \((h, m) -\) preinvex function via \( \eta \), if for all \( x, y \in [0, b^*] \), \( m \in (0,1) \) and \( t \in (0,1) \)

\[
f(x + t\eta(y, x)) \leq h(1 - t)f(x) + mh(t)f(y)
\]

the inequality holds.

**Proposition 1** Let \( f, g \) are \((h, m) -\) preinvex functions in terms of \( \eta \). Then for \( \lambda > 0, \lambda f \) and \( f + g \) are \((h, m) -\) preinvex functions.
Proof. Since f, g are \((h, m)\) –preinvex functions. Thus we can write for all \(x, y \in [0, b^*], b^* > 0, h: (0, 1) \rightarrow \mathbb{R}\) non-negative function and for all \(t \in (0, 1), m \in [0, 1]\)

\[
f(x + t\eta(y, x)) \leq h(1 - t)f(x) + mh(t)f(y)
\]

(2.1)

\[
g(x + t\eta(y, x)) \leq h(1 - t)g(x) + mh(t)g(y)
\]

(2.2)

If we add (2.1) and (2.2), then we get

\[
f(x + t\eta(y, x)) + g(x + t\eta(y, x)) \leq h(1 - t)f(x) + mh(t)f(y) + h(1 - t)g(x) + mh(t)g(y)
\]

\[
(f + g)(x + t\eta(y, x)) \leq h(1 - t)[f(x) + g(x)] + mh(t)[f(y) + g(y)]
\]

Hence \(f + g\) are \((h, m)\) –preinvex functions.

Due to \(\lambda > 0\), if we multiply \(\lambda\) in (5), we have

\[
\lambda f(x + t\eta(y, x)) \leq \lambda h(1 - t)f(x) + \lambda mh(t)f(y)
\]

\[
= h(1 - t)\lambda f(x) + mh(t)\lambda f(y)
\]

\[
= h(1 - t)(\lambda f(x) + mh(t)(\lambda f)(y)).
\]

This completes the proof.

Proposition 2 Let \(f\) and \(g\) be two \((h, m)\) –preinvex functions with respect to \(\eta\). Thus their product \(f, g\) is \((h, m)\) –preinvex function, if \(f\) and \(g\) are similarly ordered functions and

\[
h(1 - t) + mh(t) \leq 1.
\]

Proof. Since \(f\) and \(g\) are \((h, m)\) –preinvex with respect to \(\eta\), (2.1) and (2.2) are hold. If we multiply (2.1) and (2.2), we get

\[
f(x + t\eta(y, x))g(x + t\eta(y, x)) \leq [h(1 - t)f(x) + mh(t)f(y)] \ast [h(1 - t)g(x) + mh(t)g(y)]
\]

\[
(fg)(x + t\eta(y, x)) \leq h^2(1 - t)f(x)g(x) + mh(t)h(1 - t)f(x)g(y)
\]

\[
+ mh(t)h(1 - t)f(y)g(x) + m^2h^2(t)f(y)g(y)
\]

\[
= h^2(1 - t)f(x)g(x) + m^2h^2(t)f(y)g(y)
\]

Then we can rewrite (2.3) from (1.3)

\[
(fg)(x + t\eta(y, x)) \leq h^2(1 - t)f(x)g(x) + mh(t)h(1 - t)f(x)g(y)
\]

\[
+ mh(t)h(1 - t)f(y)g(x) + m^2h^2(t)f(y)g(y)
\]

Due to \(h(1 - t) + mh(t) \leq 1\), then

\[
(fg)(x + t\eta(y, x)) \leq h(1 - t)(fg)(x) + mh(t)(fg)(y).
\]

so the proof completes.

Proposition 3 Let \(h_1\) and \(h_2\) be non-negative functions defined on \([0, b^*] \subseteq \mathbb{R}, b^* > 0\) such that for all \(t \in (0, 1)\)

\[
h_1(t) \leq h_2(t).
\]

If \(f\) is \((h_1, m)\) –preinvex function, then \(f\) is \((h_2, m)\) –preinvex function.

Proof. Since \(f\) is \((h_1, m)\) –preinvex function, we have

\[
f(x + t\eta(y, x)) \leq h_1(1 - t)f(x) + mh_1(t)f(y).
\]

Due to \(h_1(t) \leq h_2(t), \) for all \(t \in (0, 1)\)

\[
f(x + t\eta(y, x)) \leq h_2(1 - t)f(x) + mh_2(t)f(y)
\]

Thus \(f\) is \((h_2, m)\) –preinvex function.
Proposition 4 Let \( h \) be a non-negative function such that for all \( t \in (0,1) \)
\[
t \leq h(t).
\]
If \( f \) is a non-negative \( m \)-preinvex function on \([0,b^*], b^* > 0\) then for all \( x,y \in [0,b^*] \), \( m \in [0,1] \) and \( t \in (0,1) \) \( f \) is \((h, m)\) –preinvex function.

Proof. Because \( f \) is non-negative \( m \)-preinvex, we have
\[
f(x + t\eta(y,x)) \leq (1-t)f(x) + mtf(y).
\]
According to \( t \leq h(t) \), we get
\[
f(x + t\eta(y,x)) \leq (1-t)f(x) + mtf(y)
\leq h(1-t)f(x) + m h(t)f(y).
\]
Hence \( f \) is \((h, m)\) –preinvex function. The proof is completed.

Theorem 1 Let \( M \subseteq [0,b^*], b^* > 0 \) is an invex set. Let \( f: M \rightarrow \mathbb{R} \) be a \( m \)-preinvex function with \( m \in (0,1) \) and \( 0 < a < a + \eta(b,a) \). Let \( \eta \) satisfies condition (C). If \( f \in L_1[a, a + \eta(b,a)] \), then the following inequality holds,
\[
f\left(\frac{2a+\eta(b,a)}{2}\right) \leq \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x) + \frac{mf(\frac{a}{m})}{2} \, dx
\leq \frac{m+1}{4} \left[ f(a) + f(b) \right] + m \frac{f(\frac{a}{m}) + f(\frac{b}{m})}{2}.
\]

(2.4)

Proof. Firstly we prove the left side of (2.4). Due to \( m \)-preinvexity of \( f \) we have for all \( x,y \in [0,\infty) \) and \( t = \frac{1}{2} \)
\[
f\left(\frac{2x + \eta(y,x)}{2}\right) \leq \frac{f(x) + mf\left(\frac{y}{m}\right)}{2}
\]
If we take \( x = a + t\eta(b,a), y = a + (1-t)\eta(b,a) \), we deduce for all \( t \in [0,1] \)
\[
f\left(\frac{2a + \eta(b,a)}{2}\right) \leq \frac{f(a + t\eta(b,a)) + mf\left(\frac{a}{m} + (1-t)\frac{\eta(b,a)}{m}\right)}{2}
= \frac{1}{2} [f(a + t\eta(b,a)) + mf\left(\frac{a}{m} + (1-t)\frac{\eta(b,a)}{m}\right)]
\]
Integrating over \( t \in [0,1] \) we get
\[
f\left(\frac{2a+\eta(b,a)}{2}\right) \leq \frac{1}{2} \left[ \int_0^1 f(a + t\eta(b,a)) \, dt + m \int_0^1 f\left(\frac{a}{m} + (1-t)\frac{\eta(b,a)}{m}\right) \, dt \right].
\]
Taking into account that
\[
\int_0^1 f(a + t\eta(b,a)) \, dt = \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x) \, dx
\]
and
\[
\int_0^1 f\left(\frac{a}{m} + (1-t)\frac{\eta(b,a)}{m}\right) \, dt = \frac{m}{\eta(b,a)} \int_{\frac{a}{m}}^{\frac{a}{m} + \frac{\eta(b,a)}{m}} f(x) \, dx = \frac{1}{\eta(b,a)} \int_a^b f\left(\frac{z}{m}\right) \, dz
\]
Now, we prove the right side of (2.4).
Due to \( m \)-preinvexity we have also for all \( t \in [0,1] \)
\[
\frac{1}{2} [f(a + t\eta(b,a)) + mf\left(\frac{a}{m} + (1-t)\frac{\eta(b,a)}{m}\right)]
\leq \frac{1}{2} [(1-t)f(a) + m(1-t)f\left(\frac{b}{m}\right) + m(1-t)f\left(\frac{a}{m}\right) + m^2 tf\left(\frac{b}{m^2}\right)].
\]
(2.5)

Integrating the inequality (2.5) over \( t \) on \([0,1] \), we deduce
If we choose \( m = 1 \) in (2.4), we obtain the following inequality of Hermite-Hadamard type for preinvex functions [19]:
\[
\frac{1}{\eta(b,a)} \int_{a}^{a+\eta(b,a)} \frac{f(x) + mf\left(\frac{x}{m}\right)}{2} \, dx \leq \frac{m+1}{4} \left[ \frac{f(a) + f(b)}{2} + m \frac{f\left(\frac{a}{m}\right) + f\left(\frac{b}{m}\right)}{2} \right].
\]

This completes the proof.

**Corollary 1** If we choose \( m = 1 \) in (2.4), we obtain the following inequality of Hermite-Hadamard type for preinvex functions [19]:
\[
f\left(\frac{2a + \eta(b,a)}{2}\right) \leq \frac{1}{\eta(b,a)} \int_{a}^{a+\eta(b,a)} f(x) \, dx \leq \frac{f(a) + f(b)}{2}.
\]

**Corollary 2** If we choose \( \eta(b,a) = b - a \) in (2.4), we obtain the following inequality of Hermite-Hadamard type for \( m \) -convex functions [20]:
\[
f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_{a}^{b} f(x) \, dx \leq \frac{f(a) + f(b)}{2}.
\]

**Corollary 3** If we choose \( \eta(b,a) = b - a, m = 1 \) in (2.4), we obtain the following inequality of Hermite-Hadamard type for convex functions;
\[
f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_{a}^{b} f(x) \, dx \leq \frac{f(a) + f(b)}{2}.
\]

**Theorem 2** Let \( f : [0, +\infty) \to \mathbb{R} \) be a \((h,m) - preinvex\) function with \( m \in (0,1], t \in [0,1]. \) Let \( \eta \) satisfies condition (C). If \( 0 < a < a + \eta(b,a) \) and \( f \in L_{1}[a, a + \eta(b,a)] \), then the following inequality holds,
\[
f\left(\frac{2a+\eta(b,a)}{2}\right) \leq \frac{h(1)}{\eta(b,a)} \int_{a}^{a+\eta(b,a)} \left[ f(x) + mf\left(\frac{x}{m}\right) \right] \, dx \leq h\left(\frac{1}{2}\right)\left[ f(a) + m f\left(\frac{a}{m}\right) \right] \int_{0}^{1} h(t) \, dt.
\]

**Proof.** From the definition of \((h,m) - preinvex\) function, we can write for all \( x,y \in [0, \infty), \) and \( t = \frac{1}{2} \),
\[
f\left(\frac{2x+\eta(y,x)}{2}\right) \leq \frac{h(1)}{2} f(x) + mh\left(\frac{1}{2}\right) f\left(\frac{y}{m}\right).
\]

If we choose \( x = a + t \eta(b,a) \) and \( y = a + (1-t) \eta(b,a) \), we get
\[
f\left(\frac{2a+\eta(b,a)}{2}\right) \leq h\left(\frac{1}{2}\right) f(a + t \eta(b,a)) + mh\left(\frac{1}{2}\right) f\left(\frac{a}{m} + (1-t) \frac{\eta(b,a)}{m}\right)
\]
and integrating on \( t \in [0,1], \)
\[
\int_{0}^{1} f\left(\frac{2a+\eta(b,a)}{2}\right) \leq h\left(\frac{1}{2}\right) \int_{0}^{1} f(a + t \eta(b,a)) \, dt + m \int_{0}^{1} f\left(\frac{a}{m} + (1-t) \frac{\eta(b,a)}{m}\right) \, dt
\]
\[
\leq h\left(\frac{1}{2}\right) \left[ \int_{a}^{a+\eta(b,a)} f(x) \, dx \right] \frac{dx}{\eta(b,a)} + m \int_{a}^{a+\eta(b,a)} f\left(\frac{x}{m}\right) \frac{dx}{\eta(b,a)} \leq h\left(\frac{1}{2}\right) \int_{a}^{a+\eta(b,a)} f(x) \, dx + m f\left(\frac{x}{m}\right) \, dx.
\]

We proved the left side of inequality.

Now we take the right side of inequality. Let we take \( x = a + t \eta(b,a) \) in the last inequality and take its integrating on \( t \in [0,1], \).
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\[ h\left(\frac{1}{2}\right) \int_a^{a + \eta(b,a)} \left[ f(x) + mf\left(\frac{x}{m}\right)\right] dx \leq \eta(b,a) \int_0^1 \left[ f(a + t\eta(b,a)) + f\left(\frac{a}{m} + t\eta(b,a)\right)\right] dt 
\leq \eta(b,a) \int_0^1 \left( h(t)f(a) + mh(1-t)f\left(\frac{b}{m}\right)\right) dt 
+ m \int_0^1 \left( h(t)f\left(\frac{a}{m}\right) + mh(1-t)f\left(\frac{b}{m}\right)\right) dt 
\leq \eta(b,a)\int_0^1 h(t)dt 
\]

The proof is completed.

**Corollary 4** If we choose \( m = 1 \) in (2.6), we obtain the following inequality of Hermite-Hadamard type for \( h \)-preinvex functions [21];

\[
\frac{1}{2h\left(\frac{1}{2}\right)} \int_a^{a + \eta(b,a)} f\left(\frac{2a + \eta(b,a)}{2}\right) dx \leq \frac{1}{\eta(b,a)} \int_a^{a + \eta(b,a)} f(x) dx \leq \int_0^1 h(t) dt 
\]

**Corollary 5** If we choose \( h(t) = t \) in (2.6), we obtain the following inequality of Hermite-Hadamard type for \( m \)-preinvex functions;

\[
f\left(\frac{2a + \eta(b,a)}{2}\right) \leq \int_a^{a + \eta(b,a)} \frac{f(x) + mf\left(\frac{x}{m}\right)}{2} dx 
\leq \frac{m+1}{4} \left[ f(a) + f(b) \right] + m \left[ \frac{f^\prime\left(\frac{a}{m}\right)}{2} + \frac{f^\prime\left(\frac{b}{m}\right)}{2} \right] 
\]

**Corollary 6** If we choose \( h(t) = t \) and \( m = 1 \) in (2.6), we obtain the following inequality of Hermite-Hadamard type for preinvex functions [19];

\[
f\left(\frac{2a + \eta(b,a)}{2}\right) \leq \int_a^{a + \eta(b,a)} f(x) dx \leq \frac{f(a) + f(b)}{2} 
\]

**Corollary 7** If we choose \( h(t) = t \) and \( \eta(b,a) = b - a \) in (2.6), we obtain the following inequality of Hermite-Hadamard type for \( m \)-convex functions [20];

\[
f\left(\frac{a+b}{2}\right) \leq \int_a^b \frac{f(x) + mf\left(\frac{x}{m}\right)}{2} dx \leq \frac{m+1}{4} \left[ f(a) + f(b) \right] + m \left[ \frac{f^\prime\left(\frac{a}{m}\right)}{2} + \frac{f^\prime\left(\frac{b}{m}\right)}{2} \right] 
\]

**Corollary 8** If we choose \( m = 1 \) and \( \eta(b,a) = b - a \) in (2.6), we obtain the following inequality of Hermite-Hadamard type for \( h \)-convex functions [22];

\[
\frac{1}{2h\left(\frac{1}{2}\right)} \int_a^{a + b} f\left(\frac{a + b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \int_0^1 h(t) dt 
\]

**Corollary 9** If we choose \( h(t) = t \), \( m = 1 \) and \( \eta(b,a) = b - a \) in (2.6), we obtain the following inequality of Hermite-Hadamard type for convex functions;

\[
f\left(\frac{a+b}{2}\right) \leq \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2} 
\]

**Corollary 10** If we choose \( \eta(b,a) = b - a \) in (2.6), we obtain the following inequality of Hermite-Hadamard type for \( (h,m) \)-convex functions [15];

\[
f\left(\frac{a+b}{2}\right) \leq \int_a^b \frac{f(x) + mf\left(\frac{x}{m}\right)}{2} dx 
\leq h\left(\frac{1}{2}\right) \left[ f(a) + mf\left(\frac{b}{m}\right) + m^2 f\left(\frac{b}{m^2}\right) \right] \int_0^1 h(t) dt 
\]
REFERENCES


