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Review Article

ON THE TRACE FORMULA FOR A DIFFERENTIAL OPERATOR OF SECOND ORDER WITH UNBOUNDED OPERATOR COEFFICIENT

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ABSTRACT

We consider the papers related to the spectrum and the trace formula of a differential operator of second order with unbounded operator coefficient. Reviewing them, we examine what the spectrum is and how to find the trace formula of the operator.
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1. INTRODUCTION

The trace formulae of differential operators can be regarded as a generalization of the traces of matrices or kernel operators. They are very important in mathematics, mathematical physics and mechanics. The trace formulae specially can be used to solve inverse problems and can be applied to index theory of some special operators. These formulae are generally referred as regularized trace formulae for operators.

Some trace formulae for differential operators with operator coefficients are found by Adıgüzelov [1], Chalilova [2], Maksudov et al. [3], Adıgüzelov et al. [4], Albayrak et al. [5], Adıgüzelov et al. [6], Gül [7] and Bakşi et al. [8]. In this work, we review calculations for a differential operator with the same boundary conditions with Bakşi et al. [8].

Let H be a separable Hilbert space and let $H_1 = L_2(H;[0,\pi])$ denotes the set of all measurable functions f with its values in H such that $\int_0^\pi \| f(x) \|_H^2 \, dx. < \infty.$

If the inner product of two arbitrary elements f, g of the space H_1 is defined by

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$$(f,g)=\int_0^{\pi}\left(f(x),g(x)\right)_H dx$$

then it is well kno wn that H_1 becomes also a separable Hilbert space.

Let consider the operators L_0 and L in H_1 defined by the differential expressions

$$L_0(y)=-y''(x)+Ay(x) \quad \text{and} \quad L(y)=L_0(y)+Q(x)y(x)$$

with the same boundary conditions as in $y'(0)=y(\pi)=0$ respectively. Suppose that A and $Q(x)$ in the above expressions satisfy the following conditions:

(1) $A:D(A)\to H$ is a self-adjoint operator. Moreover, $A\geq I$ and $A^{-1}\in\sigma_\infty(H)$ where I is the identity operator on H and $\sigma_\infty(H)$ is the set of all compacts operators from H to H.

(2) For every $x\in[0,\pi]$, $Q(x):H\to H$ is a self-adjoint compact operator. It is also a kernel operator $(Q(x)\in\sigma_1(H))$.

(3) The functions $\left\|Q^{(i)}(x)\right\|_{\sigma_{1(H)}}$ $(i=0,1,2)$ are bounded and measurable in the interval $[0,\pi]$.

(4) For every $f\in H, \int_0^{\pi}\left(Q(x)f,f\right)_H dx=0.$

After this, we will denote the norm in H_1 by $\|.\|_1$ and the sum of a kernel operator K by $trK=\text{trace}\,K.$

2. THE SPECTRUM OF OPERATOR L

Let $\gamma_1\leq\gamma_2\leq\cdots\leq\gamma_n\leq\cdots$ be the eigenvalues of the operatör A and $\varphi_1\leq\varphi_2\leq\cdots\leq\varphi_n\leq\cdots$ be the orthonormal eigenvectors corresponding to these eigenvalues. Moreover, D_0 denotes the set of the functions $y(x)$ in H_1 satisfying the conditions:

(1^0) $y(x)$ has continuous derivative of second order with respect to the norm in the space H in the interval $[0,\pi]$.

(2^0) $Ay(x)$ is continuous with respect to the norm in the space H.

(3^0) $y'(0)=y(\pi)=0$.

Easily D_0 is dense in H_1 and the operator $L_0':D_0\to H_1$ defined by $L_0'=L_0(y)$ is symmetric. The eigenvalues of this operator are

$$\left(\frac{1}{2}+k\right)^2+\gamma_j \quad (k=0,1,2,\cdots;\ j=1,2,\cdots)$$

and the ortho normal eigenvectors corresponding to these eigenvalues are

$$M_k \cos\left(k+\frac{1}{2}\right)x.\varphi_j \quad (k=0,1,2,\cdots; j=1,2,\cdots)$$

$$\text{where } M_k = \sqrt{\frac{2}{\pi}} \text{ for } k=0,1,2,\cdots.$$

Since the orthonormal eigenvectors system of the symmetric operator L_0' is close and is an orthonormal basis H_1, the operator $L_0 = \overline{L_0'}$ will be self-adjoint, and since $Q(x)$ is a bounded, self-adjoint operator from H_1 to H_1 then the operator $L = L_0 + Q$ will be also a self-adjoint operator from $D(L) = D(L_0)$ to H_1.

Let R_λ^0 and R_λ be resolvents of the operators L_0 and L respectively and let $\mu_1 \le \mu_2 \le \cdots \le \mu_n \le \cdots$ be the eigenvalues of operator L_0. Here each eigenvalue is accounted as many times as its multiplicity number. Since $\lim_{j\to\infty} \gamma_j = \infty$, we get $\lim_{n\to\infty} \mu_n = \infty$ and so we have

$$\lim_{n\to\infty} \frac{1}{\mu_n - \mu} = 0 \quad (\mu \ne \mu_n; n=1,2,3,...)$$

From the fact that for every real μ which is not an eigenvalue of L_0, the operator R_μ^0 is self-adjoint and the orthonormal eigenfunctions system

$$M_k \cos\left(k+\frac{1}{2}\right)x.\varphi_j \quad (k=0,1,2,\cdots; j=1,2,\cdots)$$

is complete, the operator R_μ^0 must be compact. Therefore, we conclude that the operator L_0 has pure discrete spectrum. On the ohter hand since Q is bounded and self-adjoint, the spectrum of operator L will be also pure discrete.

Note that if let $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \le \cdots$ be the eigenvalues of operator L, similarly as above, we can see that R_λ is a compact operator for $\lambda \ne \lambda_n (n=1,2,3,...)$. So if $\gamma_j \approx a_j^\alpha$ $(0 < a < \infty, 2 < \alpha < \infty)$ the it can be seen that as $n \to \infty$

$$\mu_n, \lambda_n \approx d_0 n^{\frac{2\alpha}{2+\alpha}}$$

where d_0 is a constant. This asymptotic approach implies that the sequence $\{\mu_n\}_{n=1}^\infty$ has a subsequence $\{\mu_{n_m}\}_{m=1}^\infty$ such that

$$\mu_k - \mu_{n_m} \ge d_1 (k^{\frac{2\alpha}{2+\alpha}} - n_m^{\frac{2\alpha}{2+\alpha}}) \quad (k = n_m, n_{m+1}, ...)$$

Here, with this property, we call the regularized trace formula of operator L to the limit

$$\lim_{m\to\infty}\sum_{k=1}^{n_m}\left(\lambda_k-\mu_k\right).$$

By knowing that R_λ^0 and R_λ are kernel operators we can write the equality

$$tr(R_\lambda - R_\lambda^0) = trR_\lambda - trR_\lambda^0 = \sum_{k=1}^{\infty}\left(\frac{1}{\lambda_k-\lambda}-\frac{1}{\mu_k-\lambda}\right).$$

If we multiply both sides of this equality by $\frac{\lambda}{2\pi i}$ and integrate on the circle $|\lambda| = b_m = 2^{-1}(\mu_{n_m}+\mu_{n_m+1})$ then it follows that

$$\sum_{k=1}^{n_m}(\lambda_k-\mu_k) = -\frac{1}{2\pi i}\int_{|\lambda|=b_m}\lambda tr(R_\lambda - R_\lambda^0)d\lambda .$$

If we substitute the expression

$$R_\lambda - R_\lambda^0 = \sum_{j=1}^{p}(-1)^j R_\lambda^0 (QR_\lambda^0)^j + (-1)^{p+1}R_\lambda (QR_\lambda^0)^{p+1}, \quad (p\ge 2)$$

into the last equation we obtain

$$\sum_{k=1}^{n_m}(\lambda_k-\mu_k) = \sum_{j=1}^{p}D_{mj} + D_m^{(p)} \tag{2.1}$$

where

$$D_{mj} = \frac{(-1)^{j+1}}{2\pi i}\int_{|\lambda|=b_m}\lambda tr\left[R_\lambda^0 (QR_\lambda^0)^j\right]d\lambda \tag{2.2}$$

and

$$D_m^p = \frac{(-1)^p}{2\pi i}\int_{|\lambda|=b_m}\lambda tr\left[R_\lambda (QR_\lambda^0)^{p+1}\right]d\lambda . \tag{2.3}$$

From the relation given in [6], one obtains

$$D_{mj} = \frac{(-1)^j}{2\pi i j}\int_{|\lambda|=b_m} tr\left[(QR_\lambda^0)^j\right]d\lambda . \tag{2.4}$$

If $\{\psi_q(x)\}_{q=1}^{\infty}$ be the system of orthonormal eigenvectors corresponding to the eigenvalues $\{\mu_q(x)\}_{q=1}^{\infty}$ of operator L_0, respectively; then we can easily have

$$\psi_q(x) = M_{k_q}\cos\left(\frac{1}{2}+k_q\right)x.\varphi_{j_k} \quad (q=1,2,\cdots) \tag{2.5}$$

Now, to give an explicit result to the limit of equation (2.1) as $m \to \infty$ we need to evaluate the expressions (2.3) and (2.4) in the sense of same limit. We will do these calculations in the next section.

3. A TRACE FORMULA FOR OPERATOR L

We start firstly with expression (2.4) for $j = 1$.

Theorem 3.1 If the operator function $Q(x)$ satisfies the conditions ((2)-(4)) and if as $j \to \infty$ $\gamma_j \approx a_j^{\alpha}$ $(0 < a < \infty, 2 < \alpha < \infty)$ then

$$\lim_{m\to\infty} D_{m1} = \frac{1}{4}\left[trQ(0) - trQ(\pi)\right]. \tag{3.1}$$

Proof: According to the formula (2.4), we get

$$D_{m1} = \frac{-1}{2\pi i}\int_{|\lambda|=b_m} tr\left[(QR_\lambda^0)\right]d\lambda\ . \tag{3.2}$$

Since $\{\psi_n(x)\}_{n=1}^{\infty}$ is an orthonormal basis in H_1, for every $\lambda \in \rho(L_0)$ we have

$$tr(QR_\lambda^0) = \sum_{q=1}^{\infty}(QR_\lambda^0\psi_q, \psi_q)_1$$

Thus, it follows that

$$D_{m1} = \frac{-1}{2\pi i}\sum_{q=1}^{\infty}(Q\psi_q, \psi_q)_1 \int_{|\lambda|=b_m}\frac{1}{\mu_q - \lambda}d\lambda$$

$$= \sum_{q=1}^{n_m}(Q\psi_q, \psi_q)_1$$

$$= \sum_{q=1}^{n_m}\int_0^{\pi}(Q(x)\psi_q(x), \psi_q(x))_H\,dx$$

$$= \frac{1}{\pi}\sum_{q=1}^{n_m}\int_0^{\pi}\cos(2k_q+1)x(Q(x)\varphi_{j_q}, \varphi_{j_q})_H\,dx$$

From here we conclude that

$$\lim_{m\to\infty} D_{m1} = \frac{1}{\pi}\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\int_0^{\pi}(Q(x)\varphi_j, \varphi_j)_H\cos(2k+1)x\,dx$$

$$= \frac{1}{4}\sum_{j=1}^{\infty}\left\{\sum_{k=1}^{\infty}\left[\frac{2}{\pi}\int_0^{\pi}(Q(x)\varphi_j, \varphi_j)_H\cos(2k+1)x\,dx\right]\cos k0\right.$$

$$\left.-\sum_{k=1}^{\infty}\left[\frac{2}{\pi}\int_0^{\pi}(Q(x)\varphi_j, \varphi_j)_H\cos(2k+1)x\,dx\right]\cos k\pi\right\}$$

The terms appear in $\{\ldots\}$ above are the values at the points 0 and π respectively of the Fourier series of the function $(Q(x)\varphi_j, \varphi_j)_H$, which have second order continuous derivative, with respect to $\{\cos kx\}_{k=1}^{\infty}$ in $[0,\pi]$. Hence, we write

$$\lim_{m\to\infty} D_{m1} = \frac{1}{4}\sum_{j=1}^{\infty}\Big((Q(0)\varphi_j, \varphi_j)_H - (Q(\pi)\varphi_j, \varphi_j)_H\Big)$$

and so the equation (3.1) follows.

Theorem 3.2 Suppose that $\gamma_j \approx a_j^{\alpha}$ as $j \to \infty$ $(0 < a < \infty, 2 < \alpha < \infty)$. If the operator function $Q(x)$ satisfies the conditions (2) and (3) then $\lim_{m\to\infty} D_{mj} = 0$ for $j \geq 2$.

Proof:: From (2.4) we have

$$\left|D_{mj}\right| \leq \frac{1}{2\pi j}\int_{|\lambda|=b_m} \|Q\|_1^j \left\|R_\lambda^0\right\|_{\sigma_{1(H_1)}} \left\|R_\lambda^0\right\|_{\sigma_{1(H_1)}}^{j-1} |d\lambda| \qquad (3.3)$$

If we choose identically $Q(x) \equiv 0$ then we have $R_\lambda = R_\lambda^0$. It means that

$$\|R_\lambda\|_1 < \frac{d_1}{4} n_m^{-\delta}, \qquad \left(\delta = \frac{\alpha-2}{\alpha+2}\right) \qquad (3.4)$$

Since

$$\left\|R_\lambda^0\right\|_{\sigma_{1(H_1)}} < const. n_m^{-\delta}, \qquad \left(\delta = \frac{\alpha-2}{\alpha+2}\right) \qquad (3.5)$$

and the inequalities (3.3) and (3.4) are hold, it follows that

$$\left|D_{mj}\right| \leq const. \int_{|\lambda|=b_m} n_m^{1-\delta} n_m^{-\delta(j-1)} |d\lambda| \leq const. \mu_{n_m} n_m^{1-\delta j}$$

Since $\mu_{n_m} \leq const. n_m^{1+\delta}$ we have

$$\left|D_{mj}\right| \leq const. \mathrm{n}_\mathrm{m}^{2\text{-}\delta(j-1)}$$

Clearly, if $j > 1 + 2\delta^{-1}$ then $\lim_{m\to\infty} D_{mj} = 0$.

By considering the inequalities

$$\left|D_{m2}\right| \leq \|Q\|_1^2 \Omega_\mathrm{m}, \qquad \Omega_\mathrm{m} = \sum_{j=n_m+1}^{\infty}\left(\mu_j - \mu_{n_m}\right)^{-1} \qquad (m = 1,2,\cdots)$$

and

$$\left|D_{m3}\right| \leq \|Q\|_1^3 \Omega_\mathrm{m}\left(\Omega_\mathrm{m} + 4d_1^{-1} n_m^{1-\delta}\right), \quad d_1 = \frac{\mathrm{d}_0}{4}, \delta = \frac{\alpha-2}{\alpha+2}$$

It follows that for $j = 2,3,\cdots,\left|2\delta^{-1}\right| + 1$

$$\lim_{m\to\infty} D_{mj} = 0.$$

Now we are ready to give the main result by next theorem.

Theorem 3.3 Suppose that $\gamma_j \approx a_j^{\alpha}$ as $j \to \infty$ $(0 < a < \infty, 2 < \alpha < \infty)$. If the operator function $Q(x)$ satisfies the conditions ((2)- (4)) then the regularized trace formula

$$\lim_{m\to\infty} \sum_{k=1}^{n_m} (\lambda_k - \mu_k) = \frac{1}{4}\left[trQ(0) - trQ(\pi)\right] \tag{3.6}$$

is satisfied.

Proof: By Theorems 3.1 and 3.2, we write

$$\lim_{m\to\infty} \sum_{k=1}^{n_m} (\lambda_k - \mu_k) = \frac{1}{4}\left[trQ(0) - trQ(\pi)\right] + \lim_{m\to\infty} D_m^{(p)} \tag{3.7}$$

To find the equality (3.6), we just need to show that

$$\lim_{m\to\infty} D_m^{(p)} = 0$$

Let us give a restriction to the magnitude of expression (2.3) :

$$\left|D_m^{(p)}\right| \le b_m \int_{|\lambda|=b_m} \|R_\lambda\|_1 \|Q\|_1^p \|R_\lambda^0\|_1^p \|Q\|_1 \|R_\lambda^0\|_{\sigma_{1(H_1)}} |d\lambda|$$

From (3.4) and (3.5) we obtain that

$$\left|D_m^{(p)}\right| \le const. b_m^2 \mathrm{n}_\mathrm{m}^{-(p+1)\delta} \mathrm{n}_\mathrm{m}^{1-\delta}$$

Since $b_m \le const. \mathrm{n}_\mathrm{m}^{1+\delta}$ then we have

$$\left|D_m^{(p)}\right| \le const. \mathrm{n}_\mathrm{m}^{-(p+2)\delta+1} \mathrm{n}_\mathrm{m}^{2(1+\delta)} = \mathrm{const}. n_m^{3-p\delta}$$

For $p > 3\delta^{-1}$, it follows that

$$\lim_{m\to\infty} D_m^{(p)} = 0$$

If we substitute this result in equation (3.7) we obtain the regularized trace formula of operator L given by equation (3.6).

4. ADDITIONAL COMMENTS

This paper is related to AHA2017.

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